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Asymptotic and Boundedness Behaviour of a Second Order Difference Equation

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Abstract

In this paper, we study the asymptotic behaviour and boundedness of the solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}\lambda^{-x_n}, \quad n = 0, 1, 2, \dots \tag{1}$$

where $\lambda > 1$ and $\alpha > 0, \beta > 0$ are the immigration rate and population growth respectively and initial conditions x_{-1}, x_0 are arbitrary positive numbers.

Key words: Equilibrium point, Local asymptotic stability, Global asymptotic stability, Positive solutions, Boundedness.

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Introduction

Difference equation containing exponential terms have many applications in biology, there are many papers dealing with such equations. Evolution of a perennial grass depends on the biomass, the litter mass and the total soil nitrogen was described by the difference equations

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}},\ L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}} \tag{2}$$

where B is the living biomass, L the litter mass, N the total soil nitrogen, t the time and constants $a, b, c, d > 0$ and $0 < k < 1$. Oscillatory and chaotic nature of (2) was discussed in [14].

Global stability, boundedness nature and periodic character of the positive solution of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}, \qquad n = 0, 1, 2, \dots \tag{3}$$

was investigated by El-Metwally et all [9], where $\alpha > 0$ and $\beta > 0$ are the immigration rate and population growth respectively and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers.

Existence, uniqueness and attractivity of prime period two solution for (3) was discussed by Fotiades et all [10].

Boundedness and global asymptotic behavior of the solution of the difference equations

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}}, \qquad n = 0, 1, 2, \dots$$

and

$$x_{n+1} = \frac{\alpha e^{-(nx_n + (n-k)x_{n-k})}}{\beta + x_n + (n-k)x_{n-k}}, \qquad n = 0, 1, 2, \dots$$

were studied by Ozturk et all [11, 12], where α and β are positive numbers $k \in \{1, 2, 3, \dots\}$ and the $x_{-k}, x_{-(k-1)}, \dots, x_{-1}, x_0$ are arbitrary numbers.

Boundedness and the persistence of the positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions concerning the biological model

$$x_{n+1} = \frac{ax_n^2}{x_n+b} + c\frac{e^{k-dx_n}}{1+e^{k-dx_n}}$$

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was established in [13], where $0 < a < 1, b, c, d, k$ are positive constants and x_0 is a real number.

Stability analysis of a nonlinear difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}},\ n = 0, 1, 2, \ldots$$

was established in [2], where α, β and initial conditions are arbitrary positive numbers.

Motivated by above studies, we generalize (3) and investigate the global attractivity and boundedness of the solutions of the difference equations (1) for $\lambda > 1$.

Preliminaries

Definition 2.1. *[7] Let $I \in \mathbb{R}$ and let $f : I \times I \to I$ be a continuous function. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \quad n = 0, 1, 2, \cdots . \tag{4}$$

for the initial conditions $x_0, x_{-1} \in I$. We say that $\bar{x}$ is an equilibrium of equation (4) if $\bar{x} = f(\bar{x}, \bar{x})$.

That is, the constant sequence $\{x_n\}_{n=-1}^{\infty}$ with $x_n = \bar{x}$ for all $n \geq -1$ is a solution of equation (4).

Definition 2.2. *[7]*

(i) The equilibrium $\bar{x}$ of equation (4) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x_{-1},\ x_0 \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all $n \geq -1$.

(ii) The equilibrium $\bar{x}$ of equation (4) is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that $x_{-1},\ x_0 \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.

(iii) The equilibrium $\bar{x}$ of equilibrium (4) is called a global attractor if for every $x_{-1}, x_0 \in I$ we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) The equilibrium $\bar{x}$ of equation (4) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) The equilibrium $\bar{x}$ of equilibrium (4) is called unstable if it is not stable.

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Definition 2.3. *[7] Let $f(u,v)$ be continuously differentiable function. Let $p = \dfrac{\partial f}{\partial u}(\bar{x},\bar{x})$ and $q = \dfrac{\partial f}{\partial v}(\bar{x},\bar{x})$ denote the partial derivatives of $f(u,v)$ evaluated at an equilibrium $\bar{x}$ of equation (4). Then the equation*

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0,1,2,\cdots \tag{5}$$

is called the linearized equation associated with the equation (4) about the equilibrium point $\bar{x}$.

