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On the oscillation of nonlinear delay fractional partial differential equations with damping term

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Abstract

In this article we will consider the oscillation of nonlinear delay fractional partial differential equations with damping subject to Neumann boundary conditions.Sufficient conditions are obtained for the oscillation of solutions by using the method of generalized Riccati transformation.

Key words: Oscillation, delay, partial differential equations, damping.

AMS classification: 35B08, 35R10, 35R12.

1 Introduction

The theory of fractional differential equations is considered as an important tool in modeling real life phenomenon. The notion of fractional differential derivative first appeared in the late 17th century. In the recent years, fractional calculus and fractional differential equations are the most rapidly growing area of research. A rigorous and encyclopedia study of fractional differential equation can be found in [9,11].

There are different concepts of fractional derivative such as Riemann Liouville and Caputo derivatives are widely used. Recently several researchers have attracted nonlinear fractional order see references[2,3,13,14,19,24]. In few decades, the problem of oscillation and non oscillation of solution of partial differential equations is one of the area of research in the qualitative theory of partial differential equations.

Partial differential equations are used to model a number of real world problems arising in various branches of science and engineering.[12,15,20,22].

Over the years, the development of oscillation theory has played a major role in

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the physical science and engineering. Well known applications of the theory of oscillations include the oscillations in building and machines, the vibrations in the operation of rocket engine, electromagnetic vibration in radio technology and optical sciences[1,4,5,6,7,8,10,16-18,21,23].

Motivated by the above observation we examine the oscillatory behavior of solutions of time fractional partial differential equations is of the form

$$\frac{\partial}{\partial t}\left[r(t)D^{\alpha}_{+,t}u(x,t)\right] + p(x,t)D^{\alpha}_{+,t}u(x,t) + q(x,t)g\int_0^t (t-s)^{-\alpha}u(x,s)ds$$

$$+ f(t,u(x,\tau(t)),\frac{\partial u}{\partial t}(x,\tau(t))) = a(t)\Delta u(x,t) + b(t)\Delta u(x,\tau(t)) \quad (1)$$

with the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0, (x,t) \in \partial\Omega X R. \quad (2)$$

Before giving results, we introduce some notations.

Let $D = \{(t,s) : -\infty < s \le t < \infty\}$. The function $H(t,s) \in C(D,R)$ is said to belong to the class X if.

(A$_1$) $H(t,t) = 0, \quad H(t,s) > 0$ for $t > s$
(A$_2$) H has partial derivatives $\frac{\partial H(t,s)}{\partial t}$ and $\frac{\partial H(t,s)}{\partial s}$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}$$

where $h_2, h_2 \in L_{loc}(D,R^+)$.

(A$_3$) $r(t) \in C^{'}([0,\infty);[0,\infty)), a \in C([0,\infty)); R_+)$;
(A$_4$) $q(x,t) \in C([0,\infty))$ and $\min_{x\in\Omega} q(x,t) = K(t)$.

By a solution of equation (1) we mean a function $u(x,t) \in C^{1+\alpha}(\bar{\Omega}X[0,\infty))$ such that

$$\int_0^t (t-s)^{\alpha}u(x,s)ds \in C^{'}(\bar{G};R), \quad D^{\alpha}_{+,t}u(x,t) \in C^{'}(\bar{G};R)$$

and satisfy as $\bar{G}$.

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2 Preliminaries

The following notations will be used for our convenience

$$U(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx, \quad where \quad |\Omega| = \int_{\Omega} dx.$$

Definition 2.1 The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\lceil(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(x,s)ds \tag{3}$$

provided the right hand side is point wise defined on R_+ where $\lceil$ is the gamma function.

Definition 2.2 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y = R_+ \to R$ on the half axis R_+ is given by

$$(I^{\alpha}_{+}y)(t) := \frac{1}{\lceil(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \quad for \quad t > 0 \tag{4}$$

provided the right hand side is point wise defined on R_+.

Definition 2.3 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y = R_+ \to R$ on the half axis R_+ is given by

$$(D^{\alpha}_{+}y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I^{\lceil\alpha\rceil-\alpha}_{+}y\right)(t)$$

$$= \frac{1}{\lceil(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1}y(s)ds \quad for \quad t > 0 \tag{5}$$

provided the right hand side is point wise defined on R_+ where $\lceil\alpha\rceil$ is the ceiling function of α.

