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ORIGINAL RESEARCH **Open Access**

Stability of a Ramanujan Type Additive Functional Equation

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Abstract

In this paper, the authors achieve the generalized Ulam - Hyers stability of a Ramanujan Type Additive Functional Equation in Paranormed Spaces and Modular spaces via classical Hyers Method.

Key words: Additive Functional Equations, Ulam - Hyers Stability, Paranormed Spaces, Modular Spaces.

AMS classification: 39B52, 32B72, 32B82

1 Introduction

The stability of a functional equation initiated from a question raised by Ulam: when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation? (see [37]). The first answer (in the case of Cauchys functional equation in Banach spaces) to Ulams question was given by Hyers in [11] .

Following his result, a abundant number of papers on the stability problems have been extensively available as generalizing Ulams problem and Hyers theorem in various directions; see for instance [3, 10, 28, 29, 31], and the references given there.

Notice that certain results on the stability of various several functional equations can be establish in [1, 4, 5, 6, 7, 12, 13, 14, 30, 32].

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The well known **Cauchy - Additive Functional Equation** is

$$R(x+y) = R(x) + R(y). \tag{1}$$

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1.1 Ramanujan Numbers

The world of mathematics is renowned for a number of interesting and fascinating numbers. Now Ramanujan Number also makes such a place in the list. Ramanujan Numbers (preciously termed as Hardy-Ramanujan Numbers) are those numbers that are the smallest positive integers that can be represented or expressed as a sum of 2 positive integers in n ways. Now lets discuss the above ways in a mathematical way.

1.2 Ramanujans 1 - Way Solution

Integers that are expressed as the sum of 2 cubes (in at least one way). Some of these numbers include :

$$\{2, 9, 16, 28, 35, 54, 65, 72, 91, 126, 128, 133, 152, 189, 217, 224, 243, 250, 280, 341,$$

$$344, 351, 370, 407, 432, 468, 513, 520, 539, 559, 576, 637, 686, 728, 730, 737, \cdots\cdots\}.$$

It is easy to verify that $2 = 1^3 + 1^3 \quad 9 = 2^3 + 1^3 \quad 16 = 2^3 + 2^3 \cdots$. Here all these numbers can be expressed as a sum of 2 cubes in a single way and so all these numbers from the above set can be expressed in this way.

The Ramanujans 1 - way solution can be converted to **Ramanujan Additive Functional Equation** of the form

$$R\left(\alpha_1^3 x_1 + \beta_1^3 y_1\right) = \alpha_1^3 R(x_1) + \beta_1^3 R(y_1). \tag{2}$$

Theorem 1.1 Assume W_1 and W_2 be real vector spaces. Suppose $R : W_1 \to W_2$ satisfies the functional equation (1) if and only of $R : W_1 \to W_2$ satisfies the functional equation (2).

Proof: With the help of oddness of R and additiveness the proof is trival.

In this paper, the authors achieve the generalized Ulam - Hyers stability of a Ramanujan Type Additive Functional Equation (2) in Paranormed Spaces and Modular spaces via classical Hyers Method.

2 Basic Concepts And Stability on Paranormed Spaces

Now, we give to adopt the usual terminologies, notations, definitions and properties of the theory of paranormed spaces given in [8, 9, 15, 18, 25, 33, 35].

Definition 2.1 Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(P4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n\, x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a ***paranormed space*** if P is a paranorm on X.

Definition 2.2 Let X be a paranormed space and let $\{x_n\}$ be a sequence in X then $\{x_n\}$ is called Cauchy if for any $\epsilon > 0$ if $P(x_n - x_m) \to 0$ for sufficiently large $m, n \in N$.

Definition 2.3 The paranorm is called ***total*** if, in addition, we have

(P5) $P(x) = 0$ implies $x = 0$.

Definition 2.4 A ***Fréchet space*** is a total and complete paranormed space.

Definition 2.5 A complete normed linear space is called ***Banach space***.

3 Basic Concepts And Stability on Modular Spaces

Now, we give to adopt the usual terminologies, notations, definitions and properties of the theory of modular spaces given in [2, 19, 20, 21, 23, 22, 24, 26, 27, 36, 39].

Definition 3.1 Let X be a linear space over a field $K(R\ or\ C)$. We say that a generalized functional $\rho : X \to [0, \infty]$ is a modular if for any $x, y \in X$,

(MS1) $\rho(x) = 0$ if and only if $x = 0$;
(MS2) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
(MS3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
(MS4) If (MS3) is replaced by $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, then the functional ρ is called a convex modular.

