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A Study on Analytical Solutions For Stochastic Differential Equations Via Martingale Processes

Syed Tahir Hussainy[1], Pathmanaban K[2]

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Abstract

In this paper, we propose some analytical solutions of stochastic differential equations related to Martingale processes. In the first resolution, the answers of some stochastic differential equations are connected to other stochastic equations just with diffusion part (or drift free). The second suitable method is to convert stochastic differential equations into ordinary ones that it is tried to omit diffusion part of stochastic equation by applying Martingale processes. Finally, solution focuses on change of variable method that can be utilized about stochastic differential equations which are as function of Martingale processes like Wiener process, exponential Martingale process and differentiable processes.

Key words: Martingale process, Itô formula, Change of variable, Differentiable process, Analytical solution

AMS classification: 60G10, 60H10, 60H30

1 Introduction

The purpose of this article is to put forward some analytical and numerical solutions to solve the Itô stochastic differential equation (SDE):

$$\begin{cases} dX(t) = \mathcal{A}(X(t),t)dt + \mathcal{B}(X(t),t)dW_t, \\ X(0) = X_0, \end{cases} \tag{1}$$

where $W(t)$ is a Wiener process and triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space under some conditions and special relations between drift and volatility.

Both the drift vector $\mathcal{A} : \mathbb{R} \times [0,T] \longrightarrow \mathbb{R}$ and the diffusion matrix $a := \mathcal{B}\mathcal{B}^T : \mathbb{R} \times [0,T] \longrightarrow \mathbb{R}$ are considered Borel measurable and locally bounded functions. It is

[1]Associate professor, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India. Email: small1maths@gmail.com

[2]Research scholar, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India.

assumed that X_0 is a non-random vector. As usual, $\mathcal{A}$ and $\mathcal{B}$ are globally Lipschitz in $\mathbb{R}$ that is:

$$|\mathcal{A}(X,t)-\mathcal{A}(Y,t)|+|\mathcal{B}(X,t)-\mathcal{B}(Y,t)|\le D|X-Y|, \quad X,Y\in\mathbb{R} \quad \text{and} \quad t\in[0,T],$$

and result in the linear growth condition:

$$|\mathcal{A}(X,t)|+|\mathcal{B}(X,t)|\le C(1+|X|).$$

These conditions guarantee the Eq. (1) has a unique t-continuous solution adapted to the filtration $\mathcal{F}_t t\ge 0$ generated by $W(t)$ and

$$E\left[\int_0^T |X(s)|^2\,\mathrm{d}s\right]<\infty. \tag{2}$$

It is generally accepted that, analytical solutions of partial and ordinary differential equations are so important particularly in physics and engineering, whereas most of them do not have an exact solution and even a limited number of these equations, (e.g., in classical form), have implicit solutions. Analytical methods and solutions, especially in stochastic differential equations, could be excessive fundamental in some cases therefore we draw to take a comparison and analyze computation error between them and different numerical methods. Numerous numerical methods can be applied to solve stochastic differential equations like Monte Carlo simulation method, finite elements and finite differences.

2 Change of measure and Martingale process

In this section under some conditions, we intend to make a Martingale process from a random one in $\mathbb{L}^2(\mathbb{R}\times[0,T])$, where T is called maturity time. The exponential Martingale process associated with $\lambda(t)$ is defined as follows:

$$\mathrm{d}Z^{\lambda}=\exp\left(\int_0^t \lambda(s)dW_s-\frac{1}{2}\int_0^t \lambda^2(s)\,ds\right). \tag{3}$$

It can be indicated by Itô formula that Z_t^{λ} is a Martingale due to the drift-free property:

$$\mathrm{d}Z_t^{\lambda}=\lambda Z_t^{\lambda}\mathrm{d}W_t, \quad Z_t^{\lambda}(0)=1. \tag{4}$$

Theorem 2.1 Suppose that stochastic processes X_t verify in differential equation:

$$\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W_t, \tag{5}$$

and let $\lambda(t) := -\mu(X_t,t)/\sigma(X_t,t)$. Therefore, $X\mathcal{Z}_t^\lambda$ is a Martingale process.