Lemma 2.4 Let y be a solution of (1) and

$$K(t) := \int_0^t (t-s)^{-\alpha} y(s)ds \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{6}$$

Then

$$K^{'}(t) = \lceil(1-\alpha)(D_+^{\alpha} y)(t)$$

Proof: From (3) and (4) for $\alpha \in$ (0,1) and $t > 0$, we obtain

$$K^{'}(t) = \Gamma(1-\alpha)\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} y(s)ds$$

$$= \Gamma(1-\alpha)\left[\frac{1}{\lceil(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1} y(s)ds\right]$$

$$= \Gamma(1-\alpha)\left(D_+^{\alpha} y\right)(t). \tag{7}$$

This completes the proof.

The following two lemmas will be useful in working with nonlinear differential equations.

Lemma 2.5 Assume that

(B_1) $\quad p \in C([t_0,\infty),[0,\infty)), \lim_{t\to\infty}\int_{\bar{t}}^t exp\left(-\int_{\bar{t}}^s p(\tau)d\tau\right)ds = \infty$

$for \quad every \quad \bar{t} \geq t_o$

(B_2) $\quad \tau(t), \sigma(t) \in C([t_0,\infty),R), \quad for \quad t \geq t_0 \quad and \quad \lim_{t\to\infty}\tau_i(t) = \infty (i = 1,2,...,m)$

(B_3) $\quad f \in C([t_0,\infty)\times R^{2m}, R)$

satisfies the one-side estimate $f(t, U(\tau(t)), U'(\tau(t)))$ sign $x \geq q(t)\mu x$, $x \geq 0$ where μ is a non negative constants and $\mu > 0$ $q \in C([t_0, \infty), [0, \infty))$ and $q(t)$ is not identically zero on any ray $(t^*, \infty) \subset [t_0, \infty)$. Then $x(t)$ is a non oscillatory solution of equation (1), we have

$$u(t) D^{\alpha}_{+,t} u(t) > 0, \quad for \quad all \quad large \quad t.$$

Lemma 2.6 Assume that $x(t) \in C^2([t_0, \infty), R)$ satisfies $u(t) > 0$, $D^{\alpha}_{+,t} u(t) > 0$

$$\frac{\partial}{\partial t}\left[D^{\alpha}_{+,t} u(t)\right] \leq 0, \quad t \geq t_0$$

then for each $0 < k < 1$, then there exists $T \geq t_0$, such that

$$x(\tau(t)) \geq K x(t) \frac{\tau_i(t)}{t}, \quad t \geq T.i = 1, 2, ..., m.$$

3 Main Results

Theorem 3.1 If the fractional differential inequality

$$\frac{\partial}{\partial t}[r(t) D^{\alpha}_{+} U(t)] + p(t) D^{\alpha}_{+}(U(t)) + Q(t) g(K(t)) + f(t, U(\tau(t)), U^{'}(\tau(t))) \leq 0 \qquad (8)$$

has no eventually positive solution, then every solutions of (1) and (2) is oscillatory in G.

Proof:Suppose that u is a non oscillatory solution of (1) and (2). Without loss of generality, we may assume that $u(x, t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$.

Integrating (1) over Ω we obtain

$$\frac{d}{dt}\left[r(t) \int_{\Omega} D^{\alpha}_{+,t} u(x, t)\right] dx + \int_{\Omega} p(x, t) D^{\alpha}_{+,t} u(x, t) dx + \int_{\Omega} q(x, t) g \int_0^t (t-s)^{-\alpha} u(x, s) ds.dx$$

$$+ f\left[t, u(x, \tau(t)), \frac{\partial u}{\partial x}(x, \tau(t))\right] dx = a(t) \int_{\Omega} \Delta u(x, t) dx + b(t) \int_{\Omega} \Delta u(x, \tau(t)) dx. \qquad (9)$$

Using Green's formula, it is obvious that

$$\int_{\Omega} \Delta u(x, t) dx = 0, \quad t \geq t_0 \quad and \quad \int_{\Omega} \Delta u(x, \tau(t)) dx = 0, \quad t \geq t_1. \qquad (10)$$