The characteristic equation of equation (5) is the equation

$$\mu^2 - p\mu - q = 0 \tag{6}$$

with characteristic roots $\mu_{\pm} = \dfrac{p \pm \sqrt{p^2+4q}}{2}$.

Theorem 2.4. *[7][Linearized Stability]*

(a) If both roots of the quadratic equation (6) lie in the open unit disk $|\mu| < 1$, then the equilibrium $\bar{x}$ of (4) is locally asymptotically stable.

(b) If atleast one of the roots of (6) has absolute value greater than one, then the equilibrium $\bar{x}$ of (4) is unstable.

(c) A necessary and sufficient condition for both roots of (6) to lie in the open unit disk $|\mu| < 1$, is $|p| < 1 - q < 2$. In this case the locally asymptotically stable equilibrium $\bar{x}$ is also called a sink.

(d) A necessary and sufficient condition for one root of (6) to have absolute value greater than one and for the other to have absolute value less than one is $p^2 + 4q > 0$ and $|p| > |1 - q|$. In this case unstable equilibrium point $\bar{y}$ is called a saddle point.

(e) A necessary and sufficient condition for a root of (6) to have absolute value equal to one is $|p| = |1 - q|$ or $q = -1$ and $|p| \leq 2$. In this case the equilibrium $\bar{y}$ is called a nonhyperbolic point.

(iv) A necessary and sufficient condition for both roots of (6) to have absolute value greater than one is $|q| > 1$ and $|p| < |1 - q|$. In this case $\bar{y}$ is called a repeller.

Theorem 2.5. *[5] Assume $\alpha \in (0,\infty), \beta \in \mathbb{R}$. Let x_n and y_n be sequence of real numbers such that x_{-1}, x_0 and $x_{-1} \leq y_{-1}, x_0 \leq y_0$ and $x_{n+1} \leq \alpha x_{n-1} + \beta$ and $y_{n+1} = \alpha y_{n-1} + \beta$. Then $x_n \leq y_n$ for $n \geq -1$.*

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Theorem 2.6. *[6] Suppose that f satisfies the following conditions.*
(i). There exists positive numbers a and b with $a < b$ such that $a \leq f(x,y) \leq b$ for all $x, y \in [a,b]$.
(ii). $f(x,y)$ is decreasing in $x \in [a,b]$ for each $y \in [a,b]$ and increasing in $y \in [a,b]$ for each $x \in [a,b]$.
(iii). Equation (4) has no solution of prime period two in $[a,b]$.
Then there exists exactly one equilibrium solution $\bar{x}$ of (4) which lies in $[a,b]$.
Moreover every solution of 4 with initial conditions $x_{-1}, x_0 \in [a,b]$ converges to $\bar{x}$.

Theorem 2.7. *[9] Assume that $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ satisfies the following conditions. $f(x,y)$ is decreasing in x for each $y \in (0,\infty)$ and $f(x,y)$ is increasing in y for each $x \in (0,\infty)$. There exists positive real numbers $0 < m < M$ and $\delta > 0$ such that all $i \geq 0$, $f(M,m) \leq m$ and $f(m, M + i\delta) \geq M + (i+1)\delta$. Let (x_n) be a positive solution of (1) such that $x_{-1} \leq m$ and $x_0 \geq m$, then $x_{2n-1} \leq m$ for all $n \geq 1$ and $\lim_{n\to\infty} x_{2n} = \infty$.*

Main Results

In this section, we discuss the local, global asymptotic stability and boundedness of the solutions of equation (1).

The equilibrium points of equation (1) are the solutions of the equation

$$\bar{x} = \alpha + \beta \bar{x} \lambda^{-\bar{x}} \tag{7}$$

Set,

$$g(x) = \alpha + \beta x \lambda^{-x} - x, \tag{8}$$

we get $g(0) = \alpha > 0$, $g'(x) = \beta\lambda^{-x}(1 - x\ln\lambda) - 1$ and when $\lambda > 1$, $\lim\limits_{x\to\infty} g(x) = -\infty$. This gives us that equation (1) has a unique equilibrium $\bar{x}$.