Proof: With attention to real function $\lambda(t)$, we have:

$$\begin{cases} dX = \mu(X,t)dt + \sigma(X,t)dW_t = -\lambda(t)\sigma(X,t)dt + \sigma(X,t)dW_t, \\ d\mathcal{Z}_t^\lambda = \mathcal{Z}_t^\lambda \lambda dW_t. \end{cases}$$

By utilizing Itô product formula, we get:

$$\begin{aligned} \mathrm{d}(X\mathcal{Z}_t^\lambda) &= X\mathrm{d}(\mathcal{Z}_t^\lambda) + \mathcal{Z}_t^\lambda \mathrm{d}X + \mathrm{d}X\mathrm{d}(\mathcal{Z}_t^\lambda) \\ &= \lambda X\mathcal{Z}_t^\lambda \mathrm{d}W_t + \mu(X,t)\mathcal{Z}_t^\lambda \mathrm{d}t + \sigma(X,t)\mathcal{Z}_t^\lambda \mathrm{d}W_t + \lambda\sigma(X,t)\mathcal{Z}_t^\lambda \mathrm{d}t. \end{aligned}$$

According to theorem assumption, we obtain:

$$\mathrm{d}(X\mathcal{Z}_t^\lambda) = \mathcal{Z}_t^\lambda (X\lambda + \sigma(X,t))\mathrm{d}W_t. \tag{6}$$

It emphasizes that $X\mathcal{Z}_t^\lambda$ is a P-Martingale.
Therefore, $\lambda(t) = \frac{-\mu(X,t)}{\sigma(X,t)}$ is the sufficient condition for following SDEs equivalence:

$$\mathrm{d}X = \mu(X,t)\mathrm{d}t + \sigma(X,t)\mathrm{d}W_t \Leftrightarrow \mathrm{d}(X\mathcal{Z}_t^\lambda) = \mathcal{Z}_t^\lambda (X\lambda(t) + \sigma(X,t))\mathrm{d}W_t. \tag{7}$$

Consequently, by solving the obtained equation in Eq. (6), we obtain the following result when $\mathcal{Z}_0^\lambda = 1$:

$$X\mathcal{Z}_t^\lambda = \int_0^t \mathcal{Z}_t^\lambda (X\lambda(s) + \sigma(X,t))\,\mathrm{d}W_t + X_0. \tag{8}$$

By taking mathematical expectation from both sides of Eq. (8):

$$E^P[X\mathcal{Z}_t^\lambda] = X_0 \Rightarrow E^P[X] = X_0(\mathcal{Z}_t^\lambda)^{-1}. \tag{9}$$

In addition, to compute the variance of this stochastic process:

$$\begin{aligned} E^P[(X Z_t^\lambda)^2] &= X_0^2 + E\left[\int_0^t (Z_s^\lambda)^2 (X\lambda(s)+\sigma(X,t))^2\,\mathrm{d}s\right] \quad (\text{ by Itǒ isometry }) \\ &= X_0^2 + \int_0^t (Z_s^\lambda)^2 E\left(\left[(X\lambda(s)+\sigma(X,t))^2\right]\right)\mathrm{d}s.\ \text{var}\ (XZ_t^\lambda) = (Z_t^\lambda)^2\ \text{var}\ (X) \\ &= \int_0^t (Z_s^\lambda)^2 E\left(\left[(X\lambda(s)+\sigma(X,t))^2\right]\right)\mathrm{d}s. \end{aligned} \tag{10}$$

Applying (6) and using numerical approximation by EM method, we have:

$$\begin{aligned} \Delta X_i Z_{t_i}^\lambda &= Z_{t_i}^\lambda (X_i\lambda(t_i)+\sigma_i)\Delta W_i. \\ X_{t_{i+1}} Z_{t_{i+1}}^\lambda &= X_{t_i} Z_{t_i}^\lambda + Z_{t_i}^\lambda (X_{t_i}\lambda(t_i)+\sigma_i)\Delta W_i. \\ X_{t_{i+1}} &= (Z_{t_{i+1}}^\lambda)^{-1} Z_{t_i}^\lambda (X_{t_i} + (X_{t_i}\lambda(t_i)+\sigma_i)\Delta W_i). \end{aligned}$$