Definition 3.2 A modular ρ defines the following vector space:

$$X_\rho = \{x \in X : \rho(\lambda x) \quad \rightarrow \quad 0 \quad as \quad \lambda \quad \rightarrow 0\},$$

and we say that X_ρ is a modular space.

Definition 3.3 Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is $\rho-$convergent to a point $x \in X_\rho$and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 3.4 Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is called $\rho-$Cauchy if for any $\epsilon > 0$one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in N$.

Definition 3.5 Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. A subset $K \subseteq X_\rho$ is called $\rho-$complete if any $\rho-$Cauchy sequence is $\rho-$convergent to a point in K.

It is said that the modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n \rightarrow \infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho-$convergent to x in modular space X_ρ .

Theorem 3.6 In modular spaces,
(1) if $x_n \xrightarrow{\rho} x$ and a is a constant vector, then $x_n + a \xrightarrow{\rho} x + a$, and
(2) if $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ then $\alpha x_n + \beta y_n \xrightarrow{\rho} \alpha x + \beta y$, where $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$.

Definition 3.7 A modularρ is said to satisfy the Δ_2-condition if there exists $k > 0$ such that $\rho(\Gamma x) \leq k\rho(x)$ for all $x \in X_\rho$.

Remark 3.8 Suppose that ρ is convex and satisfies Δ_2-condition with Δ_2- constant $k > 0$. If $k < \Gamma$, then $\rho(x) \leq k\rho(x) \leq \frac{k}{\Gamma}\rho(x)$, which implies $\rho = 0$. Therefore, we must have the Δ_2- constant $k \geq \Gamma$ if ρ is convex modular.

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On the other hand, many authors have investigated the stability using fixed point theorem of quasicontraction mappings in modular spaces without Δ_2- condition, which has been introduced by Khamsi [17]. Recently, the stability results of additive functional equations in modular spaces equipped with the Fatou property and Δ_2-condition were investigated in H. M. Kim, H. Y. Shin [16] and Sadeghi [34] who used Khamsis fixed point theorem. Also the stability of quadratic functional equations in modular spaces satisfying the Fatou property without using the Δ_2-condition was proved by Wongkum, Chaipunya and Kumam [38] .

4 Stability Theorem: Paranormed Spaces

To prove stability results, in this section let us take $(\mathcal{N}, P)$ be a Fréchet space and $(\mathcal{M}, ||\cdot||)$ be a Banach space.

Theorem 4.1 If $R : \mathcal{N} \to \mathcal{M}$ is a function satisfying the inequality

$$P\ \left(R\left(\alpha_1^3 x_1 + \beta_1^3 y_1\right) - \left\{\alpha_1^3 R\left(x_1\right) + \beta_1^3 R\left(y_1\right)\right\}\right) \leq \mathcal{B}\left(x_1, y_1\right); \forall\ \ x_1, y_1 \ \in \ \mathcal{N}. \quad (3)$$

Then there exists a unique additive mapping $\mathcal{R}_{bn} : \mathcal{N} \to \mathcal{M}$ which satisfying the functional equation (2) and the inequality

$$P\ \left(\mathcal{R}_{bn}(y) - R(y)\right) \leq \frac{1}{\Gamma} \sum_{a=\frac{1-b}{2}}^{\infty} \frac{1}{\Gamma^{ab}} \mathcal{B}\left(\Gamma^{ab} y, \Gamma^{ab} y\right); \quad \forall\ \ y \ \in \ \mathcal{N}, \quad (4)$$

where $\Gamma = \left(\alpha_1^3 + \beta_1^3\right); b = \pm 1$ and $\mathcal{R}_{bn}(y)$ is given by

$$P\ \left(\lim_{c\ \to \infty} \frac{R(\Gamma^{bc} y)}{\Gamma^{bc}} - \mathcal{R}_{bn}(y)\right) \to 0; \quad \forall\ \ y \ \in \ \mathcal{N}. \quad (5)$$

The mapping $\mathcal{B} : \mathcal{N}^2 \to [0, \infty)$ fullfilling the condition

$$\lim_{c\ \to \infty} \frac{1}{\Gamma^{bc}} \mathcal{B}\left(\Gamma^{bc} x_1, \Gamma^{bc} y_1\right) = 0; \quad \forall\ \ x_1, y_1 \ \in \ \mathcal{N}. \quad (6)$$

Proof: Fixing $x_1 = y_1 = y$ in (3), we obtain

$$P\ \left(R\left(\Gamma y\right) - \Gamma R(y)\right) \leq \mathcal{B}\left(y, y\right); \quad \forall\ \ y \ \in \ \mathcal{N}. \quad (7)$$