Theorem 3.1. *Equilibrium point $\bar{x}$ of (1) is locally asymptotically stable if*

$$\beta < \frac{-\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}} \lambda^{\frac{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{2\ln\lambda}}. \tag{9}$$

and unstable if

$$\beta > \frac{-\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}} \lambda^{\frac{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{2\ln\lambda}}. \tag{10}$$

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Proof. From Theorem 2.4, we have $\beta\lambda^{-\bar{x}} < \dfrac{1}{1+\bar{x}\ln\lambda}$.
Substituting in (7) we get, $\bar{x} < \dfrac{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{2\ln\lambda}$.
Again from (7), we get $\beta < \dfrac{-\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}\lambda^{\frac{\alpha\ln\lambda+\sqrt{\alpha^2\ln^2\lambda+4\alpha\ln\lambda}}{2\ln\lambda}}$
Also from Theorem 2.4, we have $\beta\lambda^{-\bar{x}} > \dfrac{1}{1+\bar{x}\ln\lambda}$.
Substituting in (7) we get,$\bar{x} > \dfrac{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{2\ln\lambda}$.
And from (7), we get $\beta > \dfrac{-\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}{\alpha\ln\lambda + \sqrt{\alpha^2\ln^2\lambda + 4\alpha\ln\lambda}}\lambda^{\frac{\alpha\ln\lambda+\sqrt{\alpha^2\ln^2\lambda+4\alpha\ln\lambda}}{2\ln\lambda}}$. □

Theorem 3.2. *Assume $\lambda > 1$. Every positive solution of (1) is bounded if $\beta < \lambda^\alpha$ and unbounded if $\beta > \lambda^\alpha$.*

Proof. Suppose $\beta < \lambda^\alpha$.
Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1).
Then $x_{n+1} \leq \alpha + \beta x_{n-1}\lambda^{-\alpha}$.
Consider the Initial Value Problem $y_{n+1} = \alpha + \beta y_{n-1}\lambda^{-\alpha}$ with conditions $y_0 = x_0$ and $y_1 = x_1$.
Then by Theorem (2.5), $x_n \leq y_n$ for every $n \geq 0$.

$$\lim y_{n-1} = \frac{\alpha}{1-\beta\lambda^{-\alpha}} \text{ so } \lim_{n\to\infty}\sup x_n \leq \frac{\alpha}{1-\beta\lambda^{-\alpha}}$$

$\Rightarrow \{x_n\}$ is bounded.
Suppose $\beta > \lambda^\alpha$.
Let $m = \log_\lambda \beta, M = \log_\lambda \frac{\beta m}{m-\alpha}, \delta = \alpha$.
Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1) such that $x_{-1} \leq m$ and $x_0 \geq M$.
Then from Theorem 2.7, we get $\lim_{n\to\infty} x_{2n} = \infty$ and $\lim_{n\to\infty} x_{2n+1} = \alpha$. □

Theorem 3.3. *Assume $\lambda > 1$. Suppose $\beta < \lambda^\alpha$. The set $[\alpha, \frac{\alpha}{1-\beta\lambda^{-\alpha}}]$ is invariant for $x_{n+1} = \alpha + \beta x_{n-1}\lambda^{-x_n}$*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1) with $x_0, x_1 \in [\alpha, \frac{\alpha}{1-\beta\lambda^{-\alpha}}]$.
Then $\alpha \leq x_1 \leq \alpha + \beta\frac{\alpha}{1-\beta\lambda^{-\alpha}}\lambda^{-\alpha}$.
The result follows by induction.

□

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Lemma 3.4. *We assume* $1 < \lambda < e$*. Suppose that* $\beta < \lambda^{\alpha}\left(\frac{-\alpha+\sqrt{\alpha^2+4(\ln\lambda)^2}}{2\ln\lambda}\right)$ *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of (1). Then* $\lim_{n\to\infty}\sup x_n \leq \dfrac{\alpha}{1-\beta\lambda^{-\alpha}}$.

