



Sum of Finite and Infinite Series Derived by Generalized Mixed Difference Equation

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Abstract

In this work, by introducing the generalized mixed q - ℓ difference equation with the product of generalized difference operator Δ_ℓ and the q -difference operator Δ_q , we derive mixed multi-summation formula and mixed higher order summation formula for certain functions. Suitable numerical examples verified by MATLAB are also provided.

Key words: Generalized mixed difference operator, Polynomial factorial and Summation solution.

AMS Subject classification: 39A10, 39A11, 39A13, 39A70, 49M.

1. Introduction

The modern theory of differential or integral calculus began in the 17th century with the works of Newton and Leibnitz [10]. In 1984, Jerzy Popenda [1] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, K.S.Miller and Ross [11] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. The general fractional h -difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in ([2],[3]). As application of $\Delta_h^{-\nu}$, by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$, where $\Delta_\ell u(k) = u(k+\ell) - u(k)$ [4].

In 2011, M.Maria Susai Manuel, et al.[5] have extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ which is defined as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function

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$v(k)$, $\ell > 0$. In [6], the authors have used the generalized α -difference equation $v(k + \ell) - \alpha v(k) = u(k)$, $k \in [0, \infty)$, $\ell > 0$ is fixed, and obtained a summation solution in the form

$$v(k) = \Delta_{\alpha(\ell)}^{-1} u(k) - \alpha^{[\frac{k}{\ell}]} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) = \sum_{r=1}^{[k/\ell]} \alpha^{r-1} u(k - r\ell), \quad \hat{\ell}(k) = k - [k/\ell]\ell.$$

In 2014, G.Britto Antony Xavier et al.[7] introduced a q -difference operator Δ_q , which is defined as

$$\Delta_q v(k) = v(qk) - v(k), \quad q \neq 1, \tag{1}$$

and obtained a summation solution of the q -difference equation $\Delta_q^t v(k) = u(k)$, $k \in (-\infty, \infty)$ and $q \neq 1$, in the form

$$\Delta_q^{-t} u(k) \Big|_{\frac{k}{q^m}}^k = \sum_{(r)_1 \rightarrow t}^m u\left(k \prod_{i=1}^t q^{-r_i}\right).$$

With this background, in this paper, we obtain multi-series solution for the generalized mixed q - ℓ difference equation

$$\Delta_{\ell_1 \rightarrow n} \Delta_{q_1 \rightarrow t} v(k) = u(k), \quad k \in [0, \infty), \tag{2}$$

where $\Delta_{\ell_1 \rightarrow n} \Delta_{q_1 \rightarrow t} v(k) = \Delta_{\ell_1} (\Delta_{\ell_2} (\dots \Delta_{\ell_n} (\Delta_{q_1} (\Delta_{q_2} (\dots \Delta_{q_t} (v(k)) \dots))) \dots))$.

2. Preliminaries

In this section, we present some notations, basic definitions and preliminary results which will be used for the subsequent discussions. Let $u(k)$ be a real valued function defined on $[0, \infty)$, $\hat{\ell}_i(k) = k - [k/\ell_i]\ell_i$, $[k/\ell_i]$ denotes the integer part of $\frac{k}{\ell_i}$ and m is a positive integer. For simplicity, we use the following notations:

$$\begin{aligned} \text{(i)} \quad \sum_{(h)_1 \rightarrow t}^m &= \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \dots \sum_{h_t=0}^{m_t}; & \text{(ii)} \quad \sum_{(r\ell)_1 \rightarrow i}^{[k]} &= \sum_{r_1=0}^{[\frac{k}{\ell_1}]} \sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]} \dots \sum_{r_i=0}^{[\frac{k-r_1\ell_1-r_2\ell_2-\dots-r_{i-1}\ell_{i-1}}{\ell_i}]}; \\ \text{(iii)} \quad \Delta_{\ell_1 \rightarrow n}^{-1} &= \Delta_{\ell_1}^{-1} \Delta_{\ell_2}^{-1} \Delta_{\ell_3}^{-1} \dots \Delta_{\ell_n}^{-1}; & \text{(iv)} \quad \Delta_{q_1 \rightarrow t}^{-1} &= \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} \dots \Delta_{q_t}^{-1} \quad \text{and} \\ \text{(v)} \quad \Delta_{\ell_1 \rightarrow n}^{-1} \Delta_{q_1 \rightarrow t}^{-1} u(k) &\Big|_{\frac{1}{q_1} \sim \frac{1}{q_t} \sim (\hat{\ell}_1(k) + \sum_{J=2}^n \ell_J)}^{q_1 \sim q_t \sim (k + \sum_{J=1}^n \ell_J)} &= \Delta_{\ell_1 \rightarrow n}^{-1} \Delta_{q_1 \rightarrow t}^{-1} u(k) &\Big|_{\frac{k}{q_1} \dots \frac{k}{q_t}}^{q_1 k \dots q_t k} \Big|_{\frac{k}{q_1} \dots \frac{k}{q_t}}^{(\hat{\ell}_1(k) + \sum_{J=2}^n \ell_J)}^{(k + \sum_{J=1}^n \ell_J)}. \end{aligned}$$

