

# Analysis of SEIR model with a single control for COVID-19

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## Abstract

A SEIR mathematical model for COVID - 19 with a single control vaccination was formulated to analysis the flow of state variables. Properties of Pontryagin's maximum principle is verified and found the optimal levels of control variable. Optimal values of S, E, I, R were derived by equilibrium analysis. Numerical simulations were carried out to exhibit the Susceptible, Exposed, Infectious and Recovery class with and without vaccination which directs that when the susceptible is vaccinated the flow of disease will drop eventually.

**Key words:** SEIR model, Control theory, Pontryagin's maximum principle, Equilibrium Analysis, COVID-19

**AMS classification:** 34L99, 34L30.

## 1 Introduction

Normally organisms such as bacteria and virus are not harmful but some of them cause disease. It is important to follow some control measures to control the infectious diseases. Here we consider vaccination as our control measure. Vaccination develops an immune in our body against the disease which leads us to control the infection of the disease. In India, we have vaccine for measles and for COVID - 19 developed recently.

Nowadays, mathematical model plays a key role in an analysis of spread and control of the infectious diseases. It describes the transmission process of the diseases. Mathematicians use the optimal control theory for infectious diseases to minimize the cost, infectious individuals and maximize the recovery individuals.

Many Mathematicians used the Optimal control theory for infectious diseases.

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Specifically, Holly Gaff [2] applied the optimal control for the SIR and SEIR models. Tunde Tajudeen Yusuf and Francis Benyah [9] formulated a SIR model with optimal control of vaccination and treatment. A.Huppert [3] analysed the SIR model with a single control strategy as vaccination. Jakia sultana and Chandra N.Podder [4] analysed Nipah virus infections by the optimal control theory. Gul Zaman [1] studied the behaviour of control strategies for various infectious diseases. Nita H.Shah [8] formulated a mathematical model for the Human - Bat population with control strategies. Moftah Bakush [6] studied the impact of vaccination on infectious diseases by the optimal control technique. Moussa Barro [7] has introduced an optimal control for a SIR model with time delay. A. Mhlannga [5] formulated and analyzed the two patch model for Ebola virus disease.

The first COVID - 19 case was reported on 7 March 2020 in Tamilnadu, India. Chennai, the capital of the state being worst affected. The COVID - 19 outbreak was reported on 1 April 2020 in Vellore district of Tamilnadu. In this paper, we have used the data of COVID - 19, Vellore district has been collected from the press release of the Vellore Collectorate and formulated a SEIR model for COVID - 19 with a single control, vaccination. We have analysed the model by optimal control theory and attained that the vaccination decrease the flow of the disease in the state variables.

## 2 Formulation of Mathematical model and Control set

The SEIR model for COVID-19 with a single control is given by the following system of ordinary differential equations.

$$\begin{aligned}\frac{dS}{dt} &= \omega N(t) - \beta S(t)I(t) - \mu S(t) - vS(t) \\ \frac{dE}{dt} &= \beta S(t)I(t) - \mu E(t) - \alpha E(t) \\ \frac{dI}{dt} &= \alpha E(t) - \gamma I(t) - \mu I(t) - dI(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) + vS(t)\end{aligned}\tag{1}$$

where  $S(t), E(t), I(t), R(t)$  are the Susceptible, Exposed, Infectious and Recovery state respectively,  $N(t)$  - Total population,  $\omega$  - Average birth rate,  $\mu$  - Average death rate,  $\beta$  - Transition infectious rate,  $\gamma$  - Recovery rate,  $d$  - disease induced death rate,  $\alpha$  - exposed rate. We assume the model with single control  $v$  - vaccination.