By using Jensen's inequality and (B_2), we have

$$\int_\Omega q(x,t) g \int_0^t (t-s)^{-\alpha} u(x,s) ds + \int_\Omega f(t, u(x,\tau(t)), \frac{\partial u}{\partial x}(x,\tau(t))) dx$$

$$\geq |\Omega| \, Q(t) g(K(t)) + \int_\Omega f(t, u(x,\tau(t)), \frac{\partial u}{\partial x}(x,\tau(t))) dx$$

$$= |\Omega| \, Q(t) g(K(t)) + f(t, \int_\Omega u(x,\tau(t)) \int_\Omega \frac{\partial u}{\partial x}(x,\tau(t))) dx$$

$$= |\Omega| \left[Q(t) g(K(t)) + f(t, U(t), U^{'} \tau((t))) \right] \quad (11)$$

combining (9) and (11) and using definition, we have

$$\frac{d}{dt}(r(t) D_+^\alpha U(t)) + p(t) D_+^\alpha (U(t)) + Q(t) g(K(t)) + f(t, U(\tau(t)), U^{'}(\tau(t))) \leq 0.$$

Therefore $u(t)$ is eventually positive solution of (8). This contradicts the hypothesis and completes the proof.

Theorem 3.2 In equation (1) suppose the condition $(B_1) - (B_3)$ hold. If each sufficiently large $T \geq t_0$, there exist $h(t) \in C^{'}([t_0, \infty), R) H \in X$ and $a, b, c \in R$ such that $T \leq a < c < b$ and

$$\frac{1}{H(b,c)} \int_c^b H(b,s)\Psi(s) - \frac{1}{4} c(s) \left[\frac{h_2(b,s) + \sqrt{H(b,s)} \left(2h(s) + \frac{p(s)}{r^2(s)} \right)}{q^{'}(K(s))\Gamma(1-\alpha} \right]^2 r(s) ds +$$

$$\frac{1}{H(c,a)} \int_a^c H(s,a)\Psi(s) - \frac{1}{4} c(s) \left[\frac{h_2(s,a) - \sqrt{H(s,a)} \left(2h(s) + \frac{p(s)}{r^2(s)} \right)}{q^{'}(K(s))\Gamma(1-\alpha)} \right]^2 r(s) ds \quad (12)$$

where

$$c(t) = \exp\left(-2 \int^t m(s) ds \right),$$

$$\Psi(t) = c(t) \Bigg\{ \frac{p(t) m(t)}{r^2(t)} + q(t)\mu u(t) + Q(t) g(K(t)) + q^{'}(K(t))\Gamma(1-\alpha)$$

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$$\left[m^{'}(t)-\frac{m^2(t)}{r(t)}+\frac{2\omega(t)m(t)}{r(t)}\right]\Bigg\} \tag{13}$$

then equation (1) is oscillatory.
Proof:Let $u(t)$ be a nonoscillatory solution of equation (1) without loss of generality, we may assume that $u(t)>0$ on $[T_0,\infty)$ for $T_0\geq t_0$. As $\lim_{t\to\infty}\tau(t)=\infty$, there exist $T'\geq T_0$ such that $\tau(t)\geq T_0, t\geq T^{'}$. Hence $x(\tau(t))>0, t\geq T$. By lemma (2.5) we can obtain $\frac{\partial}{\partial t}\left[D^{\alpha}_{+,t}u(t)\right]\leq 0$ for $t\geq T^{''}$.

Define

$$w(t)=c(t)\left[\frac{r(t)D^{\alpha}_{+,t}(u(t))}{g(K(t))}+h(t)\right],\quad t\geq T. \tag{14}$$

From (1), we have

$$w^{'}(t)=-2h(t)w(t)+c(t)\left[\frac{-p(t)r(t)D^{\alpha}_{+}u(t)}{g[K(t)]}-f(t,U(\tau(t),U^{'}(\tau(t)))-Q(t)g(K(t))\right]$$

$$c(t)\left[h^{'}(t)-\frac{D^{\alpha}_{+}U(t)r(t))^2}{g(K(t))^2}g^{'}(K(t))\Gamma(1-\alpha)\right] \tag{15}$$

$$\leq -2h(t)w(t)+c(t)\left\{\frac{-p(t)}{r^2(t)}\left[\frac{w(t)}{c(t)}-h(t)\right]-q(t)\mu u(t)+Q(t)g(K(t))\right\}$$

$$+c(t)\left\{h^{'}(t)-\frac{\left(\frac{w(t)}{c(t)}-h(t)\right)^2q^{'}(K(t))\Gamma(1-\alpha)}{r(t)}\right\}$$