It follows from (7) and $(P3)$ for a positive integer c, we arrive

$$P\ \left(\tfrac{1}{\Gamma^c}R\left(\Gamma^c y\right)-R\left(y\right)\right)=P\ \left(\sum_{a=0}^{c-1}\tfrac{1}{\Gamma^{a+1}}\left[\Gamma\ R\left(\Gamma^a y\right)-R\left(\Gamma^{a+1}y\right)\right]\right)$$
$$\leq \sum_{a=0}^{c-1}\ \tfrac{1}{\Gamma^{a+1}}\ P\ \left(\Gamma\ R\left(\Gamma^a y\right)-R\left(\Gamma^{a+1}y\right)\right)$$
$$\leq \frac{1}{\Gamma}\sum_{a=0}^{c-1}\frac{1}{\Gamma^a}\mathcal{B}\left(\Gamma^a y,\Gamma^a y\right);\quad \forall\ y\ \in\ \mathcal{N}. \tag{8}$$

It is ensure that the sequence

$$\left\{\frac{1}{\Gamma^c}R\left(\Gamma^c y\right)\right\}$$

is a Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is complete, there exists a limit function $\mathcal{R}_n:\mathcal{N}\to\mathcal{M}$ defined by

$$P\left(\lim_{c\ \to\infty}\frac{R(\Gamma^c y)}{\Gamma^c}-\mathcal{R}_n(y)\right)\to 0;\qquad y\in\mathcal{N}. \tag{9}$$

Also, by continuity of multiplication, we have

$$P\left(\lim_{c\ \to\infty}\frac{t_c R(\Gamma^c y)}{\Gamma^c}-t\mathcal{R}_n(y)\right)\to 0;\qquad y\in\mathcal{N}.$$

Indeed, suppose if we replace y as $\Gamma^d y$ and divided by Γ^d in (8), we get

$$P\ \left(\frac{1}{\Gamma^c\Gamma^d}R\left(\Gamma^c\Gamma^d y\right)-\frac{1}{\Gamma^d}R\left(\Gamma^d y\right)\right)=\frac{1}{\Gamma^d}P\ \left(\frac{1}{\Gamma^c}R\left(\Gamma^c\Gamma^d y\right)-R\left(\Gamma^d y\right)\right)$$
$$\leq\frac{1}{\Gamma}\sum_{a=0}^{c-1}\frac{1}{\Gamma^{a+d}}\mathcal{B}\left(\Gamma^{a+d}y,\Gamma^{a+d}y\right)$$
$$\to\quad 0\quad as\quad d\quad to\quad \infty. \tag{10}$$

Letting $p\to\infty$ in (8) and using (9), we arrive (4). If we replacing (x_1,y_1) as $(\Gamma^c x_1,\Gamma^c y_1,)$ and divided by Γ^c in (3) , we arrive

$$P\ \left(\frac{1}{\Gamma^c}\left\{R\left(\alpha_1^3\ \Gamma^c x_1+\beta_1^3\ \Gamma^c y_1\right)-\alpha_1^3 R\left(\Gamma^c x_1\right)-\beta_1^3 R\left(\Gamma^c y_1\right)\right\}\right)\leq\frac{1}{\Gamma^c}\mathcal{B}\left(\Gamma^c y_1,\Gamma^c y_2\right)$$

for all $x_1,y_1\in\mathcal{N}$. Letting $c\to\infty$ in the above inequality and using (9), $(P5)$ and we see that $\mathcal{R}_n(y)$ satisfies the Ramanujan Type additive functional equation (2) . It is clear that the existence of $\mathcal{R}_n(y)$ is unique.

Indeed, suppose that assume $\mathcal{R}_1(y)$ be another additive mapping satisfying (2) and (9). So, one can easy to verify that for a positive integer d, we observe

$$\mathcal{R}_1(\Gamma^d y) = \Gamma^d \mathcal{R}_1(y); \mathcal{R}_n(\Gamma^d y) = \Gamma^d \mathcal{R}_n(y) \tag{11}$$

and

$$\mathcal{R}_1(y) = \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y); \mathcal{R}_n(y) = \frac{1}{\Gamma^d}\mathcal{R}_n(\Gamma^d y) \tag{12}$$