The transmission of SEIR model with a single control for COVID-19 is given as

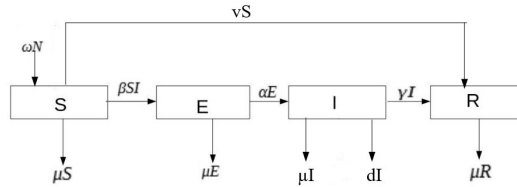


Figure 1: SEIR model with a single control for COVID-19

For this system of equations, we define an objective functional similar as in [6]

$$J = \min_{v \in U} AI(t) + \frac{1}{2} \int_0^T Bv^2 dt \quad (2)$$

which is a free problem system subject to infected.

Where  $AI(t)$  is the number of infected individuals,  $B$  is the weight parameter.

The control set is defined as

$$U = \{v : 0 \leq v \leq 1, t \in [0, 1]\} \quad (3)$$

### 3 Pontryagin's Maximum Principle

Pontryagin's maximum principle is a powerful method to compute the optimal value of the control variable. To use this, we have to prove the properties  $P_1$ ,  $P_2$ ,  $P_3$  and  $P_4$ [2].

$P_1$ : The set of controls and the corresponding state variables is non empty.

$P_2$ : The control set  $U$  is convex and closed.

$P_3$ : The R.H.S. of the state system is bounded by a linear function in the state and control variables.

$P_4$ : The integrand of the objective functional is convex on  $U$  and is bounded below by  $-k_2 + k_1 |U|^\eta$ , with  $k_1 > 0, k_2 > 0, \eta > 1$ .

To prove this we need the following theorem and lemma.

**Theorem 3.1** Existence and Uniqueness Theorem: Let  $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$  and  $f(x, y)$  be continuous on a domain  $\{D = (x, y) / x_0 - a < x < x_0 + a; y_0 - b < y < y_0 + b\}$  with the Lipschitz condition  $\|f(x, y_1) - f(x, y_2)\| \leq K \|y_1 - y_2\|$  where  $K$  is a positive integer, then there exist bounded solution in  $D$ . Further more, if  $f(x, y)$

is Lipschitz continuous with respect to  $y$  on a rectangle  $R = \{(x, y) / x_0 - c < x < x_0 + c; y_0 - b < y < y_0 + b; c < a\}$  then there is a unique solution  $y(t)$  in  $R$ .

**Lemma 3.2** If  $f(x, y)$  has continuous partial derivative  $\frac{\partial f_i}{\partial y_j}$  on a bounded closed domain  $R$ , is a set of real numbers. Then it satisfies a Lipschitz condition in  $R$ .

To prove  $P_1$ :

From (1),

$$G_1 = \omega N - \beta SI - \mu S - vS \quad (4)$$

$$G_2 = \beta SI - \mu E - \alpha E \quad (5)$$

$$G_3 = \alpha E - \gamma I - \mu I - dI \quad (6)$$

$$G_4 = \gamma I - \mu R + vS \quad (7)$$

From (4),  $|\frac{\partial G_1}{\partial S}| = \beta I + \mu + v < \infty$ ,  $|\frac{\partial G_1}{\partial E}| = 0 < \infty$ ,  $|\frac{\partial G_1}{\partial I}| = \beta S < \infty$ ,  $\frac{\partial G_1}{\partial R} = 0 < \infty$

From (5),  $|\frac{\partial G_2}{\partial S}| = \beta I < \infty$ ,  $|\frac{\partial G_2}{\partial E}| = \mu + \alpha < \infty$ ,  $|\frac{\partial G_2}{\partial I}| = \beta S < \infty$ ,  $|\frac{\partial G_2}{\partial R}| = 0 < \infty$

From (6),  $|\frac{\partial G_3}{\partial S}| = 0 < \infty$ ,  $|\frac{\partial G_3}{\partial E}| = \alpha < \infty$ ,  $|\frac{\partial G_3}{\partial I}| = \gamma + \mu + d < \infty$ ,  $\frac{\partial G_3}{\partial R} = 0 < \infty$

From (7),  $|\frac{\partial G_4}{\partial S}| = v < \infty$ ,  $|\frac{\partial G_4}{\partial E}| = 0 < \infty$ ,  $|\frac{\partial G_4}{\partial I}| = \gamma < \infty$ ,  $|\frac{\partial G_4}{\partial R}| = \mu < \infty$

As the partial derivatives are continuous and bounded, it satisfies the Lipschitz condition.

