

A New Divergence Measure of Interval-valued Pythagorean Fuzzy Sets and its Application

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Abstract

As the extension of the Fuzzy sets (FSs) theory, the Interval-valued Pythagorean Fuzzy Sets (IVPFS) was introduced which play an important role in handling the uncertainty. The Pythagorean fuzzy sets (PFSs) proposed by Yager in 2013 can deal with more uncertain situations than intuitionistic fuzzy sets because of its larger range of describing the membership grades. How to measure the distance of Interval-valued Pythagorean fuzzy sets is still an open issue. Jensen–Shannon divergence is a useful distance measure in the probability distribution space. In order to efficiently deal with uncertainty in practical applications, this paper proposes a new divergence measure of Interval-valued Pythagorean fuzzy sets, which is based on the belief function in Dempster–Shafer evidence theory, and is called IVPFSDM distance. It describes the Interval-Valued Pythagorean fuzzy sets in the form of basic probability assignments (BPAs) and calculates the divergence of BPAs to get the divergence of IVPFSs, which is the step in establishing a link between the IVPFSs and BPAs. Since the proposed method combines the characters of belief function and divergence, it has a more powerful resolution than other existing methods.

Key words: Interval-Valued Pythagorean Fuzzy Set, Dempster-Shafer Evidence Theory, Basic Probability Assignment, Medical Diagnosis.

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1 Introduction

Distance measure plays a vital role in pattern recognition, information fusion, decision-making, and other fields. With the development of fuzzy mathematics, medical diagnosis has attracted more and more attention from the research society of applied computer mathematics. A general distance measurement of PFSs was proposed by Chen [2], which is an extension of Euclidean distance and Hamming distance, and generates a reasonable result in multiple-criteria decision analysis. Wei

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and Wei [19] proposed a PFS measurement method based on cosine function and applied it to medical diagnosis to achieve an ideal result. Later on, Xiao[9] presented a distance measurement of PFSs based on divergence, called PFSJS distance. Among these methods of measure distance of PFSs, membership, non-membership and hesitancy are calculated based on some weights. It is well known that the hesitancy expresses the uncertainty of membership and non-membership, so finding a proper method to distribute hesitancy to membership and non-membership can more reasonably handle the distance of PFSs. The basic probability assignment (BPA) in evidence theory presented by Dempster-Shafer [3] unifies uncertainty in a new set, and can handle various uncertainty reasonably. Song [16] presented a divergence measure of belief function based on Kullback–Leibler (KL) [10] divergence and Deng entropy[4], which has better results when dealing with the distance between BPAs with greater uncertainty. The distance measurement in Interval-valued Pythagorean Fuzzy Set(IVPFS) is still an open issue, which attracts many researchers to explore the distance measurement of Interval -valued Pythagorean Fuzzy Sets(IVPFSs) and its related applications.

This motivates us to propose a new method to describe the Basic Probability Assignment(BPA) in the form of Interval-valued Pythagorean Fuzzy Set(IVPFS), and uses an improved divergence measurement of Basic Probability Assignments(BPAs) based on Jensen–Shannon divergence to measure the distance of Interval-valued Pythagorean Fuzzy Set(IVPFS).

2 Preliminaries

Definition 2.1 Let X be a limited universe of discourse. An intuitionistic fuzzy set (IFS) M in X is defined by $M = \{(x, \mu_M(x), v_M(x)) | x \in X\}$, where $\mu_M(x) : X \rightarrow [0, 1]$ represents the degree of support for membership of the $x \in X$ of IFS and $v_M(x) : X \rightarrow [0, 1]$ represents the degree of support for non-membership of the $x \in X$ of IFS, with the condition that $0 \leq \mu_M(x) + v_M(x) \leq 1$ and the hesitancy function $\pi_M(x)$ of IFS reflecting the uncertainty of membership and non-membership is defined by

$$\pi_{M(x)} = 1 - \mu_{M(x)} - v_{M(x)}. \quad (1)$$

Definition 2.2 Let X be a limited universe of discourse. A Pythagorean fuzzy set (PFS) M in X is defined by $M = \{(x, M_Y(x), M_N(x)) | x \in X\}$, where $M_Y(x) : X \rightarrow [0, 1]$ represents the degree of support for membership of the $x \in X$ of PFS, and

$M_N(x) : X \rightarrow [0, 1]$ represents the degree of support for non-membership of the $x \in X$ of PFS, with the condition that $0 \leq M_\gamma^2(x) \leq 1$ and the hesitancy function $M_H(x)$ of PFS reflecting the uncertainty of membership and non-membership is defined by

$$M_H(x) = \sqrt{1 - M_\gamma^2(x) - M_N^2(x)}.$$

The membership and the non-membership of PFS also can be expressed by another way. A pair of values $r(x)$ and $d(x)$ for each $x \in X$ are used to represent the membership and non-membership as follows:

$$M_Y(x) = r_{M(x)}(x) \cos(\theta M(x)) \quad M_N(x) = r_{M(x)} \sin(\theta M(x)),$$

Where $\theta(x) = (1 - d(x))\pi/2 = \text{Arctan}(M_N(x)/M_\gamma(x))$. Here we see that $\theta(x)$ is expressed as radians and $\theta(x) \in [0, \frac{\pi}{2}]$.