From Lemma (2.6) there exists $T^{'''}\geq T^{''}$, making latter inequality yields

$$w^{'}(t)\leq -\Psi(t)-w(t)\left[2h(t)+\frac{p(t)}{r^2(t)}\right]-\frac{w^2(t)q^{'}(K(t))\Gamma(1-\alpha)}{r(t)c(t)} \tag{16}$$

where $\Psi(t)$ is defined by (13)

Setting $T\geq T^{'''}$, multiplying (16) by $H(t,s)$ integrating it with respect to s from

c to t for $t \in [c, b)$using (A_1) and (A_2), we get

$$\int_c^t H(t,s)\Psi(s)ds \leq -\int_c^t H(t,s)w^{'}(s)ds - \int_c^t H(t,s)\omega(s)\left[2h(s) + \frac{p(s)}{r^2(s)}\right]ds$$

$$-\int_c^t H(t,s)w^2(s)\left[\frac{q^{'}(K(s))\Gamma(1-\alpha)}{r(s)c(s)}\right]ds$$

$$= H(t,t)w(t) - \int_c^t [-\frac{\partial H}{\partial s}(t,s)w(s)ds + H(t,s)w(s)(2h(s) + \frac{p(s)}{r^2(s)}$$

$$+H(t,s)w^2(s)\frac{q^{'}(K(s))\Gamma(1-\alpha)}{r(s)c(s)}]ds$$

$$= H(t,t)w(t) - \int_c^t [h_2(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w(s)\left[2h(s) + \frac{p(s)}{r^2(s)}\right]$$

$$+ H(t,s)w^2(s)\frac{q^{'}(K(s))\Gamma(1-\alpha)}{r(s)c(s)}]ds \tag{17}$$

$$= H(t,t)w(t) - \int_c^t \left\{w(s)\sqrt{\frac{H(t,s)q^{'}(K(s))\Gamma(1-\alpha)}{r(s)c(s)}}\right.$$

$$\left. +\frac{1}{2}\frac{\left[h_2(t(s)) + \sqrt{H(t,s)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)\right]^2}{q^{'}(K(s)\Gamma(1-\alpha)}r(s)c(s)\right\}^2 ds$$

$$+\frac{1}{4}\int_c^t \left[\frac{h_2(t(s)) + \sqrt{H(t,s)}\left(2h(s)) + \frac{p(s)}{r^2(s)}\right)}{q^{'}(K(s))\Gamma(1-\alpha)}\right]^2 r(s)c(s)ds$$

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$$\leq H(t,t)w(t) + \frac{1}{4}\int_c^t r(s)c(s)\left[\frac{h_2(t,s)+\sqrt{H(t,s)}\left(2h(s)+\frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2$$

Let $t \to \bar{b}$ in the above, dividing by $H(b,c)$ on both sides, we get

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_2(b,s)+\sqrt{H(b,s)}\left(2h(s)+\frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds \leq w(c) \tag{18}$$

Similarly, multiply (16) by $H(s,t)$, integrate it with respect to s from t to c for $t \in (a,c]$ and we (A_1) and (A_2), we get

$$\int_t^c H(s,t)\Psi(s)ds \leq -\int_t^c H(s,t)w'(s)ds - \int_t^c H(s,t)w(s)\left[2h(s)+\frac{p(s)}{r^2(s)}\right]ds$$

$$-\int_t^c \frac{H(s,t)w^2(s)q'(K(s))\Gamma(1-\alpha)}{r(s)c(s)}ds.$$

$$= -H(c,t)w(c) - \int_t^c \Bigg\{ -\frac{\partial H(s,t)}{\partial t}w(s) + H(s,t)w(s)\left(2h(s)+\frac{p(s)}{r^2(s)}\right)$$

