ISSN: 2456-8686, Volume 1, Issue 2, 2017:46-53
DOI : http://doi.org/10.26524/cm15

Journal of Computational Mathematica

Open Access

Stability of Orthogonally Quintic Functional Equation in Multi-Banach Spaces

Murali R[1*] and Antony Raj A[2]
[1,2]Department of Mathematics, Sacred Heart College,
Tirupattur, TamilNadu, India.

Abstract

In this paper, we establish the Hyers-Ulam stability of the orthogonally quintic functional equation in Multi-Banach Spaces.

Key words: Hyers-Ulam stability, Multi-Banach Spaces, orthogonally quintic functional equation, Fixed Point Method.

AMS Subject classification: 39B82, 39B52, 47H10, 46H25

1. Introduction

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutionsthe set of functions that satisfy the equation. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a closed-form expression for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Theorem 1.1 [10] Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for

---

[1*]shcrmurali@yahoo.co.in,

each given element $x \in X$, either

$$d(T^n, T^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer m such that

1. $d(T^n, T^{n+1}x) < \infty, \forall n \geq m$;
2. the sequence $\{T^n x\}$ converges to a fixed point u^* of T;
3. u^* is the unique fixed point of T in the set $Y = \{u \in X : d(T^m x, u) < \infty\}$;
4. $d(u, u^*) \leq \dfrac{1}{1-\alpha} d(u, Tu)$ for all $u \in Y$.

Definition 1.2 [3] A Multi- norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

1. $\left\|\left(x_{\sigma(1)}...x_{\sigma(k)}\right)\right\|_k = \|(x_1...x_k)\|_k$, for $\sigma \in \Psi_k, x_1...x_k \in \wp$;
2. $\|\alpha_1 x_1...\alpha_k x_k\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|(x_1...x_k)\|_k$ for $\alpha_1...\alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;
3. $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ...x_{k-1})\|_{k-1}$, for $x_1, ...x_{k-1} \in \wp$;
4. $\|(x_1...x_{k-1}, x_{k-1})\|_k = \|(x_1...x_{k-1})\|_{k-1}$ for $x_1...x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed space.

Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed spaces, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [3].

$$(a) \, \|(x, ...x)\|_k = \|x\|, \text{for} x \in \wp,$$

$$(b) \max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|, \text{for} x_1, ..., x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - Banach space.

Definition 1.3 [9] Suppose that X is a vector space (algebraic module) with $\dim X \geq 2$, and $\perp$ is a binary relation on X with the following properties:

1. Totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
2. Independence: If $x, y \in X - 0$ and $x \perp y$, then x and y are linearly independent;
3. Homogeneity: If $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
4. Thalesian properity: If P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$ which is the set of non-negative real numbers, then there exists $y_0 \in P$ such that

---

[1*] shcrmurali@yahoo.co.in,

$x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (resp., module). By an orthogonality normed space (normed module) we mean an orthogonality space (resp., module) having a normed (resp., normed module) structure.

In this paper, we achieve the Hyers - Ulam stability in orthogonally quintic functional equation in Multi-Banach Spaces

$$\begin{aligned} Df(x,y) = f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y) - 5f(x-y) \\ -10f(y) - f(3x) + 3f(2x) + 27f(x). \end{aligned} \tag{1}$$

Theorem 1.4 Let X be an orthogonality space and let $\big((Y^k, \|.\|) : K \in \mathbb{N}\big)$ be a multi-Banach Suppose that β is a nonnegative real number and $f : X \to Y$ is a mapping satisfying

$$\sup_{k \in \mathbb{N}} \|(Df(x_1, y_1), ..., Df(x_k, y_k))\|_k \leq \beta \tag{2}$$

$x_1, ..., x_k, y_1, ..., y_k \in P$ and $x_i \perp y_i$ $(i = 1, 2...k)$ and $f(0) = 0$. Then there exists a unique orthogonally quintic mapping $Q_5 : X \to Y$ such that

$$\sup_{k \in \mathbb{N}} \|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\|_k \leq \frac{1}{31}\beta \tag{3}$$

$x_1, x_2, ..., x_k \in X$. Proof: Letting $y_1 = y_2 =, ..., = y_k = 0$ in (2), we obtain that

$$\sup_{k \in \mathbb{N}} \|(32f(x_1) - f(2x_1), ..., 32f(x_k) - f(2x_k))\| \leq \beta \tag{4}$$

for all $x_1, ..., x_k \in X, x_i \perp 0$ where$(i = 1, 2, ..., k)$. Dividing on both side by 32 in (4), we get

$$\sup_{k \in \mathbb{N}} \left\| \left(f(x_1) - \frac{1}{32} f(2x_1), ..., f(x_k) - \frac{1}{32} f(2x_k) \right) \right\| \leq \frac{1}{32}\beta \tag{5}$$

Let $\Lambda = \{g : P \to Q | g(0) = 0\}$ and introduce the generalized metric d defined on λ by

$$d(l, m) = \inf \left\{ \lambda \in [0, \infty] | \sup_{k \in \mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \lambda \quad \forall x_1, ..., x_k \in X \right\}$$

Then it is easy to show that Λ, d is a generalized complete metric space. See [8].