Proof. The proof follows from $\frac{-\alpha+\sqrt{\alpha^2+4(\ln\lambda)^2}}{2\ln\lambda} < 1$ and Theorem 3.2. □

Lemma 3.5. *We assume* $1 < \lambda < e$*. Suppose that* $\beta \leq \min\left(\lambda^{\alpha}\left(\frac{-\alpha+\sqrt{\alpha^2+4(\ln\lambda)^2}}{2\ln\lambda}\right), \frac{1-\sqrt{1-(\ln\lambda)^4}}{\lambda^{-\alpha}(\ln\lambda)^2}\right)$*. Then (1) has no positive solution of prime period two.*

Proof. Let $x, y \in (\alpha, \infty)$.
Then from (1), we write $x = \alpha + \beta x\lambda^{-y}, y = \alpha + \beta y\lambda^{-x}$.
Then $y = \log_\lambda \frac{\beta x}{x-\alpha}$ and $x = \log_\lambda \frac{\beta y}{y-\alpha}$ then

$$\log_\lambda \frac{\beta x}{x-\alpha}(1-\beta\lambda^{-x}) = \alpha = \log_\lambda \frac{\beta y}{y-\alpha}(1-\beta\lambda^{-y}).$$

Set $F(z) = \log_\lambda \frac{\beta z}{z-\alpha}(1-\beta\lambda^{-z}) - \alpha$.
Since $\bar{x}$ is a solution of (1) , $F(\bar{x}) = 0$ and $\bar{x} > \alpha$.
We show that $\bar{x}$ is the only solution of (1).
To show that $F(z)$ has exactly one z-intercept greater that α.
Let $z > \alpha$ be such that $F(z) = 0$.
The proof will follow by showing that $F'(z) < 0$

$$F'(z) = \frac{-\alpha\ln\lambda(1-\beta\lambda^{-z})}{z(z-\alpha)} + \log_\lambda(\frac{\beta z}{z-\alpha})\beta\lambda^{-z}\ln\lambda.$$

Clearly $\log_\lambda \dfrac{\beta z}{z-\alpha}(1-\beta\lambda^{-z}) - \alpha = F(z) = 0$

$$\log_\lambda \frac{\beta z}{z-\alpha} = \frac{\alpha}{1-\beta\lambda^{-z}}$$

$\Rightarrow F'(z) = \alpha\ln\lambda\left(\dfrac{-[1-\beta\lambda^{-z}]}{(z-\alpha)z} + \dfrac{\beta\lambda^{-z}}{1-\beta\lambda^{-z}}\right)$.
Claim : $\beta d^2 - \beta\alpha d < \lambda^d + \beta^2\lambda^{-d} - 2\beta$ for all $d \geq \alpha$.
For $d \geq \alpha$, set $G(d) = \lambda^d + \beta^2\lambda^{-d} - 2\beta$ and $H(d) = \beta d^2 - \beta\alpha d$.
Then, $H(\alpha) = 0$ and $G(\alpha) = \lambda^{-\alpha}(\lambda^{\alpha} - \beta)^2 > 0$.