Lemma 2.1 [8] Let s_r^n and S_r^n be the Stirling numbers of first and second kinds respectively, $n \in N(1)$. If $k_q^{(n)} = \prod_{i=0}^{n-1} (k - iq)$ and $\left(\frac{1}{k}\right)_q^{(n)} = \prod_{i=0}^{n-1} \left(\frac{1}{k} - iq\right)$, $q \neq 0$, then

$$k_q^{(n)} = \sum_{r=1}^n s_r^n q^{n-r} k^r, \quad k^n = \sum_{r=1}^n S_r^n q^{n-r} k_q^{(r)} \quad \text{and} \quad \left(\frac{1}{k}\right)_q^{(n)} = \sum_{r=1}^n s_r^n q^{n-r} \left(\frac{1}{k}\right)^r. \quad (3)$$

Lemma 2.2 [9] For $k \in (0, \infty)$ and $\ell > 0$, we have

$$\Delta_\ell^{-n} k_\ell^{(m)} = \frac{k_\ell^{(m+n)}}{\ell^n (m+n)^{(n)}} \quad (4)$$

and

$$\Delta_\ell^{-1} u(k + \ell) - \Delta_\ell^{-1} u(\hat{\ell}(k)) = \sum_{r=0}^{[k/\ell]} u(k - r\ell). \quad (5)$$

3. Main Results

The purpose of this section is obtaining the sum of mixed q - ℓ multi-series by equating summation and closed form solutions of mixed difference equation (2).

Theorem 3.1 [Finite mixed summation formula] Let $q \neq 0$, $\ell > 0$, $m \in N(1)$ and $u(k)$ be a real valued function defined on $[0, \infty)$. Then

$$\sum_{r=0}^{[k/\ell]} \sum_{h=0}^m u\left(\frac{k - r\ell}{q^h}\right) = \Delta_\ell^{-1} \Delta_q^{-1} u(k) \Big|_{\frac{1}{q^m} \sim \hat{\ell}(k)}^{q \sim (k+\ell)} \quad (6)$$

is a solution of the mixed difference equation $\Delta_\ell \Delta_q v(k) = u(k)$.

Proof: From (1) and by taking $\Delta_q v(k) = u(k)$, we have

$$v(qk) = u(k) + v(k). \quad (7)$$

Replacing k by k/q in (7), we get

$$v(k) = u\left(\frac{k}{q}\right) + v\left(\frac{k}{q}\right). \quad (8)$$

Again replacing k by k/q^h , $h = 1, 2, \dots, (m-1)$ in (8) repeatedly and substituting

the resultant expressions in (7), we arrive

$$u(k) + u\left(\frac{k}{q}\right) + u\left(\frac{k}{q^2}\right) + \cdots + u\left(\frac{k}{q^m}\right) = v(qk) - v\left(\frac{k}{q^m}\right),$$

$$\text{i.e., } \sum_{h=0}^m u\left(\frac{k}{q^h}\right) = \Delta_q^{-1}u(qk) - \Delta_q^{-1}u\left(\frac{k}{q^m}\right). \quad (9)$$

For $r = 1, 2, 3, \dots, [k/\ell]$, replacing k by $k - r\ell$ in (9) and adding all the resultant expressions, we find that

$$\sum_{r=1}^{[k/\ell]} \sum_{h=0}^m u\left(\frac{k-r\ell}{q^h}\right) = \sum_{r=1}^{[k/\ell]} \Delta_q^{-1}u(q(k-r\ell)) - \sum_{r=1}^{[k/\ell]} \Delta_q^{-1}u\left(\frac{k-r\ell}{q^m}\right). \quad (10)$$

Adding (9) and (10) and applying (5), we arrive

$$\sum_{r=0}^{[k/\ell]} \sum_{h=0}^m u\left(\frac{k-r\ell}{q^h}\right) = \Delta_\ell^{-1} \Delta_q^{-1} \left\{ u(q(k+\ell)) - u(q\hat{\ell}(k)) - \left(u\left(\frac{k+\ell}{q^m}\right) - u\left(\frac{\hat{\ell}(k)}{q^m}\right) \right) \right\},$$

which completes the proof of the Theorem.