Hence there exist a unique solution of (1).

Thus the set of controls and the corresponding state variables is non empty,  $P_1$  is satisfied.

To prove  $P_2$ :

By the definition of  $U$ , it is closed.

Here  $0 \leq \alpha v + (1 - \alpha)v \leq 1 \implies U$  is convex.

Hence  $P_2$  is satisfied.

To prove  $P_3$ :

From (4)-(7),

$$G_1 \leq \omega N - vS$$

$$G_2 \leq \beta SI$$

$$G_3 \leq \alpha E$$

$$G_4 \leq \gamma I + vS$$

The state system can be rewrite in the matrix form,

$$\bar{G}(t, \bar{X}, U) \leq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta S & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix} \begin{bmatrix} S \\ E \\ I \\ R \end{bmatrix} + \begin{bmatrix} -S \\ 0 \\ 0 \\ S \end{bmatrix} [v(t)]$$

which can be written as the linear combination of control v,

$$\begin{aligned} |\bar{G}(t, \bar{X}, U)| &\leq \|\bar{m}\| |\bar{X}| + \|\bar{S}\| |v(t)| \\ &\leq K_2 [|\bar{X}| + |v(t)|] \end{aligned}$$

As  $\bar{S}$  is bounded,  $K_2$  is upper bound.

Hence R.H.S. is bounded by the sum of state and control variables,  $P_3$  is satisfied.

To prove  $P_4$ :

The control and state variables are non negative and the control variable is also convex and closed.

Then the integrand of the objective functional (2) is convex on the set U.

Now,

$$J = AI(t) + \frac{Bv^2}{2} \geq -AI(t) + \frac{Bv^2}{2} = -K_2 + K_1v^2$$

where  $K_1 = \frac{B}{2}$

Here  $K_2 > 0$  as it depends on  $I$ ,  $K_1 > 0$  and  $\eta = 2 > 1$

Hence  $P_4$  is satisfied.

Therefore, the control system satisfies the properties of Pontryagin's maximum principle.

## 4 Existence of Optimal Control

As the properties for Pontryagin's maximum principle are satisfied, which converts (1) into a problem of minimizing.

An Hamiltonian H is defined as

$$\begin{aligned}
 H(S, E, I, R) = & AI(t) + \frac{B}{2}v^2 + \lambda_S(\omega N - \beta SI - \mu S - vS) \\
 & + \lambda_E(\beta SI - \mu E - \alpha E) + \lambda_I(\alpha E - \gamma I - \mu I - dI) \\
 & + \lambda_R(\gamma I - \mu R + vS)
 \end{aligned} \tag{8}$$

and the adjoint system is given by

$$\begin{aligned}
 \frac{d\lambda_S}{dt} &= \lambda_S(\beta SI + \mu S + vS) - \lambda_E\beta I - v\lambda_R \\
 \frac{d\lambda_E}{dt} &= \lambda_E(\mu + \alpha) - \alpha\lambda_I \\
 \frac{d\lambda_I}{dt} &= \lambda_S\beta S - \lambda_E\beta S + \lambda_I(\gamma + \mu + d) - \gamma\lambda_R - A \\
 \frac{d\lambda_R}{dt} &= \mu\lambda_R
 \end{aligned}$$

with the final conditions  $\lambda_S(T) = \lambda_E(T) = \lambda_I(T) = \lambda_R(T) = 0$  for free problem. Differentiate (8) with respect to  $v$ ,

$$\begin{aligned}
 \left[ \frac{\partial H}{\partial v} \right]_{v=v^*} &= 0 \\
 v^* &= \frac{S(\lambda_S - \lambda_R)}{B}
 \end{aligned}$$

Hence the optimal control is

$$v^* = \min \left( 1, \max \left( 0, \frac{S(\lambda_S - \lambda_R)}{B} \right) \right)$$

## 5 Equilibrium Analysis

Now set  $s(t) = \frac{S(t)}{N}$ ,  $e(t) = \frac{E(t)}{N}$ ,  $i(t) = \frac{I(t)}{N}$ ,  $r(t) = \frac{R(t)}{N}$  and  $r(t) = 1 - s(t) - e(t) - i(t)$ .