Direct calculations would lead to the conclusion that:

$$R_{t_i} = (Z_{t_{i+1}}^\lambda)^{-1} Z_{t_i}^\lambda = \exp\left(-\int_{t_i}^{t_{i+1}} \lambda(s)\,\mathrm{d}W_s + \frac{1}{2}\int_{t_i}^{t_{i+1}} |\lambda^2(s)|\,\mathrm{d}s\right).$$

So the following Milstein recursive method is inferred as a good numerical method to find $X(t_{i+1})$:

$$X_{t_{i+1}} = R_{t_i}(X_{t_i} + (X_{t_i}\lambda(t_i)+\sigma_i)\Delta W_i) + \frac{1}{2} R_{t_i}^2 \lambda(t_i)\,(X_{t_i}\lambda(t_i)+\sigma_i)\,(\Delta^2 W_i - \Delta t_i). \tag{11}$$

In example, we compare this method with usual Milstein method in the case that a stochastic differential equation contains drift and volatility both parts and indicate that this method could be better in some cases.

3 Change of variable method

This section intends to analyze the change of variable method like, to get explicitly the solution of arbitrary SDE:

$$\mathrm{d}X = \mathcal{A}(X,t)\mathrm{d}t + \mathcal{B}(X,t)\mathrm{d}W_t, \quad X(0) = x.$$

By finding appropriate variables $u(Y) = X$ and their conditions so that Y is the answer of a well-known SDEs related to Martingale processes.

$$\mathrm{d}Y = f(X,t)\mathrm{d}t + g(X,t)\mathrm{d}W_t, \quad y(0) = y.$$

For more explanation and different conditions under which they are possible. Now we consider following various cases.

Case 1 Consider the following SDE:

$$\mathrm{d}Y = a(t)\mathrm{d}t + b(t)\mathrm{d}W_t. \tag{12}$$

Applying Itô formula for $u(Y) = X$, to (12), we get:

$$\begin{cases} u'(a(t)) + \frac{1}{2}u''b^2(t) = \mathcal{A}(u(Y),t), \\ u'b(t) = \mathcal{B}(u(Y),t). \end{cases} \tag{13}$$

Thus, it concludes that:

$$\frac{a(t)}{b(t)}\mathcal{B} + \frac{1}{2}\mathcal{B}\mathcal{B}' = \mathcal{A} \Rightarrow \frac{\mathcal{A}}{\mathcal{B}} - \frac{1}{2}\mathcal{B}' = \frac{a(t)}{b(t)}. \tag{14}$$

Finally, the equation $\frac{\partial}{\partial Y}\left(\frac{\mathcal{A}}{\mathcal{B}} - \frac{1}{2}\mathcal{B}'\right) = 0$ is necessary condition to solve an equation via change of variable in (12) $\left(\mathcal{B}' = \frac{\partial \mathcal{B}}{\partial X}\right)$.

Case 2 Consider the exponential Martingale process SDE (3):

$$\begin{cases} \mathrm{d}Y = \lambda(t)Y\mathrm{d}W_t, \\ Y(0) = Y_0. \end{cases} \tag{15}$$

Applying Itô formula for $u(Y) = X$, to (15), we acquire:

$$\begin{cases} u'\lambda Y = \mathcal{B}(u,t) = \lambda(t)Y\hat{\mathcal{B}}(u) \quad \text{or} \quad u' = \hat{\mathcal{B}}(u), \\ \frac{1}{2}u''\lambda^2Y^2 = \mathcal{A}(u,t). \end{cases} \tag{16}$$

So from the last equality, we have $\frac{\mathcal{B}'}{\lambda(t)} - \frac{2\mathcal{A}}{\mathcal{B}} = \lambda(t)$. Therefore, $\frac{\partial}{\partial u}\left(\mathcal{B}'_u - \frac{2\lambda(t)\mathcal{A}}{\mathcal{B}}\right) = 0$ is necessary condition to solve SDE, with this change of variable.