$$+\frac{H(s,t)q'(K(s))\sqrt{(1-\alpha)}}{r(s)c(s)}w^2(s)ds\Bigg\}$$

$$= -H(c,t)w(c) + \int_t^c \Bigg\{ \left[h_1(s,t)\sqrt{H(s,t)} - H(s,t)\left(2h(s)+\frac{p(s)}{r^2(s)}\right)\right]w(s)$$

$$-\frac{H(s,t)q'(K(s))\Gamma(1-\alpha)}{r(s)c(s)}w^2(s)ds\Bigg\} \tag{19}$$

$$= -H(c,t)w(c) - \int_t^c \left\{ w(s)\sqrt{\frac{q'(K(s))\Gamma(1-\alpha)H(s,t)}{r(s)c(s)}} + \frac{1}{2}\frac{h_1(s,t) - \sqrt{H(s,t)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{\sqrt{\frac{q'(K(s))\Gamma(1-\alpha)}{r(s)c(s)}}} \right\}^2$$

$$+\frac{1}{4}\int_t^c c(s)\left[\frac{h_1(s,t) - \sqrt{H(s,t)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)$$

$$\leq -H(c,t)w(c) + \frac{1}{4}\int_t^c c(s)\left[\frac{h_1(s,t) - \sqrt{H(s,t)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds$$

Let $t \to a^+$ in the above, dividing both side by $H(c,a)$, we get

$$\frac{1}{H(c,s)}\int_a^c H(s,a)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_1(s,a) - \sqrt{H(s,a)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds \leq -w(c). \tag{20}$$

Now, we state that $x(t)$ has atleast one zero (a,b). Otherwise adding (18) and (15) would yield an inequality which contradicts the assumptions (8). Pick a sequence $T \leq \tau_1 \leq \tau_2 < \ldots$ Satisfying $\tau_n \to \infty$ as $n \to \infty$ for each $n \leq N$, there exists $a_n, c_n, b_n \in R$ such that $\tau_n \leq a_n < c_n < b_n$ and

$$\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n} H(b_n,s)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_2(b_n,s) + \sqrt{H(b_n,s)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds +$$

$$\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_1(s,a_n) - \sqrt{H(s,a_n)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q'(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds > 0 \tag{21}$$

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According to the above result, any solution $x(t)$ of equation (1) has atleast one zero (a_n, b_n). Taking into to account that $a_n \to +\infty$ and $b_n \to +\infty$ as $n \to \infty$. We see that every solution has arbitrary large zero. Thus, every solution of equation (1) is oscillatory which completes the proof.

Theorem 3.3 Assume that $(B_1)-(B_2)$ hold suppose for each sufficiently large $t \geq t_0$ there exist $H \in x$ and $g(t) \in C^{'}([t_0, \infty), R)$ such that

$$\limsup_{t\to\infty} \int_l^t H(s,l)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_1(s,l) - \sqrt{H(s,l)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q^{'}(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds > 0 \tag{22}$$

and

$$\limsup_{t\to\infty} \int_l^t H(t,s)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_2(t,s) + \sqrt{H(t,s)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q^{'}(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds > 0 \tag{23}$$

where $c(t)$, $\Psi(t)$ we defined in Theorem 3.1 then every solution of equation (8) is oscillatory.

Proof:For any sufficiently large $T \geq t_0$, let $a = T$ in (22) we choose $l = a$, then there exist $c > a$ such that

$$\int_a^c H(s,a)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_1(s,a) - \sqrt{H(s,a)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q^{'}(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds > 0 \tag{24}$$

Setting $l = c$ in (23). Then there exist $b > c$ such that

$$\int_c^b H(b,s)\Psi(s) - \frac{1}{4}c(s)\left[\frac{h_2(b,s) + \sqrt{H(b,s)}\left(2h(s) + \frac{p(s)}{r^2(s)}\right)}{q^{'}(K(s))\Gamma(1-\alpha)}\right]^2 r(s)ds > 0 \tag{25}$$

Combining (24) and (25), we get (8). The conclusion then comes from Theorem 3.1 which completes the proof.

4 Conclusion

In this article, we have investigated oscillation results for partial differential equations with Neumann boundary conditions. By using Riccati transformation, we establish the solution for the oscillatory. Our newly obtained results in this paper have improved and extended some of the results already prevailing in the existing literature.

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