for all $y \in \mathcal{N}$. Now,

$$\begin{aligned}
&P\ (\mathcal{R}_n(y) - \mathcal{R}_1(y)) \\
&= P\ \left(\frac{1}{\Gamma^d}\mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y)\right) \\
&\leq P\ \left(\frac{1}{\Gamma^d}\ \mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d}R(\Gamma^d y)\right) + P\ \left(\frac{1}{\Gamma^d}R(\Gamma^d y) - \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y)\right) \\
&\leq \frac{1}{\Gamma^d}\ P\ \left(\mathcal{R}_n(\Gamma^d y) - R(\Gamma^d y)\right) + \frac{1}{\Gamma^d}\ P\ \left(R(\Gamma^d y) - \mathcal{R}_1(\Gamma^d y)\right) \\
&\leq \frac{2}{\Gamma}\sum_{a=0}^{\infty}\frac{1}{\Gamma^a\Gamma^d}\mathcal{B}\left(\Gamma^{ab}\Gamma^d y, \Gamma^{ab}\Gamma^d y\right) \\
&\rightarrow \quad 0 \quad as \quad d \quad to \quad \infty.
\end{aligned} \tag{13}$$

for all $y \in \mathcal{N}$. Again, taking y as $\frac{y}{\Gamma}$ in (7), we have

$$P\ \left(R\,(y) - \Gamma R\left(\frac{y}{\Gamma}\right)\right) \leq \mathcal{B}\left(\frac{y}{\Gamma}, \frac{y}{\Gamma}\right); \quad \forall\ y\ \in\ \mathcal{N}. \tag{14}$$

In widely for a positive integer p, we arrive

$$P\ \left(R\,(y) - \Gamma^c R\left(\frac{y}{\Gamma^c}\right)\right) \leq \frac{1}{\Gamma}\sum_{a=1}^{bc}\Gamma^a\ \Phi\left(\frac{y}{\Gamma^a}, \frac{y}{\Gamma^a}\right); \quad \forall\ y\ \in\ \mathcal{N}. \tag{15}$$

The rest of proof is similar to that of previous one. This completes the proof of the theorem.

Corollary 4.2 If $R : \mathcal{N} \to \mathcal{M}$ is a function satisfying the inequality

$$P\ \left(R\left(a_1^3x_1+b_1^3y_1\right)-\left\{a_1^3R\left(x_1\right)+b_1^3R\left(y_1\right)\right\}\right) \le \begin{cases} \mathcal{T}, \\ \mathcal{T}\left\{P(x_1)^r+P(y_1)^r\right\}, \\ \mathcal{T}\left\{P(x_1)^{r_1}+P(y_1)^{r_2}\right\}, \\ \mathcal{T}\ \ P(x_1)^rP(y_1)^r \end{cases} \tag{16}$$

for all $x_1, y_1 \in \mathcal{N}$ and $\mathcal{T} > 0$. Then there exists a unique additive mapping $\mathcal{R}_{bn} : \mathcal{N} \to \mathcal{M}$ which satisfying the functional equation (2) and the functional inequality

$$P\ \left(\mathcal{R}_{bn}(y)-R(y)\right) \le \begin{cases} \dfrac{\mathcal{T}}{|\Gamma-1|}; \\ \dfrac{2\mathcal{T}P(y)^r}{|\Gamma-\Gamma^r|}; & r \neq 1; \\ \dfrac{\mathcal{T}P(y)^{r_1}}{|\Gamma-\Gamma^{r_1}|}+\dfrac{\mathcal{T}P(y)^{r_2}}{|\Gamma-\Gamma^{r_2}|}; & r_1; r_2 \neq 1; \\ \dfrac{\mathcal{T}P(y)^{2r}}{|\Gamma-\Gamma^r|}; & 2r \neq 1; \end{cases} \tag{17}$$

for all $y \in \mathcal{N}$.

5 Stability Theorem: Without Using The Δ_2- Condition

To prove stability results, in Sections 5 and 6, let us take $\mathcal{N}$ be an linear space and $\mathcal{M}_\rho$ be an $\rho-$complete convex modular space.

Theorem 5.1 Assume $\mathcal{M}_\rho$ satisfy the Fatou property. If $R : \mathcal{N} \to \mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho\left(R\left(\alpha_1^3x_1+\beta_1^3y_1\right)-\left\{\alpha_1^3R\left(x_1\right)+\beta_1^3R\left(y_1\right)\right\}\right) \le \mathcal{B}\left(x_1,y_1\right); \forall\ \ x_1,y_1\ \ \in\ \ \mathcal{N}. \tag{18}$$

Then there exists a unique additive mapping $\mathcal{R}_n : \mathcal{N} \to \mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality

$$\rho\left(\mathcal{R}_n(y)-R(y)\right) \le \frac{1}{\Gamma}\ \sum_{a=0}^{\infty}\frac{1}{\Gamma^a}\mathcal{B}\left(\Gamma^a y,\Gamma^a y\right); \quad \forall\ \ y\ \ \in\ \ \mathcal{N}, \tag{19}$$

where $\Gamma = (\alpha_1^3 + \beta_1^3)$ and $\mathcal{R}_n(y)$ is given by

$$\lim_{c \to \infty} \rho \left(\frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) = 0; \quad \forall \ y \ \in \ \mathcal{N}. \tag{20}$$