Theorem 3.2 [Multi mixed-summation formula] Let $k \in [\sum_{i=1}^n \ell_i, \infty)$, $q_i \neq 0$, $\ell_i > 0$ and m is any positive integer. Then equation (2) has a solution of the form

$$\sum_{i=1}^{n-1} \sum_{(r\ell)_{1 \rightarrow i}}^{[k]} \Delta_{\ell_{i+1 \rightarrow n}}^{-1} \Delta_{q_{1 \rightarrow t}}^{-1} u(k) \Big|_{\substack{q_1 \sim q_t \sim (\hat{\ell}_{i+1}(k - \sum_{j=1}^i r_j \ell_j) + \sum_{j=i+2}^n \ell_j) \\ \frac{1}{q_1} \sim \frac{1}{q_t}}} \\ + \sum_{(r\ell)_{1 \rightarrow n}}^{[k]} \sum_{(h)_{1 \rightarrow t}}^m u\left(\frac{k - \sum_{j=1}^n r_j \ell_j}{\prod_{p=1}^t q_p^{h_p}}\right) = \Delta_{\ell_{1 \rightarrow n}}^{-1} \Delta_{q_{1 \rightarrow t}}^{-1} u(k) \Big|_{\substack{q_1 \sim q_t \sim (k + \sum_{j=1}^n \ell_j) \\ \frac{1}{q_1} \sim \frac{1}{q_t} \sim (\hat{\ell}_1(k) + \sum_{j=2}^n \ell_j)}}, \quad (11)$$

where $\Delta_{\ell_{1 \rightarrow n}}^{-1} \Delta_{q_{1 \rightarrow t}}^{-1} u(k) \Big|_{\substack{q_1 \sim q_t \sim (\hat{\ell}_{i+1}(k - \sum_{j=1}^i r_j \ell_j) + \sum_{j=i+2}^n \ell_j) \\ \frac{1}{q_1} \sim \frac{1}{q_t}}}$

$$= \Delta_{\ell_{1 \rightarrow n}}^{-1} \Delta_{q_{1 \rightarrow t}}^{-1} u(k) \Big|_{\substack{\frac{k}{q_1} \\ \frac{k}{q_2} \\ \dots \\ \frac{k}{q_t}}} \Big|_{\substack{q_1 k \\ q_2 k \\ \dots \\ q_t k}} \Big|_{\substack{(\hat{\ell}_{i+1}(k - \sum_{j=1}^i r_j \ell_j) + \sum_{j=i+2}^n \ell_j)}}.$$

Proof: Replacing q, m by q_2, m_2 in (9), we get

$$u(k) + u\left(\frac{k}{q_2}\right) + u\left(\frac{k}{q_2^2}\right) + \cdots + u\left(\frac{k}{q_2^{m_2}}\right) = \Delta_{q_2}^{-1}u(q_2k) - \Delta_{q_2}^{-1}u\left(\frac{k}{q_2^{m_2}}\right). \quad (12)$$

Again replacing k by $k/q_1^{h_1}$ for $h_1 = 1, 2, \dots, m_1$ in (12), we find

$$u\left(\frac{k}{q_1^{h_1}}\right) + u\left(\frac{k}{q_1^{h_1}q_2}\right) + u\left(\frac{k}{q_1^{h_1}q_2^2}\right) + \dots + u\left(\frac{k}{q_1^{h_1}q_2^{m_2}}\right) = \Delta_{q_2}^{-1}u\left(\frac{q_2k}{q_1^{h_1}}\right) - \Delta_{q_2}^{-1}u\left(\frac{k}{q_1^{h_1}q_2^{m_2}}\right). \quad (13)$$

Adding (12) and (13) for $h_1 = 1, 2, \dots, m_1$ and applying (9), we arrive

$$\sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} u\left(\frac{k}{q_1^{h_1}q_2^{h_2}}\right) = \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u(q_1q_2k) - u\left(\frac{q_2k}{q_1^{m_1}}\right) \right\} - \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u\left(\frac{q_2k}{q_1^{m_1}}\right) - u\left(\frac{k}{q_1^{h_1}q_2^{m_2}}\right) \right\}. \quad (14)$$

For $r_1 = 1, 2, 3, \dots, [k/\ell_1]$, replacing k by $k - r_1\ell_1$ in (14), we find that

$$\sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} u\left(\frac{k - r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\right) = \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u(q_1q_2(k - r_1\ell_1)) - u\left(\frac{q_2(k - r_1\ell_1)}{q_1^{m_1}}\right) \right\} - \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u\left(\frac{q_2(k - r_1\ell_1)}{q_1^{m_1}}\right) - u\left(\frac{k - r_1\ell_1}{q_1^{h_1}q_2^{m_2}}\right) \right\}. \quad (15)$$

