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A Study On The Cross Product Of Fuzzy Numbers

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Abstract

In fuzzy arithmetic, the multiplication operation based on Zadeh's extension principle owns several unnatural properties both from theoretical and practical points of view. To overcome some of these shortcomings, a new operation called cross product has been introduced recently. We show the main properties of the cross product. We also present a comparative study of the traditional multiplication and the cross product in geological applications, especially for estimating of resources of solid mineral deposits.

Key words: Fuzzy number, cross product, solid mineral deposit estimation.

AMS classification: 03E72, 16Y80, 03B52, 20N25.

1 Introduction

In this section we study the theoretical properties of the cross product of fuzzy numbers. Let $R_F^* = \{u \in R_F : u \text{ is positive or negative}\}$. Firstly we begin with a theorem which was obtained by using the stacking theorem.

Theorem 1.1 If u and v are positive fuzzy numbers then $w = u \odot v$ defined by $[w]^r = [\underline{w}^r, \overline{w}^r]$, where $\underline{w}^r = \underline{u}^r \underline{v}^1 + \underline{u}^1 \underline{v}^r - \underline{u}^1 \underline{v}^1$ and $\overline{w}^r = \overline{u}^r \overline{u}^1 + \overline{u}^1 \overline{v}^r - \overline{u}^1 \overline{v}^1$, for every $r \in [0,1]$, is a positive fuzzy number.

Corollary 1.2 Let u and v be two fuzzy numbers.

(i) If u is positive and v is negative then $u \odot v = -(u \odot (-v))$ is a negative fuzzy number;

(ii) If u is negative and v is positive then $u \odot v = -((-u) \odot v)$ is a negative fuzzy number;

(iii) If u and v are negative then $u \odot v = (-u) \odot (-v)$ is a positive fuzzy Number.

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Definition 1.3 The binary operation $\odot$ on R_F^* introduced by Theorem 1.1 and Corollary 1.2 is called cross product of fuzzy numbers.

Remark 1.4 1) The cross product is defined for any fuzzy numbers in $R_F^{\wedge} = \{u \in R_F^*;$ there exists an unique $x_0 \in \mathbf{R}$ such that $u(x_0) = 1\}$, so implicitly for any triangular fuzzy number. In fact, the cross product is defined for any fuzzy number.
2) The below formulas of calculus can be easily proved $(r \in [0,1])$:

$$\underline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1,$$

$$\overline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1$$

If u is positive and v is negative, $\underline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1$, $\overline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1$
If u is negative and v is positive. In the last possibility, if u and v are negative then

$$\underline{(u \odot v)}^r = \overline{u}^r \overline{v}^1 + \overline{u}^1 \overline{v}^r - \overline{u}^1 \overline{v}^1, \ \overline{(u \odot v)}^r = \underline{u}^r \underline{v}^1 + \underline{u}^1 \underline{v}^r - \underline{u}^1 \underline{v}^1$$

3) The cross product extends the scalar multiplication of fuzzy numbers. Indeed, if one of operands is the real number k identified with its characteristic function then $\underline{k}^r = \overline{k}^r = k, \forall r \in [0,1]$ and following the above formulas of calculus we get the result.

The main algebraic properties of the cross product are the following.

Theorem 1.5 If $u, v, w \in R_F^*$ then

(i) $(-u) \odot v = u \odot (-v) = -(u \odot v)$;

(ii) $u \odot v = v \odot u$;

(iii) $(u \odot v) \odot w = u \odot (v \odot w)$;

(iv) If u and v have the same sign then $(u \oplus v) \odot w = (u \odot w) \oplus (v \odot w)$;

(v) $(u \odot v)^{\odot n} = u^{\odot n} \odot v^{\odot n}, \forall n \in N^*$, where $a^{\odot n} = \underbrace{a \odot \ldots \odot a}_{n \text{ times}}$ for any $a \in R_F^*$.

Remark 1.6 1) If u is positive and v negative (or u is negative and v positive) then the property of distributivity in (iv) is not verified even if u and v are real numbers.

2) The above properties (i)-(iii) hold for the usual product "$\cdot$" based on the extension

principle. The property (iv) holds in a weaker form: If u and v are on the same side of 0 then for any $w, w \prec 0$ or $0 \prec w$ we have $(u \oplus v) \cdot w = (u \cdot w) \oplus (v \cdot w)$

The so-called $L - R$ fuzzy numbers are considered important in fuzzy arithmetic. These and their particular cases triangular and trapezoidal fuzzy numbers are used almost exclusively in applications.

Definition 1.7 Let $L, R : [0, +\infty) \to [0,1]$ be two continuous, decreasing functions fulfilling $L(0) = R(0) = 1, L(1) = R(1) = 0$, invertible on $[0,1]$. Moreover, let a^1 be any real number and suppose $\underline{a}, \bar{a}$ be positive numbers. The fuzzy set $u : \mathbf{R} \to [0,1]$ is an $L - R$ fuzzy number if $u(t) = \begin{cases} L\left(\frac{a^1 - t}{\underline{a}}\right), & \text{for } t \leq a^1 \\ R\left(\frac{t - a^1}{\bar{a}}\right), & \text{for } t > a^1 \end{cases}$.