The mapping $\mathcal{B} : \mathcal{N}^2 \to [0, \infty)$ fullfilling the condition

$$\lim_{c \to \infty} \frac{1}{\Gamma^c} \mathcal{B} \left(\Gamma^c x_1, \Gamma^c y_1 \right) = 0; \quad \forall \ x_1, y_1 \ \in \ \mathcal{N}. \tag{21}$$

Proof: Fixing $x_1 = y_1 = y$ in (18), we obtain

$$\rho \left(R \left(\Gamma y \right) - \Gamma R(y) \right) \leq \mathcal{B} \left(y, y \right); \quad \forall \ y \ \in \ \mathcal{N}. \tag{22}$$

Without using the Δ_2-condition it follows from (22) and $(MS3)$x for a positive integer c, we arrive

$$\begin{aligned} \rho \left(\frac{1}{\Gamma^c} R \left(\Gamma^c y \right) - R \left(y \right) \right) &= \rho \left(\sum_{a=0}^{c-1} \frac{1}{\Gamma^{a+1}} \left[\Gamma \ R \left(\Gamma^a y \right) - R \left(\Gamma^{a+1} y \right) \right] \right) \\ &\leq \sum_{a=0}^{c-1} \ \frac{1}{\Gamma^{a+1}} \ \rho \left(\Gamma \ R \left(\Gamma^a y \right) - R \left(\Gamma^{a+1} y \right) \right) \\ &\leq \frac{1}{\Gamma} \sum_{a=0}^{c-1} \frac{1}{\Gamma^a} \mathcal{B} \left(\Gamma^a y, \Gamma^a y \right); \quad \forall \ y \ \in \ \mathcal{N}. \end{aligned} \tag{23}$$

It is ensure that the sequence

$$\left\{ \frac{1}{\Gamma^c} R \left(\Gamma^c y \right) \right\}$$

is a $\rho-$Cauchy sequence in $\mathcal{M}_\rho$. Since $\mathcal{M}_\rho$ is $\rho-$complete, there exists a $\rho-$ limit function $\mathcal{R}_1 : \mathcal{N} \to \mathcal{M}_\rho$ defined by

$$\rho - \lim_{c \to \infty} \frac{R(\Gamma^c y)}{\Gamma^c} = \mathcal{R}_n(y) \ \ or \ \ \lim_{c \to \infty} \rho \left(\frac{R(\Gamma^c y)}{\Gamma^c} - \mathcal{R}_n(y) \right) = 0; \qquad y \in \mathcal{N}. \tag{24}$$

Indeed, suppose if we replace y as $\Gamma^d y$ and divided by Γ^d in (23), we get

$$\begin{aligned}\rho\left(\frac{1}{\Gamma^c\Gamma^d}R\left(\Gamma^c\Gamma^d y\right)-\frac{1}{\Gamma^d}R\left(\Gamma^d y\right)\right)&=\frac{1}{\Gamma^d}\rho\left(\frac{1}{\Gamma^c}R\left(\Gamma^c\Gamma^d y\right)-R\left(\Gamma^d y\right)\right)\\&\leq\frac{1}{\Gamma}\sum_{a=0}^{c-1}\frac{1}{\Gamma^{a+d}}\mathcal{B}\left(\Gamma^{a+d}y,\Gamma^{a+d}y\right)\\&\to\quad 0\quad as\quad d\quad to\quad\infty.\end{aligned}\tag{25}$$

It follows from the Fatou property that the inequality

$$\rho\left(\mathcal{R}_n(y)-R(y)\right)\leq\liminf_{c\ \to\infty}\rho\left(\frac{R(\Gamma^c y)}{\Gamma^c}-\mathcal{R}_n(y)\right)\leq\frac{1}{\Gamma}\sum_{a=0}^{\infty}\frac{1}{\Gamma^a}\mathcal{B}\left(\Gamma^a y,\Gamma^a y\right);y\in\mathcal{N}.$$