Adding (14) and (15), we obtain

$$\sum_{r_1=0}^{[k/\ell_1]} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} u\left(\frac{k - r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\right) = \sum_{r_1=0}^{[k/\ell_1]} \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u(q_1q_2(k - r_1\ell_1)) - u\left(\frac{q_2(k - r_1\ell_1)}{q_1^{m_1}}\right) \right\} - \sum_{r_1=0}^{[k/\ell_1]} \Delta_{q_1}^{-1}\Delta_{q_2}^{-1} \left\{ u\left(\frac{q_2(k - r_1\ell_1)}{q_1^{m_1}}\right) - u\left(\frac{k - r_1\ell_1}{q_1^{h_1}q_2^{m_2}}\right) \right\}. \quad (16)$$

Now, applying (5) in (16), we get

$$\sum_{r_1=0}^{[k/\ell_1]} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} u\left(\frac{k - r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\right) = \Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}u(k) \Big|_{\frac{1}{q_1^{m_1}} \sim \frac{1}{q_2^{m_2}}}^{q_1 \sim q_2 \sim (k+\ell_1)}. \quad (17)$$

Replacing r_1, ℓ_1 by r_2, ℓ_2 in (16) and repeating the above procedure, we get

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]} \sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} u\left(\frac{k - r_1\ell_1 - r_2\ell_2}{q_1^{h_1}q_2^{h_2}}\right) + \sum_{r_1=0}^{[\frac{k}{\ell_1}]} \Delta_{\ell_2}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}u(k) \Big|_{\frac{1}{q_1^{m_1}} \sim \frac{1}{q_2^{m_2}}}^{q_1 \sim q_2 \sim \hat{\ell}_2(k-r_1\ell_1)}$$

$$= \Delta_{\ell_1}^{-1} \Delta_{\ell_2}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} u(k) \Big|_{\substack{q_1 \sim q_2 \sim (k+\ell_1+\ell_2) \\ \frac{1}{q_1} \sim \frac{1}{q_2} \sim (\hat{\ell}_1(k)+\ell_2)}} \quad (18)$$

Also replacing $q_1, q_2, m_1, m_2, h_1, h_2$ by $q_2, q_3, m_2, m_3, h_2, h_3$ in (14), and then k by $k/q_1^{h_1}$ for $h_1 = 1, 2, \dots, m_1$ and so adding all the resultant expressions, we arrive

$$\begin{aligned} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \sum_{h_3=0}^{m_3} u\left(\frac{k}{q_1^{h_1} q_2^{h_2} q_3^{h_3}}\right) &= \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} u(q_1 q_2 q_3 k) - \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} u\left(\frac{q_2 q_3 k}{q_1^{m_1}}\right) \\ &- \sum_{h_1=0}^{m_1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} u\left(\frac{q_3 k}{q_1^{h_1} q_2^{m_2}}\right) - \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \Delta_{q_3}^{-1} u\left(\frac{k}{q_1^{h_1} q_2^{h_2} q_3^{m_3}}\right). \end{aligned} \quad (19)$$

Again using the above steps on (19) as we use to get the equations from (15) to (18), we obtain

$$\begin{aligned} \sum_{r_1=0}^{\lfloor \frac{k}{\ell_1} \rfloor} \sum_{r_2=0}^{\lfloor \frac{k-r_1 \ell_1}{\ell_2} \rfloor} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \sum_{h_3=0}^{m_3} u\left(\frac{k-r_1 \ell_1-r_2 \ell_2}{q_1^{h_1} q_2^{h_2} q_3^{h_3}}\right) &+ \sum_{r_1=0}^{\lfloor \frac{k}{\ell_1} \rfloor} \Delta_{\ell_2}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} u(k) \Big|_{\substack{q_1 \sim q_2 \sim q_3 \sim \ell_2(k-r_1 \ell_1) \\ \frac{1}{q_1} \sim \frac{1}{q_2} \sim \frac{1}{q_3}}} \\ &= \Delta_{\ell_1}^{-1} \Delta_{\ell_2}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} u(k) \Big|_{\substack{q_1 \sim q_2 \sim q_3 \sim (k+\ell_1+\ell_2) \\ \frac{1}{q_1} \sim \frac{1}{q_2} \sim \frac{1}{q_3} \sim (\hat{\ell}_1(k)+\ell_2)}}. \end{aligned} \quad (20)$$