Symbolically, we write $u = \left(a^1, \underline{a}, \bar{a}\right)_{L,R}$, where a^1 is called the mean value of $u, \underline{a}, \bar{a}$ are called the left and the right spread. If u is an $L - R$ fuzzy number then

$$[u]^r = \left[a^1 - L^{-1}(r)\underline{a}, a^1 + R^{-1}(r)\bar{a}\right]$$

Theorem 1.8 If u and v are strict positive $L - R$ fuzzy numbers then $u \odot v$ is a strict positive $L - R$ fuzzy number.

Since we are interested mainly in the applications of the cross product we may restrict our attention to positive fuzzy numbers, however in other cases some similar properties can be obtained.
The cross product verifies the following metric property.

Theorem 1.9 If u, v have the same sign and $w \in R_F^*$ then

$$D(w \odot u, w \odot v) \leq K_w D(u,v), \text{ where } K_w = \max\left\{\left|\overline{w}^1\right|, \left|\underline{w}^1\right|\right\} + \overline{w}^0 - \underline{w}^0.$$

Definition 1.10 Let u be a fuzzy number. The crisp number $\Delta_L^r(u) = \underline{u}^1 - \underline{u}^r$ is called $r-$ error to left of u and the crisp number $\Delta_R^r(u) = \bar{u}^r - u^{-1}$ is called r -error to right of u, where $r \in [0,1]$. The sum $\Delta^r(u) = \Delta_L^r(u) + \Delta_R^r(u)$ is called r -error of u.

If u expresses the fuzzy concept A then $\Delta_L^r(u)$ and $\Delta_R^r(u)$ can be interpreted as the values of tolerance of level r from the concept A to left and to right, respectively.

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For example, if the triangular fuzzy number $u = (5,7,9)$ expresses "early morning" then $\Delta_L^{\frac{1}{2}}(u) = 1$ (one hour) is the tolerance of level $\frac{1}{2}$ of u towards night from the concept of "early morning" and $\Delta_R^{\frac{1}{4}}(u) = 0.5$ (30 minutes) is the tolerance of level $\frac{1}{4}$ of u towards moon from the concept of "early morning".

A new argument in the use of addition of fuzzy numbers as extension (by Zadeh's principle) of real addition is the validity of the formula $\Delta^r(u \oplus v) = \Delta^r(u) + \Delta^r(v)$ which is consistent to the classical error theory. It is an immediate consequence of the obvious formulas $\Delta_L^r(u \oplus v) = \Delta_L^r(u) + \Delta_L^r(v)$ and $\Delta_R^r(u \oplus v) = \Delta_R^r(u) + \Delta_R^r(v)$

Now, let us study the relative error of the cross product.

Definition 1.11 Let u be a fuzzy number such that $\underline{u}^1 \neq 0$ and $\bar{u}^1 \neq 0$. The crisp numbers $\delta_L^r(u) = \frac{\Delta_L^r(u)}{|\underline{u}^1|}$ and $\delta_R^r(u) = \frac{\Delta_R^r(u)}{|\bar{u}^1|}$ are called relative r -errors of u to left and to right. The quantity $\delta^r(u) = \delta_L^r(u) + \delta_R^r(u)$ is called relative $r-$ error of u.

Theorem 1.12 If u and v are strict positive or strict negative fuzzy numbers then

$$\delta^r(u \odot v) = \delta^r(u) + \delta^r(v)$$

2 Applications of the Cross Product in Geology

Recently, fuzzy arithmetic has found several applications in geology. In the above cited work the usual (Zadeh's extension principle based) product is used for estimation of resources of solid mineral deposits. In this section we propose an alternative study of the same problem, by using the cross product. The reasons of the possible usefulness of the cross product are the following.

Firstly, in this case the shape of the result of the product is conserved, i.e. the product of triangular numbers is triangular and the product of trapezoidal numbers is trapezoidal. Secondly, the 1-level sets are better taken into account by the use of cross product. Also, the consistency of the cross product with the classical error theory motivates this study.

As we perform resource estimation on several bauxite deposits in Hungary. In the same way as with the traditional methods, the tonnage of the resources is obtained by the product of the deposit area, the average thickness and the average bulk-density of the studied ore or mineral commodity. Large deposits can be split into blocks, preferably along natural boundaries, such as tectonic lines. We present the results obtained by the usual multiplication and the results obtained by using the cross product.

Furthermore, if we defuzzify the two results obtained by the two different producttype operations we conclude that the results are different. Also, we observe that after defuzzification (by centroid method) the result of the cross product in the study of the Óbarok deposit is smaller than that of the usual product i.e. the cross product leads to a more pessimistic result than the usual multiplication in this case. So, the risks of an investment at this site can be more realistically evaluated.

3 Conclusion

In this paper, we discussed about the concept of Cross product of fuzzy numbers and the applications of cross product in Geology.

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