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Hence if suffices to show that $G'(d) > H'(d)$ for all $d \geq \alpha$.
Here $G'(d) = \lambda^d \ln\lambda - \beta^2\lambda^{-d}\ln\lambda$ and $H'(d) = 2\beta d - \beta\alpha$.
Since $\beta \leq \lambda^{\alpha}\left(\frac{-\alpha+\sqrt{\alpha^2+4\ln\lambda^2}}{2\ln\lambda}\right)$, we get $G'(\alpha) \geq H'(\alpha)$
and hence it is enough to show that $G''(d) > H''(d)$ for all $d \geq \alpha$.
Now, $G''(d) = (\ln\lambda)^2\lambda^d + \beta^2\lambda^{-d}(\ln\lambda)^2$ and $H''(d) = 2\beta$.
Since $\beta \leq \lambda^{\alpha}\left(\frac{-\alpha+\sqrt{\alpha^2+4\ln\lambda^2}}{2\ln\lambda}\right)$, we get $G''(\alpha) \geq H''(\alpha)$
and hence it is enough to show that $G'''(d) > H'''(d)$ for all $d \geq \alpha$.
Here $G'''(d) = (\ln\lambda)^3\lambda^d - \beta^2\lambda^{-d}(\ln\lambda)^3 = \lambda^{-d}(\ln\lambda)^3(\lambda^{2d} - \beta^2) > 0$ and $H'''(d) = 0$.
Hence it is clear that $G'''(d) > H'''(d)$ for all $d \geq \alpha$ and so the claim is true.
In particular since $z > \alpha$, $\beta z^2 - \beta\alpha z < \lambda^z + \beta^2\lambda^{-z} - 2\beta$
ie, $\beta z(z-\alpha) < (1-\beta\lambda^{-z})(\lambda^z - \beta)$
$\frac{(1-\beta\lambda^{-z})}{z(z-\alpha)} > \frac{\beta}{\lambda^z-\beta} = \frac{\beta\lambda^{-z}}{1-\lambda^z\beta}$
$\Rightarrow F'(z) < 0$. □

Theorem 3.6. *We assume* $1 < \lambda < e$. *Suppose that* $\beta \leq \min\left(\lambda^{\alpha}\left(\frac{-\alpha+\sqrt{\alpha^2+4(\ln\lambda)^2}}{2\ln\lambda}\right), \frac{1-\sqrt{1-(\ln\lambda)^4}}{\lambda^{-\alpha}(\ln\lambda)^2}\right)$. *The equilibrium* $\bar{x}$ *of (1) is globally asymptotically stable.*

Proof. By a simple calculation we can see that

$$\min\left(\lambda^{\alpha}\frac{-\alpha+\sqrt{\alpha^2+4(\ln\lambda)^2}}{2\ln\lambda}, \frac{1-\sqrt{1-(\ln\lambda)^4}}{\lambda^{-\alpha}(\ln\lambda)^2}\right)$$

$$< \frac{-\alpha\ln\lambda+\sqrt{\alpha^2\ln^2\lambda+4\alpha\ln\lambda}}{\alpha\ln\lambda+\sqrt{\alpha^2\ln^2\lambda+4\alpha\ln\lambda}}\lambda^{\frac{\alpha\ln\lambda+\sqrt{\alpha^2\ln^2\lambda+4\alpha\ln\lambda}}{2\ln\lambda}}.$$

Hence by Theorem 3.1, $\bar{x}$ is locally asymptotically stable.
It remains to show that $\bar{x}$ is a global attractor of (1)
For $x, y \in (0,\infty)$, set $Let f(x,y) = \alpha + \beta y\lambda^{-x}$
Then $f : (0,\infty)\times(0,\infty) \to (0,\infty)$ is a continuous function, $f(x,y)$ is decreasing in x for each $y \in (0,\infty)$ and $f(x,y)$ is increasing in y for each $x \in (0,\infty)$
We shall show that there exists positive numbers a and b with $a < b$ such that the following statements are true.
(i). $a < f(x,y) < b$ for all $x, y \in [a,b]$.
(ii). Every positive solution (x_n) of (1) eventually lies in (a,b).
Let $\epsilon > 0$, be given. Set $a = \alpha$ and $b = \frac{\alpha+\epsilon}{1-\beta\lambda^{-\alpha}}$ Then,
$f(b,a) = f(\frac{\alpha+\epsilon}{1-\beta\lambda^{-\alpha}}, \alpha) = \alpha + \beta\alpha\lambda^{\frac{-\alpha+\epsilon}{1-\beta\lambda^{-\alpha}}} > \alpha = a$

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and $f(a,b) < b$.
So as $f(x,y)$ is decreasing in x and increasing in y, if follows that $a < f(x,y) < b$ for all $x, y \in [a,b]$.
Clearly every positive solution x_n of (1) is eventually greater than $a = \alpha$ and eventually less than $b = \frac{\alpha+\epsilon}{1-\beta\lambda^{-\alpha}}$. The proof follows by Lemmas 3.4, 3.5 and Theorem 2.6.

□

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