Similarly, replacing k by $k/q_1^{h_1}$ for $h_1 = 1, 2, \dots, m_1$, $(t-3)$ times in (19) and repeating the above procedure, we get

$$\sum_{i=1}^{t-1} \sum_{(h_1)_{1 \rightarrow i}}^m \Delta_{q_1}^{-1} u\left(\frac{\prod_{p=i+2}^t q_p k}{\prod_{p=1}^i q_p^{h_p} q_{i+1}^m}\right) + \sum_{(h_1)_{1 \rightarrow t}}^m u\left(\frac{k}{\prod_{p=1}^t q_p^{h_p}}\right) = \Delta_{q_1}^{-1} u\left(\prod_{p=1}^t q_p k\right) - \Delta_{q_1}^{-1} u\left(\prod_{p=2}^t \frac{q_p k}{q_1^{m_1}}\right) \quad (21)$$

Now, replacing k by $k - r_1 \ell_1$ for $r_1 = 1, 2, 3, \dots, \lfloor k/\ell_1 \rfloor$ and applying the above procedure ' n' ' times successively, we get the desired result.

Remark 3.3 Note that, throughout this paper, to evaluate the expressions consisting of $\Delta_{\ell_i}^{-1} \Delta_{q_i}^{-1}$, first apply all the limits in terms of ' q_i ', then operating $\Delta_{q_i}^{-1}$ and then operating $\Delta_{\ell_i}^{-1}$ and at last apply limits in terms of ' ℓ_i '.

Corollary 3.4 Let $u(k) = k^2, k \in [0, \infty)$. Then we obtain

$$\sum_{r_1=0}^{\lfloor k/\ell_1 \rfloor} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \left(\frac{k-r_1 \ell_1}{q_1^{h_1} q_2^{h_2}}\right)^2 = \Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 \Big|_{\substack{q_1 \sim q_2 \sim (k+\ell_1) \\ \frac{1}{q_1} \sim \frac{1}{q_2} \sim (\hat{\ell}_1(k))}} \quad (22)$$

Proof: The proof is trivial by putting $t = 2$ and $n = 1$ in (11).

The example given below illustrates Corollary (3.4).

Example 3.5 Taking $m_1 = 2, m_2 = 1, \ell_1 = 6, q_1 = 3, q_2 = 4$ and $k = 10$ in (22), we

$$\text{get } \sum_{r_1=0}^{\lfloor k/\ell_1 \rfloor} \sum_{h_1=0}^{m_1} \sum_{h_2=0}^{m_2} \left(\frac{k - r_1 \ell_1}{q_1^{h_1} q_2^{h_2}} \right)^2 = \sum_{r_1=0}^1 \sum_{h_1=0}^2 \sum_{h_2=0}^1 \left(\frac{k - r_1 \ell_1}{q_1^{h_1} q_2^{h_2}} \right)^2 = 138.46604942.$$

By (1), we find $\Delta_{q_2}^{-1} k^2 = \frac{k^2}{(q_2^2 - 1)}$ and so $\Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 = \frac{k^2}{(q_1^2 - 1)(q_2^2 - 1)}$.

Also, by using the formula for k^n from (3), we obtain

$$\Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 = \Delta_{\ell_1}^{-1} \frac{\ell_1 k_{\ell_1}^{(1)} + k_{\ell_1}^{(2)}}{(q_1^2 - 1)(q_2^2 - 1)}. \quad (23)$$

Using the notation given in the Preliminaries section and by Corollary (3.4), we write

$$\begin{aligned} \Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 \Big|_{\substack{q_1 \sim q_2 \sim (k+\ell_1) \\ \frac{1}{q_1^{m_1}} \sim \frac{1}{q_2^{m_2}} \sim (\hat{\ell}_1(k))}} &= \Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 \Big|_{\substack{q_1 k \Big|_{\frac{k}{q_1^{m_1}}} \Big|_{\frac{k}{q_2^{m_2}}} \Big|_{\hat{\ell}_1(k)} \\ q_2 k \Big|_{\frac{k}{q_2^{m_2}}} \Big|_{\hat{\ell}_1(k)}}} \\ &= \Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \left(q_1^2 - \frac{1}{q_1^{2m_1}} \right) \left(q_2^2 - \frac{1}{q_2^{2m_2}} \right) k^2 \Big|_{\hat{\ell}_1(k)}. \end{aligned}$$

Using (4) in (23) and by applying the limits for k , the above equation becomes

$$\begin{aligned} \Delta_{\ell_1}^{-1} \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} k^2 \Big|_{\substack{q_1 \sim q_2 \sim (k+\ell_1) \\ \frac{1}{q_1^{m_1}} \sim \frac{1}{q_2^{m_2}} \sim (\hat{\ell}_1(k))}} &= \frac{\left(q_1^2 - \frac{1}{q_1^{2m_1}} \right) \left(q_2^2 - \frac{1}{q_2^{2m_2}} \right)}{(q_1^2 - 1)(q_2^2 - 1)} \left(\frac{k_{\ell_1}^{(2)}}{2} + \frac{k_{\ell_1}^{(3)}}{3\ell_1} \right) \Big|_{\hat{\ell}_1(k)} \\ &= 138.46604942. \end{aligned}$$