∴ The equation (1) is reduced to the following equations:

$$\frac{ds}{dt} = \omega - \beta si - \mu s - vs \tag{9}$$

$$\frac{de}{dt} = \beta si - \mu e - \alpha e \tag{10}$$

$$\frac{di}{dt} = \alpha e - \gamma i - \mu i - di \tag{11}$$

The steady states are  $G_0(0, 0, 0)$ ,  $G_1(\bar{s}, \bar{e}, 0)$ ,  $G_2(s^*, e^*, i^*)$ .

Case(1): Always steady state  $G_0(0, 0, 0)$  exists always.

Case(2): For  $G_1(\bar{s}, \bar{e}, 0)$

Let  $\bar{s}, \bar{e}$  be the positive solutions of  $\frac{ds}{dt} = 0, \frac{de}{dt} = 0$ .

From (9) and (10),

$G_1(\bar{s}, \bar{e}, 0) = G_1(\frac{\omega}{\mu+v}, 0, 0)$ , which is the disease free equilibrium.

Case(3): For  $G_2(s^*, e^*, i^*)$ .

Let  $s^*, e^*, i^*$  be the positive solutions of  $\frac{ds}{dt} = 0, \frac{de}{dt} = 0, \frac{di}{dt} = 0$

From (9),

$$s^* = \frac{\omega}{(\mu + \beta i^* + v)} \tag{12}$$

From (10),

$$s^* = \frac{(\mu + \alpha)e^*}{\beta i^*} \tag{13}$$

From (11),

$$e^* = \frac{(\gamma + \mu + d)i^*}{\alpha} \tag{14}$$

using (14) in (13),

$$s^* = \frac{(\mu + \alpha)(\gamma + \mu + d)}{\alpha\beta}$$

using  $s^*$  in (12),

$$i^* = \frac{\omega\alpha}{(\mu + \alpha)(\gamma + \mu + d)} - \frac{\mu}{\beta} - \frac{v}{\beta}$$

From (14),

$$e^* = \frac{\omega}{\mu + \alpha} - \frac{\mu(\gamma + \mu + d)}{\alpha\beta} - \frac{v(\gamma + \mu + d)}{\alpha\beta}$$

Hence, the endemic equilibrium is

$$(s^*, e^*, i^*) = \left( \frac{(\mu + \alpha)(\gamma + \mu + d)}{\alpha\beta}, \frac{\omega}{\mu + \alpha} - \frac{\mu(\gamma + \mu + d)}{\alpha\beta} - \frac{v(\gamma + \mu + d)}{\alpha\beta}, \frac{\omega\alpha}{(\mu + \alpha)(\gamma + \mu + d)} - \frac{\mu}{\beta} - \frac{v}{\beta} \right)$$

## 6 Numerical Analysis

As we found the optimal value of control variable and state variables, we analyse the flow of state variables to study the model with and without vaccination. In Vellore district, the birth rate is  $\omega = 0.0477$  and death rate is  $\mu = 0.0203$ . The transmission rate of COVID-19 is  $\beta = 0.011$ , the recovery rate is  $\gamma = 0.96$  and the

death rate is  $d = 0.017$ .