Case 3 Consider the well-known equation:

$$\begin{cases} dY = a(t)Y dt + b(t)Y dW_t, \\ Y(0) = Y_0. \end{cases} \tag{17}$$

Which is Black-Scholes equation with exact solution

$$Y_0 = \exp\left(\int_0^t b(s)dW_s + \int_0^t \left(a(s) - \frac{1}{2}b^2(s)\right) ds\right).$$

Applying Itô formula for $u(Y) = X$, to (17), we get:

$$\begin{cases} u'a(t)Y + \frac{1}{2}u''b^2(t)Y^2 = \mathcal{A}(u,t), \\ u'Yb(t) = \mathcal{B}(u,t) = b(t)Y\hat{\mathcal{B}}(u). \end{cases} \tag{18}$$

For this reason, $u' = \hat{\mathcal{B}}(u)$ and we have:

$$\frac{a(t)}{b(t)} = \frac{\mathcal{A}}{\mathcal{B}} - \frac{1}{2}(\mathcal{B}'_u - b(t)) = \gamma(u,t). \tag{19}$$

It means that $\frac{\partial}{\partial u}\gamma(u,t) = 0$, is a necessary condition to solve the initial stochastic differential equation by this change of variable.
Case 4 Another appropriate and prominent case is as follows:

$$\begin{cases} dY_t = f(Y_t,t)dt + c(t)Y_t dW_t, \\ Y(0) = Y_0. \end{cases} \tag{20}$$

This kind of equations, applying Itô formula on $X_t = Y_t Z_t^c(t)^{-1}$, is converted to a ordinary differential equations.

Theorem 3.1 The stochastic differential equations in (20) given by continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $C : \mathbb{R} \to \mathbb{R}$ can be written as:

$$d(Y_t(Z_t^c(t))^{-1}) = (Z_t^c(t))^{-1} f(Y_t,t)dt, \tag{21}$$

where $Z_t^c(t)$ is an exponential Martingale process.

To be more precise, using change of variable $V = X(Z_t^{c(t)})^{-1}$, it is enough to solve

$$\begin{cases} X_t' = (Z_t^{c(t)})^{-1} f(X_t Z_t^{c(t)}), \\ X(0) = X_0 \end{cases} \tag{22}$$

Applying Itô formula for $u(Y) = M_t$, in (20) we get:

$$\mathrm{d}M_t = M_t' \mathrm{d}Y + \frac{1}{2} M_t'' (\mathrm{d}Y)^2.$$

$$\begin{cases} f(Y,t)M_t' + \frac{1}{2} M_t'' c^2(t) Y^2 = \mathcal{A}(M_t, t), & (1) \\ c(t) Y M_t' = \mathcal{B}(M_t, t), \; u(Y_0) = M_0. & (2) \end{cases} \tag{23}$$

According to (23), we have $\mathcal{B}(M_t, t) = c(t)\hat{\mathcal{B}}(M_t)$. Besides, if the new stochastic differential equation is related to a Martingale process, we have $\mathcal{A}(M_t, t) = 0$ and:

$$f(Y,t) = -\frac{c^2(t)Y}{2}(\hat{\mathcal{B}}(M_t)' - 1). \tag{24}$$

Again, applying Itô formula for $\phi(M_t) = V_t$ to Martingale equation contributes to

$$\mathrm{d}M_t = \mathcal{B}(M_t, t)\mathrm{d}W_t = c(t)\hat{\mathcal{B}}(M_t)\mathrm{d}W_t,$$

we can achieve to a novel group of stochastic differential equation that its solution is as a function of a Martingale process.

4 Conclusions

In this paper, a couple of analytical solutions of some determined set of stochastic differential equations was indicated via making the Martingale process from a stochastic process. Converting stochastic differential equations to ordinary ones as another suitable method was posed. Indeed, it is tried to omit diffusion part of stochastic equation by applying Martingale processes. In addition, change of variable method on SDEs related to Martingale processes was discussed.

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