Thus, we arrive (19). If we replacing (x_1, y_1) as $(\Gamma^c x_1, \Gamma^c y_1,)$ and divided by Γ^c in (18) , we arrive

$$\rho\left(\frac{1}{\Gamma^c}\left\{R\left(\alpha_1^3\ \Gamma^c x_1+\beta_1^3\ \Gamma^c y_1\right)-\alpha_1^3 R\left(\Gamma^c x_1\right)-\beta_1^3 R\left(\Gamma^c y_1\right)\right\}\right)\leq\frac{1}{\Gamma^c}\mathcal{B}\left(\Gamma^c y_1,\Gamma^c y_2\right)$$

for all $x_1, y_1 \in \mathcal{N}$. By convexity of ρ that

$$\begin{aligned}&\rho\left(\frac{1}{4}\mathcal{R}_n\left(\alpha_1^3\ x_1+\beta_1^3\ y_1\right)-\frac{1}{4}\alpha_1^3\mathcal{R}_n\left(x_1\right)-\frac{1}{4}\beta_1^3\mathcal{R}_n\left(y_1\right)\right)\\&\leq\frac{1}{4}\rho\left(\mathcal{R}_n\left(\alpha_1^3\ x_1+\beta_1^3\ y_1\right)-\frac{1}{\Gamma^c}\ R\left(\alpha_1^3\ \Gamma^c x_1+\beta_1^3\ \Gamma^c y_1\right)\right)\\&\quad+\frac{1}{4}\rho\left(-\mathcal{R}_n\left(\alpha_1^3\ x_1\right)+\frac{1}{\Gamma^c}\ R\left(\alpha_1^3\ \Gamma^c x_1\right)\right)+\frac{1}{4}\rho\left(-\mathcal{R}_n\left(\beta_1^3\ y_1\right)+\frac{1}{\Gamma^c}\ R\left(\beta_1^3\ \Gamma^c y_1\right)\right)\\&\quad\quad+\frac{1}{4}\rho\left(\frac{1}{\Gamma^c}\ R\left(\alpha_1^3\ \Gamma^c x_1+\beta_1^3\ \Gamma^c y_1\right)-\frac{1}{\Gamma^c}\ R\left(\alpha_1^3\ \Gamma^c x_1\right)-\frac{1}{\Gamma^c}\ R\left(\beta_1^3\ \Gamma^c y_1\right)\right)\end{aligned}$$

for all $x_1, y_1 \in \mathcal{N}$. Taking p tends to infinity in the above inequality, we arrive

$$\rho\left(\frac{1}{4}\mathcal{R}_n\left(\alpha_1^3\ x_1+\beta_1^3\ y_1\right)-\frac{1}{4}\alpha_1^3\mathcal{R}_n\left(x_1\right)-\frac{1}{4}\beta_1^3\mathcal{R}_n\left(y_1\right)\right)=0$$

for all $x_1, y_1 \in \mathcal{N}$. Thus, $\mathcal{R}_n(y)$ satisfies the Ramanujan Type additive functional equation (2) . It is clear that the existence of $\mathcal{R}_n(y)$ is unique.

Indeed, suppose that assume $\mathcal{R}_1(y)$ be another additive mapping satisfying (2)

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and (24). So, one can easy to verify that for a positive integer d, we observe

$$\mathcal{R}_1(\Gamma^d y) = \Gamma^d \mathcal{R}_1(y); \mathcal{R}_n(\Gamma^d y) = \Gamma^d \mathcal{R}_n(y) \tag{26}$$

and

$$\mathcal{R}_1(y) = \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y); \mathcal{R}_n(y) = \frac{1}{\Gamma^d}\mathcal{R}_n(\Gamma^d y) \tag{27}$$

for all $y \in \mathcal{N}$. Now,

$$\begin{aligned}
&\rho\left(\frac{1}{2}\mathcal{R}_n(y) - \frac{1}{2}\mathcal{R}_1(y)\right)\\
&\leq \frac{1}{2}\,\rho\left(\frac{1}{\Gamma^d}\mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y)\right)\\
&\leq \frac{1}{2}\,\rho\left(\frac{1}{\Gamma^d}\,\mathcal{R}_n(\Gamma^d y) - \frac{1}{\Gamma^d}R(\Gamma^d y)\right) + \frac{1}{2}\,\rho\left(\frac{1}{\Gamma^d}R(\Gamma^d y) - \frac{1}{\Gamma^d}\mathcal{R}_1(\Gamma^d y)\right)\\
&\leq \frac{1}{2}\frac{1}{\Gamma^d}\,\rho\left(\mathcal{R}_n(\Gamma^d y) - R(\Gamma^d y)\right) + \frac{1}{2}\frac{1}{\Gamma^d}\,\rho\left(R(\Gamma^d y) - \mathcal{R}_1(\Gamma^d y)\right)\\
&\leq \frac{1}{\Gamma}\sum_{a=0}^{\infty}\frac{1}{\Gamma^a\Gamma^d}\mathcal{B}\left(\Gamma^a\Gamma^d y, \Gamma^a\Gamma^d y\right)\\
&\rightarrow \quad 0 \quad as \quad d \quad to \quad \infty.
\end{aligned} \tag{28}$$

for all $y \in \mathcal{N}$.