Theorem 3.6 [q - ℓ mixed higher order Summation Formula] Let $k \in [0, \infty)$, $q \neq 0$ and $\ell > 0$. Then the equation $\Delta_\ell^n \Delta_q^t v(k) = u(k)$ has a solution of the form

$$\begin{aligned} &\sum_{r=0}^{\lfloor k/\ell \rfloor} \sum_{h=0}^m \binom{h+t-1}{t-1} \binom{r+n-1}{n-1} u\left(\frac{k-r\ell}{q^h}\right) \\ &+ \sum_{r=1}^{n-1} \frac{\left(\left[\frac{k}{\ell}\right] + r\right)^{(r)}}{r!} \Delta_\ell^{-(n-r)} \Delta_q^{-t} u\left(q^{t-1}k\right) \Big|_{\frac{1}{q^m}}^{q \sim (\hat{\ell}(k) + (n-r-1)\ell)} \\ &+ \sum_{h=1}^{t-1} \frac{(m+h)^{(h)}}{h!} \Delta_\ell^{-n} \Delta_q^{-(t-h)} u\left(\frac{q^{t-(h+1)}k}{q^m}\right) \Big|_{\hat{\ell}(k) + (n-1)\ell}^{k+n\ell} \\ &- \sum_{h=1}^{t-1} \sum_{r=1}^{n-1} \frac{(m+h)^{(h)}}{h!} \frac{\left(\left[\frac{k}{\ell}\right] + r\right)^{(r)}}{r!} \Delta_\ell^{-(n-r)} \Delta_q^{-(t-h)} u\left(\frac{q^{t-(h+1)}k}{q^m}\right) \Big|_{\hat{\ell}(k) + (n-r-1)\ell} \end{aligned}$$

$$= \Delta_\ell^{-n} \Delta_q^{-t} u(q^{t-1}k) \Big|_{\frac{1}{q^m} \sim (\hat{\ell}(k)+(n-1)\ell)}^{q \sim (k+n\ell)} \quad (24)$$

Proof: From (9), we have

$$u(k) + u\left(\frac{k}{q}\right) + u\left(\frac{k}{q^2}\right) + \cdots + u\left(\frac{k}{q^m}\right) = \Delta_q^{-1}u(qk) - \Delta_q^{-1}u\left(\frac{k}{q^m}\right). \quad (25)$$

Replacing k by k/q^h for $h = 1, 2, \dots, m$ in (25) and then summing all the resultant expressions with (25), we arrive

$$\sum_{h=0}^m \binom{h+1}{1} u\left(\frac{k}{q^h}\right) = \Delta_q^{-2}u(q^2k) - \Delta_q^{-2}u\left(\frac{qk}{q^m}\right) - (m+1)\Delta_q^{-1}u\left(\frac{k}{q^m}\right). \quad (26)$$

Also replacing k by $k - r\ell$ for $r = 1, 2, \dots, [k/\ell]$ in (26) and then adding all the resultant expressions with (26), we arrive

$$\sum_{r=0}^{[k/\ell]} \left\{ \Delta_q^{-2}u(q^2(k-r\ell)) - \Delta_q^{-2}u\left(\frac{q(k-r\ell)}{q^m}\right) - (m+1)\Delta_q^{-1}u\left(\frac{k-r\ell}{q^m}\right) \right\}. \quad (27)$$

By (5), (27) becomes

$$\sum_{h=0}^m \binom{h+1}{1} \sum_{r=0}^{[k/\ell]} u\left(\frac{k-r\ell}{q^h}\right) = \Delta_\ell^{-1} \Delta_q^{-2} \left\{ u(q^2(k+\ell)) - u(q^2\hat{\ell}(k)) \right\} - \Delta_\ell^{-1} \Delta_q^{-2} \left\{ u\left(\frac{q(k+\ell)}{q^m}\right) - u\left(\frac{q\hat{\ell}(k)}{q^m}\right) \right\} - (m+1)\Delta_\ell^{-1} \Delta_q^{-1} \left\{ u\left(\frac{k+\ell}{q^m}\right) - u\left(\frac{\hat{\ell}(k)}{q^m}\right) \right\}, \quad (28)$$

which yields

$$\sum_{h=0}^m \binom{h+1}{1} \sum_{r=0}^{[k/\ell]} u\left(\frac{k-r\ell}{q^h}\right) = \Delta_\ell^{-1} \Delta_q^{-2} u(qk) \Big|_{\frac{1}{q^m} \sim \hat{\ell}(k)}^{q \sim (k+\ell)} - (m+1)\Delta_\ell^{-1} \Delta_q^{-1} u\left(\frac{k}{q^m}\right) \Big|_{\hat{\ell}(k)}^{k+\ell}.$$