Property 2.3 Let B and C be two PFSs in X, then

1. $B \subseteq C$ if $\forall x \in X, B_Y(x) = C_Y(x)$ and $B_N(x) = C_N(x)$,
2. $B = C$ if $\forall x \in X, B_Y(x) = C_Y(x)$ and $B_N(x) = C_N(x)$,
3. $B \cap C = \{hx, \min[B_Y(x), C_Y(x)], \max[B_N(x), C_N(x)]\} / x \in X\}$,
4. $B \cup C = \{(x, \max[B_Y(x), C_Y(x)], \min[B_N(x), C_N(x)])\} / x \in X\}$,
5. $B.C = \left\{ \left\langle x, B_\gamma(x)C_\gamma(x), (B_N^2(x) + C_N^2(x) - B_N^2(x)C_N^2(x))^{\frac{1}{2}} \right\rangle / x \in X \right\}$,
6. $B^n = \{(x, B_\gamma^n(x), (1 - (1 - B_N^2(x))^n)^{\frac{1}{2}}) / x \in X\}$

Dempster-Shafer Evidence Theory

The Dempster-Shafer evidence theory [18, 3] is proposed to deal with conditions that are weaker than Bayes probability [34, 5], which can directly express uncertainty and unknown information [25, 6], so that it has been widely used in various applications, FMEA [20, 9, 31], evidential reasoning [32, 33], evaluation [7], target recognition [15], industrial alarm system [21, 22].

Definition 2.4 Θ is the set of N elements which represent mutually exclusive and exhaustive hypotheses. Θ is the frame of discernment

$$\Theta = \{H_1, H_2, \dots, H_i, \dots, H_N\}$$

The power set of Θ is denoted by 2^Θ and

$$2^\Theta = \{\emptyset, \{H_1\}, \dots, \{H_n\}, \{H_1, H_2\}, \dots, \{H_1, \dots, H_N\}\}, \text{ where } \emptyset \text{ is an empty set}$$

Definition 2.5 A mass function m, also called as BPA, is a mapping of 2^Θ , defined as follows: $m : 2^\Theta \rightarrow [0, 1]$, which satisfies the following conditions:

$$m(\emptyset) = 0 \quad \sum m(A) = 1 \quad 0 = \sum m(A) = 1 \quad A \in 2^\Theta$$

The mass $m(A)$ represents how strongly the evidence supports A

Definition 2.6 Given two probabilities distribution $A = \{A(x_1), A(x_2), \dots, A(x_n)\}$ and $B = \{B(x_1), B(x_2), \dots, B(x_n)\}$.

Kullback- Leibler divergence between A and B is defined as:

$$Div_{KL}(A, B) = \sum_{i=1}^n A(x_i) \log_2 \left(\frac{A(x_i)}{B(x_i)} \right); \text{ With } \sum_{i=1}^n A(x_i) = \sum_{i=1}^n B(x_i) = 1$$

The Kullback–Leibler also has some disadvantages of its properties, and one of them is that it doesn't satisfy the commutative property:

$$Div_{KL}(A, B) \neq Div_{KL}(B, A).$$

In order to realize the commutation in the distance measure, the Jensen–Shannon divergence is an adaptive choice.

Definition 2.7 Given two probabilities distribution $A = \{A(x_1), A(x_2), \dots, A(x_n)\}$ and $B = \{B(x_1), B(x_2), \dots, B(x_n)\}$. Jensen-Shannon divergence between A and B is defined as:

$$JS_{AB} = \frac{Div_{KL}(A, \frac{A+B}{2}) + Div_{KL}(B, \frac{A+B}{2})}{2} = H\left(\frac{A+B}{2}\right) - \frac{H(A)}{2} - \frac{H(B)}{2}$$

$$\text{Where } H(A) = - \sum_{i=1}^n A(x_i) \log A(x_i) = \sum_{i=1}^n A(x_i) = 1$$

Song's divergence is used to measure the belief function, which is capable of processing uncertainty efficiently in a highly fuzzy environment by applying the thinking of Deng entropy

Definition 2.8 Given two basic probability assignments (BPAs) m_1 and m_2 , the divergence between m_1 and m_2 is defined as follows:

$$D_{SD}(m_1, m_2) = \sum_i \frac{1}{2^{|F_i|} - 1} m_1(F_i) \log \left(\frac{m_1(F_i)}{m_2(F_i)} \right), \text{ Where the } F_i \text{ holds on } \sum_{i=1}^{2^\theta - 1} m(F_i) = 1$$

which is the power subset of frame of discernment Θ and $|F_i|$ is the cardinal number of F_i . It is obvious that $D_{SD}(m_1, m_2) \neq D_{SD}(m_2, m_1)$. In order to realize the commutative property, a divergence measurement based on Song's divergence is defined as follows:

$$D_{SD}(m_1, m_2) = \tilde{D}_{SD}(m_2, m_1) = \frac{D_{SD}(m_1+m_2) + D_{SD}(m_2+m_1)}{2}$$

Because of thinking of the number of subsets of the mass function and averagely distributing the BPAs to these subsets, Song's divergence is more reasonable than others when the basic probability assignments of non-singleton powers sets F_i are

larger. The Euclidean distance and the Hamming distance are the most widely applied distances, and Chen proposed a generalized distance measure of PFS, which is the extension of Hamming distance and Euclidean distance.