Corollary 5.2 Let $\mathcal{N}$ be a normed space with norm $||\cdot||$ and $\mathcal{M}_\rho$ satisfy the Fatou property. If $R : \mathcal{N} \rightarrow \mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho\left(R\left(a_1^3 x_1 + b_1^3 y_1\right) - \left\{a_1^3 R\left(x_1\right) + b_1^3 R\left(y_1\right)\right\}\right) \leq \begin{cases} \mathcal{T}, \\ \mathcal{T}\left\{||x_1||^r + ||y_1||^r\right\}, \\ \mathcal{T}\left\{||x_1||^{r_1} + ||y_1||^{r_2}\right\}, \\ \mathcal{T} \quad ||x_1||^r||y_1||^r \end{cases} \tag{29}$$

for all $x_1, y_1 \in \mathcal{N}$ and $\mathcal{T} > 0$. Then there exists a unique additive mapping $\mathcal{R}_n : \mathcal{N} \rightarrow \mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality

$$\rho\left(\mathcal{R}_n(y)-R(y)\right)\leq\begin{cases}\dfrac{\mathcal{T}}{(\Gamma-1)}; & \\ \dfrac{2\mathcal{T}||y||^r}{(\Gamma-\Gamma^r)}; & r<1; \\ \dfrac{\mathcal{T}||y||^{r_1}}{(\Gamma-\Gamma^{r_1})}+\dfrac{\mathcal{T}||y||^{r_2}}{(\Gamma-\Gamma^{r_2})}; & r_1;r_2<1; \\ \dfrac{\mathcal{T}||y||^{2r}}{(\Gamma-\Gamma^r)}; & 2r<1;\end{cases} \tag{30}$$

for all $y\in\mathcal{N}$.

6. Stability Theorem: Using The Δ_2- Condition

Theorem 6.1 Assume $\mathcal{M}_\rho$ satisfy the Fatou property. If $R:\mathcal{N}\to\mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho\left(R\left(a_1^3x_1+b_1^3y_1\right)-\left\{a_1^3R\left(x_1\right)+b_1^3R\left(y_1\right)\right\}\right)\leq\mathcal{B}\left(x_1,y_1\right);\forall\ \ x_1,y_1\ \in\ \mathcal{N}. \tag{31}$$

Then there exists a unique additive mapping $\mathcal{R}_n:\mathcal{N}\to\mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality

$$\rho\left(\mathcal{R}(y)-R(y)\right)\leq\frac{1}{k}\sum_{a=1}^{\infty}\left(\frac{k^2}{\Gamma}\right)^a\mathcal{B}\left(\frac{y}{\Gamma^a},\frac{y}{\Gamma^a}\right);\quad\forall\ y\ \in\ \mathcal{N}, \tag{32}$$

where $\Gamma=(a_1^3+b_1^3)$ and $\mathcal{R}_n(y)$ is given by

$$\lim_{c\ \to\infty}\rho\left(\Gamma^cR\left(\frac{y}{\Gamma^c}\right)-\mathcal{R}_n(y)\right)=0;\quad\forall\ y\ \in\ \mathcal{N}. \tag{33}$$

The mapping $\mathcal{B}:\mathcal{N}^2\to[0,\infty)$ fullfilling the condition

$$\lim_{c\ \to\infty}k^c\mathcal{B}\left(\frac{x_1}{\Gamma^c},\frac{y_1}{\Gamma^c}\right)=0;\forall\ \ x_1,y_1\ \in\ \mathcal{N}. \tag{34}$$

Proof: Fixing $x_1=y_1=y$ and again replace $y=\dfrac{y}{\Gamma}$ in (31), we reach

$$\rho\left(R\left(y\right)-\Gamma R\left(\frac{y}{\Gamma}\right)\right)\leq\mathcal{B}\left(\frac{y}{\Gamma},\frac{y}{\Gamma}\right);\quad\forall\ y\ \in\ \mathcal{N}. \tag{35}$$

Using the Δ_2-condition it follows from (35) and the convexity of the modular ρ that,

$$\rho\left(R\left(y\right)-\Gamma R\left(\frac{y}{\Gamma}\right)\right)\leq\frac{k}{\Gamma}\,\mathcal{B}\left(\frac{y}{\Gamma},\frac{y}{\Gamma}\right);\quad\forall\;\;y\;\;\in\;\;\mathcal{N}. \tag{36}$$