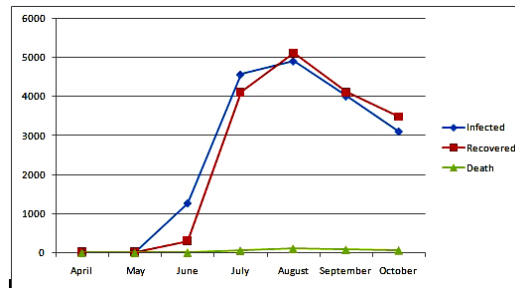


Figure 2: Transmission of COVID-19 in Vellore district

Figure (2) represents the transmission of Confirmed cases, Recovered cases and Death cases of COVID-19 in Vellore district as on 31.10.2020. Figure (3) shows the flow of Susceptible, Exposed, Infectious and Recovery for the model without Vaccination. The flow of disease for the state variables is higher without vaccination.

We study the flow of variable of susceptible. Figure (4) shows that the

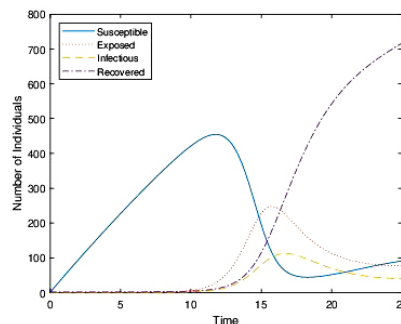


Figure 3: Flow of Variables with respect to Time without vaccination

susceptible with vaccination is less than without vaccination, which shows that vaccination can decrease the spread of the disease. On the study of variable of exposed figure(5) shows that the Exposed with vaccination is reduced from day 1 and without vaccination is higher.

We study the flow of variable of infectious. It is clear from figure (6) that the Infectious with vaccination is reduced from day 1 while without vaccination is higher. On the study of variable of recovered, figure (7) shows that the recovery with vaccination is higher than without vaccination. Hence the control variable vaccination plays an important role in control the spread of COVID - 19.



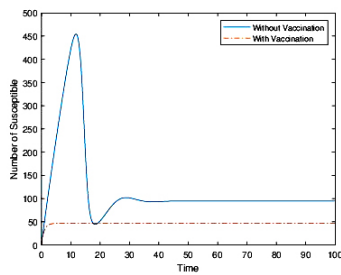


Figure 4: Susceptible class with and without vaccination

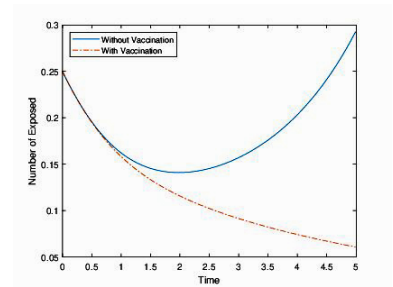


Figure 5: Exposed class with and without vaccination

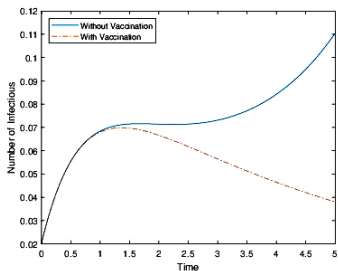


Figure 6: Infectious class with and without vaccination

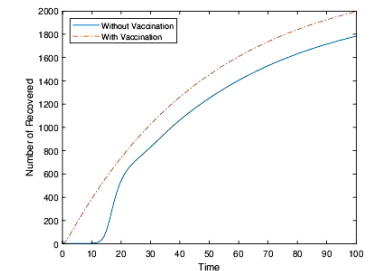


Figure 7: Recovered class with and without vaccination

## 7 Conclusion

A SEIR model with single control measure vaccination is formulated. By Pontryagin's maximum principle and by equilibrium analysis, the optimal levels of controls and variables were analysed. Numerical simulations were given for transition of COVID-19 in Vellore district, the flow of variables with respect to time without vaccination and the Susceptible, Exposed, Infectious and recovery class with and without vaccination which justifies that the flow of the disease decreases when the susceptible is vaccinated.

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