[1*] shcrmurali@yahoo.co.in,

We define an operator $\mathcal{J} : P \to P$ by

$$\mathcal{J}l(x) = \frac{1}{32}l(2x) \qquad x \in X.$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l, m \in \Lambda$, let $\lambda \in [0, \infty]$ be an arbitary constant with $d(l,m) \leq \lambda$. From the definition d, if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \lambda \quad x_1, ..., x_k \in X.$$

Therefore,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1) - \mathcal{J}m(x_1), ..., \mathcal{J}l(x_k) - \mathcal{J}m(x_k))\|_k \\
&\qquad \leq \sup_{k\in\mathbb{N}} \left\| \left(\frac{1}{32}l(2x_1) - \frac{1}{32}m(2x_1), ..., \frac{1}{32}l(2x_k) - \frac{1}{32}m(2x_k) \right) \right\|_k \\
&\qquad \leq \frac{1}{32}\lambda
\end{aligned}$$

$x_1, ..., x_k \in X$.

Hence,it holds that

$$d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{32}\lambda d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{32}d(l,m) \qquad \forall l, m \in \Lambda.$$

This Means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $L = \frac{1}{32}$.

By (5), we have $d(\mathcal{J}f, f) \leq \frac{1}{32}\beta < \infty$. According to Theorem (1.1), we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $Q_5 : P \to Q$ such that

$$Q_5(2x) = 32Q_5(x) \qquad \forall x \in X.$$

Moreover, we have $d\left(\mathcal{J}^n f, Q_5\right) \to 0$, which implies

$$Q_5(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{32^n}$$

for all $x \in X$.

---

[1]*shcrmurali@yahoo.co.in,

Also, $d(f, Q_5) \leq \dfrac{1}{1-L} d(\mathcal{J}f, f)$ implies the inequality

$$\begin{aligned} d(f, Q_5) &\leq \frac{1}{1-\dfrac{1}{32}} d(\mathcal{J}f, f) \\ &\leq \frac{1}{31}\beta. \end{aligned}$$

Considering Definition, we have $2^n x \perp 2^n y$. Set $x_1 =, \cdots, = x_k = 2^n x$, $y_1 =, \cdots, = y_k = 2^n y$ in (2) and divide both sides by 32^n. Then, using property (a) of multi-norms, we obtain

$$\begin{aligned} \|DQ_5(x,y)\| &= \lim_{n\to\infty} \frac{1}{32^n} \|Df(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{\beta}{32^n} = 0 \end{aligned}$$

for all $x, y \in X$. Hence Q_5 is Quintic.
The uniqueness of Q_5 follows from the fact that Q_5 is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\|_k \leq \ell$$

for all $x_1, ..., x_k \in X$. This completes the proof of the Theorem.

Theorem 1.5 Let $\phi : X^{2k} \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\phi(x_1, y_1, .., x_k, y_k) \leq 32\alpha\phi\left(\frac{x_1}{2}, \frac{y_1}{2}, .., \frac{x_k}{2}, \frac{y_k}{2}\right) \tag{6}$$

for all $x_i, y_i \in X$ with $x_i \perp y_i$, where $i = 1, .., k$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|Df(x_1, y_1, .., x_k, y_k)\| \leq \phi(x_1, y_1, .., x_k, y_k) \tag{7}$$

for all $x_i, y_i \in X$ with $x_i \perp y_i$, where $i = 1, .., k$. Then there exists a unique orthogonally quintic mapping $Q_5 : X \to Y$ such that

$$\|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\| \leq \frac{\alpha}{1-\alpha}\phi(x_1, 0, ..., x_k, 0) \tag{8}$$

---

^{1*}shcrmurali@yahoo.co.in,

for all $x_i \in X$, where $i = 1, .., k$. Proof: Taking $y_i = 0$ in (7), we get

$$\|(32f(x_1) - f(2x_1), .., 32f(x_k) - f(2x_k))\| \leq \phi\left(x_1, 0, .., x_k, 0\right) \tag{9}$$

for all $x_i \in X$, since $x_i \perp 0$, where $i = 1, .., k$. So

$$\left\|\left(f(x_1) - \frac{1}{32}f(2x_1), ..., f(x_k) - \frac{1}{32}f(2x_k)\right)\right\| \leq \alpha\phi\left(x_1, 0, .., x_k, 0\right) \tag{10}$$

for all $x_i \in X$, where $i = 1, .., k$. Consider the set $G : h : X \to Y$ and introduce the generalized metric on G.

$$d(g, h) = inf\left\{\mu \in \mathbb{R}_+ : \|(g(x_1) - h(x_1), .., g(x_k) - h(x_k))\| \leq \mu\phi\left(x_1, 0, .., x_k, 0\right) \quad \forall x_i \in X\right\}$$

where $i = 1, .., k$. It is easy to prove that (G, d) is complete.See [8]. It follows from (10) that $d(f, Jf) \leq \alpha$. The rest of the proof is similar to the proof of Theorem 1.1.