Generalizing for a positive integer $c>0$, we obtain

$$\rho\left(R\left(y\right)-\Gamma^{c}R\left(\frac{y}{\Gamma^{c}}\right)\right)\leq\frac{1}{k}\,\sum_{a=1}^{c}\left(\frac{k}{\Gamma}\right)^{a}\mathcal{B}\left(\frac{y}{\Gamma^{a}},\frac{y}{\Gamma^{a}}\right);\quad\forall\;\;y\;\;\in\;\;\mathcal{N}. \tag{37}$$

So, for all $c,d\geq 0$ with $c\geq d$, we have

$$\rho\left(\Gamma^{c}R\left(\frac{y}{\Gamma^{c}}\right)-\Gamma^{d}R\left(\frac{y}{\Gamma^{d}}\right)\right)\leq\frac{1}{k}\,\left(\frac{\Gamma}{k}\right)^{d}\sum_{a=d+1}^{c}\left(\frac{k}{\Gamma}\right)^{a}\mathcal{B}\left(\frac{y}{\Gamma^{a}},\frac{y}{\Gamma^{a}}\right);\forall\;\;y\;\;\in\;\;\mathcal{N}. \tag{38}$$

Thus the sequence $\left\{\Gamma^{c}R\left(\frac{y}{\Gamma^{c}}\right)\right\}$ is a $\rho-$Cauchy sequence in $\mathcal{M}_\rho$. Since $\mathcal{M}_\rho$ is $\rho-$complete, there exists a $\rho-$ limit function $\mathcal{R}_n:\mathcal{N}\to\mathcal{M}_\rho$ defined by

$$\rho-\lim_{c\;\to\infty}\Gamma^{c}R\left(\frac{y}{\Gamma^{c}}\right)=\mathcal{R}_n(y)\;\;\;or\;\;\;\lim_{c\;\to\infty}\rho\left(\Gamma^{c}R\left(\frac{y}{\Gamma^{c}}\right)-\mathcal{R}(y)\right)=0;y\in\mathcal{N}. \tag{39}$$

Letting $d=0$ and $c\to\infty$ in (38) and using (39), we arrive (32). The rest of proof is similar to that of Theorem 5.1. This completes the proof of the theorem.

Corollary 6.2 Let $\mathcal{N}$ be a normed space with norm $||\cdot||$ and $\mathcal{M}_\rho$ satisfy the Fatou property. If $R:\mathcal{N}\to\mathcal{M}_\rho$ is a function satisfying the inequality

$$\rho\left(R\left(a_1^3x_1+b_1^3y_1\right)-\left\{a_1^3R\left(x_1\right)+b_1^3R\left(y_1\right)\right\}\right)\leq\begin{cases}\mathcal{T},\\ \mathcal{T}\left\{||x_1||^{r}+||y_1||^{r}\right\},\\ \mathcal{T}\left\{||x_1||^{r_1}+||y_1||^{r_2}\right\},\\ \mathcal{T}\;\;||x_1||^{r}||y_1||^{r}\\ \mathcal{T}\;\;||x_1||^{r_1}||y_1||^{r_2}\end{cases} \tag{40}$$

for all $x_1,y_1\in\mathcal{N}$ and $\mathcal{T}>0$. Then there exists a unique additive mapping $\mathcal{R}_n:\mathcal{N}\to\mathcal{M}_\rho$ which satisfying the functional equation (2) and the functional inequality

$$\rho\left(\mathcal{R}_n(y) - R(y)\right) \leq \begin{cases} \dfrac{\mathcal{T}\ k}{\Gamma - k^2}; \\ \dfrac{2\ \mathcal{T}\ k}{\Gamma^{1+r} - k^2}; \\ \dfrac{\mathcal{T}\ k}{\Gamma^{1+r_1} - k^2} + \dfrac{\mathcal{T}\ 2k}{\Gamma^{1+r_2} - k^2}; \\ \dfrac{\mathcal{T}\ k}{\Gamma^{1+2r} - k^2}; \\ \dfrac{\mathcal{T}\ k}{\Gamma^{1+r_1+r_2} - k^2}; \end{cases} \tag{41}$$

for all $y \in \mathcal{N}$ with $r; r_1; r_2; 2r; r_1 + r_2 > log_2 \frac{k^2}{\Gamma}$.

7 Conclusion

In this paper with the help of classical Hyers Method, we analyze the generalized Ulam - Hyers stability of a Ramanujan Type Additive Functional Equation in Paranormed Spaces and Modular spaces. The stability results in these two spaces are varying due to their respective definitions.

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