Again replacing k by $k - r\ell$ for $r = 1, 2, \dots, [k/\ell]$ in (28) and summing all the resultant expressions, we obtain

$$\sum_{h=0}^m \binom{h+1}{1} \sum_{r=0}^{[k/\ell]} \binom{r+1}{1} u\left(\frac{k-r\ell}{q^h}\right) = \Delta_\ell^{-2} \Delta_q^{-2} u(qk) \Big|_{\frac{1}{q^m} \sim (\hat{\ell}(k)+\ell)}^{q \sim (k+2\ell)}$$

$$\begin{aligned}
 & -(m+1)\Delta_\ell^{-2}\Delta_q^{-1}u\left(\frac{k}{q^m}\right)\Big|_{\hat{\ell}(k)+\ell}^{k+2\ell} - \left(\left[\frac{k}{\ell}\right]+1\right)\Delta_\ell^{-1}\Delta_q^{-2}u(qk)\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)} \\
 & \quad + (m+1)\left(\left[\frac{k}{\ell}\right]+1\right)\Delta_\ell^{-1}\Delta_q^{-1}u\left(\frac{k}{q^m}\right)\Big|_{\hat{\ell}(k)}. \tag{29}
 \end{aligned}$$

Replacing k by k/q^h for $h = 1, 2, \dots, m$ in (26) and then summing all the resultant expressions with (26), we arrive

$$\begin{aligned}
 \sum_{h=0}^m \binom{h+2}{2} u\left(\frac{k}{q^h}\right) &= \Delta_q^{-3}u(q^3k) - \Delta_q^{-3}u\left(\frac{q^2k}{q^m}\right) - (m+1)\Delta_q^{-2}u\left(\frac{qk}{q^m}\right) \\
 &\quad - \frac{(m+2)^{(2)}}{2}\Delta_q^{-1}u\left(\frac{k}{q^m}\right). \tag{30}
 \end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}
 \sum_{h=0}^m \binom{h+t-1}{t-1} u\left(\frac{k}{q^h}\right) &+ \sum_{h=1}^{t-1} \frac{(m+h)^{(h)}}{h!}\Delta_q^{-(t-h)}u\left(\frac{q^{t-(h+1)}k}{q^m}\right) \\
 &= \Delta_q^{-t}u(q^tk) - \Delta_q^{-t}u\left(\frac{q^{t-1}k}{q^m}\right). \tag{31}
 \end{aligned}$$

And then replacing k by $k - r\ell$ for $r = 1, 2, \dots, [k/\ell]$ in (31) and using the above procedure with Δ_ℓ^{-1} , 'n' times repeatedly, we get the proof of the theorem.

Corollary 3.7 For any bounded real valued function $u(k)$ and $q \neq 0$, we obtain

$$\begin{aligned}
 \sum_{r=0}^{[k/\ell]} \sum_{h=0}^m \binom{h+3}{3} \binom{r+1}{1} u\left(\frac{k-r\ell}{q^h}\right) &+ \left(\left[\frac{k}{\ell}\right]+1\right)\Delta_\ell^{-1}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)} \\
 &+ \sum_{h=1}^3 \frac{(m+h)^{(h)}}{h!}\Delta_\ell^{-2}\Delta_q^{-(4-h)}u\left(\frac{q^{4-(h+1)}k}{q^m}\right)\Big|_{\hat{\ell}(k)+\ell}^{k+2\ell} - \sum_{h=1}^3 \frac{(m+h)^{(h)}}{h!}\left(\left[\frac{k}{\ell}\right]+1\right) \\
 &\quad \Delta_\ell^{-1}\Delta_q^{-(4-h)}u\left(\frac{q^{4-(h+1)}k}{q^m}\right)\Big|_{\hat{\ell}(k)} = \Delta_\ell^{-2}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}}^{q\sim(k+2\ell)} \sim_{\hat{\ell}(k)+\ell}. \tag{32}
 \end{aligned}$$

Proof: The proof is obvious by putting $t = 4$ and $n = 2$ in (24). The following example is a verification of Corollary 3.7.