Corollary 1.6 Let θ be a positive real number and p a real number with $p > 5$. Let $f : X \to Y$ be a mapping satisfying

$$\|(Df(x_1, y_1, .., x_k, y_k))\| \leq \theta\left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right) \tag{11}$$

for all $x_i, y_i \in X$, since $x_i \perp y_i$, where $i = 1, .., k$. Then there exists a unique orthogonally quintic mapping $Q_5 : X \to Y$ such that

$$\|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\| \leq \frac{2^p\theta}{32 - 2^p}\left(\|x_1\|^p, ..., \|x_k\|^p\right)$$

for all $x_i \in X$, where $i = 1, .., k$.
Proof: The proof follows from Theorem1.5 by taking
$\phi(x_1, y_1, .., x_k, y_k) = \theta\left(\|x_1\|^p + \|y_1\|^p, .., \|x_k\|^p + \|y_k\|^p\right)$ for all $x_i, y_i \in X$, since $x_i \perp y_i$, where $i = 1, .., k$. Then we can choose $\alpha = 2^{p-5}$ and we get the desired result.

References

[1] Aoki T, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Jpn, 2, 1950, 64-66.

[2] Choonkil Park, Jian Lian CUI, Madjid Eshaghi GORDJI, Orthogonality and Quintic Functional Equations, Acta Mathematica Sinica, 29, 2013, 1381-1390.

---

^{1*}shcrmurali@yahoo.co.in,

[3] Dales HG and Moslehian, Stability of mappings on multi-normed spaces, Glasgow Mathematical Journal, 49, 2007, 321-332.

[4] Fridoun Moradlou, Approximate Euler-Lagrange-Jensen type Additive mapping in Multi-Banach Spaces: A Fixed point Approach, Commun. Korean Math. Soc., 28, 2013, 319-333.

[5] Hyers DH, On the stability of the linear functional equation. Proc. Natl. Acad. Sci., USA, 27, 1941, 222-224.

[6] Jun K and Kim H, The Generalized Hyers-Ulam-Rassias stability of a cubic functional equation, J. Math. Anal. Appl., 274, 2002, 867-878.

[7] Lee S, Im S and Hwang I, Quartic functional equations, J. Math. Anal. Appl., 307, 2005, 387-394.

[8] Mihet D and Radu V, On the stability of the additive Cauchy functional equation in random normed spaces, Journal of mathematical Analysis and Applications, 343, 2008, 567-572.

[9] Ratz J, On Orthogonally Additive Mappings, Aequationes Mathematicae, 28, 1985, 35-49.

[10] Radu V, The fixed point alternative and the stability of functional equations, Fixed Point Theory, 4, 2003, 91-96.

[11] Sattar Alizadeh, Fridoun moradlou, Approximate a quadratic mapping in Multi-Banach Spaces, A Fixed Point Approach, Int. J. Nonlinear Anal. Appl., 7, 2016, 63-75.

[12] Tian Zhou Xu, John Michael Rassias and Wan Xin Xu, Generalized Ulam - Hyers Stability of a General Mixed AQCQ functional equation in Multi-Banach Spaces: A Fixed point Approach, European Journal of Pure and Applied Mathematics, 3, 2010, 1032-1047.

[13] Xiuzhong Wang, Lidan Chang, Guofen Liu, Orthogonal Stability of Mixed Additive-Quadratic Jenson Type Functional Equation in Multi-Banach Spaces, Advances in Pure Mathematics, 5, 2015, 325-332.

[14] Rassias TM, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc., 72, 1978, 297-300.

[15] Czerwik S, Functional Equations and Inequalities in Several Variables, World Scientific Publishing Co., Singapore, New Jersey, London, (2002).

---

[1]*shcrmurali@yahoo.co.in,

[16] Hyers DH, Isac G and Rassias TM, Stability of Functional Equations in Several Variables, Birkhuser, Basel, (1998).

[17] Liguang Wang, Bo Liu and Ran Bai, Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach, Fixed Point Theory and Applications, (2010), 9 pages.

[18] Ulam SM, A Collection of the Mathematical Problems, Interscience, New York, (1960).

[19] Wilansky A, Modern Methods in Topological Vector Space, Mc Graw-Hill, New York, (1978).

[20] Zhihua Wang, Xiaopei Li and Themistocles M Rassias, Stability of an Additive-Cubic-Quartic Functional Equation in Multi-Banach Spaces, Abstract and Applied Analysis, (2011), 11 pages.

[1*] shcrmurali@yahoo.co.in,