Example 3.8 Consider $u(k) = k_q^{(2)}$, $m = 2$, $k = 15$, $\ell = 8$ and $q = 5$ in (32). Then using the procedure in Example (3.5), we get

$$\begin{aligned} \Delta_\ell^{-1} \Delta_q^{-4} u(q^3 k) \Big|_{\frac{1}{q^m}}^{q \sim \hat{\ell}(k)} &= \Delta_\ell^{-1} \Delta_q^{-4} \{ (q^8 - q^2)k^2 - (q^5 - q^2)k \} \Big|_{\frac{1}{q^m}}^{\hat{\ell}(k)} \\ &= \left\{ \frac{(q^8 - q^2)}{(q^2 - 1)^4} \left(\frac{k_\ell^{(2)}}{2} + \frac{k_\ell^{(3)}}{3\ell} \right) - \frac{(q^5 - q^2)k_\ell^{(2)}}{2\ell(q - 1)^4} \right\} \Big|_{15}^7, \\ \sum_{h=1}^3 \frac{(m+h)^{(h)}}{h!} \Delta_\ell^{-2} \Delta_q^{-(4-h)} u \left(\frac{q^{4-(h+1)}k}{q^m} \right) \Big|_{\hat{\ell}(k)+\ell}^{k+2\ell} &= \left\{ \left(\frac{3}{(q^2 - 1)^3} + \frac{6}{q^2(q^2 - 1)^2} \right. \right. \\ &\quad \left. \left. + \frac{10}{q^4(q^2 - 1)} \right) \left(\frac{k_\ell^{(3)}}{6\ell} + \frac{k_\ell^{(4)}}{12\ell^2} \right) - \left(\frac{3q}{(q - 1)^3} + \frac{6}{(q - 1)^2} + \frac{10}{q(q - 1)} \right) \frac{k_\ell^{(3)}}{6\ell^2} \right\} \Big|_{15}^{31}, \\ \sum_{h=1}^3 \frac{(m+h)^{(h)}}{h!} \Delta_\ell^{-1} \Delta_q^{-(4-h)} u \left(\frac{q^{4-(h+1)}k}{q^m} \right) \Big|_{\hat{\ell}(k)}^{\hat{\ell}(k)} &= \left\{ \left(\frac{3}{(q^2 - 1)^3} + \frac{6}{q^2(q^2 - 1)^2} \right. \right. \\ &\quad \left. \left. + \frac{10}{q^4(q^2 - 1)} \right) \left(\frac{k_\ell^{(2)}}{2} + \frac{k_\ell^{(3)}}{3\ell} \right) - \left(\frac{3q}{(q - 1)^3} + \frac{6}{(q - 1)^2} + \frac{10}{q(q - 1)} \right) \frac{k_\ell^{(2)}}{2\ell} \right\} \Big|_{15}^7 \end{aligned}$$

and

$$\Delta_\ell^{-2} \Delta_q^{-4} u(q^3 k) \Big|_{\frac{1}{q^m}}^{q \sim (k+2\ell)} = \left\{ \frac{(q^8 - q^2)}{(q^2 - 1)^4} \left(\frac{k_\ell^{(3)}}{6\ell} + \frac{k_\ell^{(4)}}{12\ell^2} \right) - \frac{(q^5 - q^2)k_\ell^{(3)}}{6\ell^2(q - 1)^4} \right\} \Big|_{15}^{31}.$$

Putting these values in (32) and by applying the limits for 'k', we get

$$\begin{aligned} \sum_{r=0}^1 \sum_{h=0}^2 \binom{h+3}{3} \binom{r+1}{1} \left(\frac{k-8r}{5^h} \right)_5^{(2)} + 2 \Delta_\ell^{-1} \Delta_q^{-4} (q^3 k)_5^{(2)} \Big|_{\frac{1}{q^m}}^{q \sim \hat{\ell}(k)} \\ + \sum_{h=1}^3 \frac{(2+h)^{(h)}}{h!} \Delta_\ell^{-2} \Delta_q^{-(4-h)} \left(\frac{q^{4-(h+1)}k}{q^m} \right)_5^{(2)} \Big|_{\hat{\ell}(k)+\ell}^{k+2\ell} - 2 \sum_{h=1}^3 \frac{(2+h)^{(h)}}{h!} \\ \Delta_\ell^{-1} \Delta_q^{-(4-h)} \left(\frac{q^{4-(h+1)}k}{q^2} \right)_5^{(2)} \Big|_{\hat{\ell}(k)}^{\hat{\ell}(k)} = \Delta_\ell^{-2} \Delta_q^{-4} (q^3 k)_5^{(2)} \Big|_{\frac{1}{q^m}}^{q \sim (k+2\ell)} = 37.6315646. \end{aligned}$$

4. Conclusion

In this paper, we have derived multi-series summation formula for mixed q - ℓ difference equation. Moreover, solution for higher order mixed difference equation is also obtained. Consequently, relevant examples are being given to verify the results.

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