Definition 2.9 Let X be a limited universe of discourse, and M and N are two PFSs. Chen's distance measure between PFSs M and N denoted as $D_C(M, N)$ is defined as $D_C(M, N) = [\frac{1}{2}(|M_\gamma^2(x) - N_\gamma^2(x)|^\beta + |M_N^2(x) - N_N^2(x)|^\beta + |M_H^2(x) - N_H^2(x)|^\beta)]^{\frac{1}{\beta}}$. When $\beta = 1$ is called the distance parameter. As the extension of the Hamming distance and Euclidean distance, if $\beta = 1$ and $\beta = 2$, the Chen's distance is equal to the Hamming distance and Euclidean distance respectively

- If $\beta = 2$

$$D_C(M, N) = D_{HM}(M, N) [\frac{1}{2}(|M_\gamma^2(x) - N_\gamma^2(x)|^\beta + |M_N^2(x) - N_N^2(x)|^\beta + |M_H^2(x) - N_H^2(x)|^\beta)]$$

- If $\beta = 1$

$$D_C(M, N) = D_{HM}(M, N) [\frac{1}{2}(|M_\gamma^2(x) - N_\gamma^2(x)| + |M_N^2(x) - N_N^2(x)| + |M_H^2(x) - N_H^2(x)|)]^{\frac{1}{2}}$$

In the application of distance measure, the universe of discourse always has many properties. Xiao extended them as the normalized distance and proposed a divergence measure of PFSs called PFSJS based on the Jensen–Shannon divergence, which is the first work to calculate the distance of PFSs using divergence.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a limited universe of discourse, two PFSs

$M = \{(x_i, M_Y(x_i), M_N(x_i)) | x_i \in X\}$ and $N = \{(x_i, N_Y(x_i), N_N(x_i)) | x_i \in X\}$ are in X .

Definition 2.10 S The normalized Hamming distance denoted as $D_{Hm}(M, N)$ is defined as:

$$\tilde{D}_{HM}(M, N) = \frac{1}{2n} \sum_{i=1}^n (|M_\gamma^2(x_i) - N_\gamma^2(x_i)| + |M_N^2(x_i) - N_N^2(x_i)| + |M_H^2(x_i) - N_H^2(x_i)|)$$

The normalized Euclidean distance denoted as $D_E(M, N)$ is defined as:

$$\tilde{D}_E(M, N) = \frac{1}{n} \sum_{i=1}^n [\frac{1}{2} |M_\gamma^2(x_i) - N_\gamma^2(x_i)|^2 + |M_N^2(x_i) - N_N^2(x_i)|^2 + |M_H^2(x_i) - N_H^2(x_i)|^\beta]^{\frac{1}{\beta}}$$

The normalized Chen's distance denoted as $D_E(M, N)$ is defined as:

$$\widetilde{D}_C(M, N) = \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{2} |M_\gamma^2(x) - N_\gamma^2(x)|^2 + |M_N^2(x) - N_N^2(x)|^2 + |M_H^2(x) - N_H^2(x)|^\beta \right]^{\frac{1}{\beta}},$$

where $\beta \geq 1$

According to the existing methods for measuring the PFSs' distance, what they have in common is that the weights of membership $A_Y(x)$ non-membership $A_N(x)$ and hesitancy $A_H(x)$ are considered to be the same when calculating distances. As is well known, the hesitancy represents the uncertainty of membership degree and non-membership degree, and the belief function in evidence theory can handle the uncertainty in a more proper way. Hence, if the ability of evidence theory to handle uncertainty is combined with the high resolution of divergence in distance measurement, the PFSs' distance measurement will be further optimized. In the next section, a new divergence measure of PFSs is proposed based on belief function, which describes the PFSs in the form of BPAs and measures the distance of PFSs by calculating the divergence of BPAs.

3. A New Divergence Measure Of IVPFSS

In this section, a new divergence measure of IVPFSSs, called IVPFSDM distance, is proposed. The first subsection shows how IVPFS reasonably expressed in the form of BPA. A new improved method of BPAs' divergence measure is introduced in the second subsection, and then the IVPFSDM distance and its properties is proposed. In the last subsection, some examples are used to prove its properties and demonstrate its feasibility by comparing with existing other methods.

3.1 IVPFS Is Expressed in the Form of BPA

In the evidence theory, the basic probability assignment (BPA) $m(A)$ represents the degree of evidence supporting A , the elements of power set of frame of discernment (Θ) should satisfy $\sum_{A \in 2^\Theta} \theta m(A) = 1$. Thus, the method of representing IPFS in the form of BPA is shown as follows:

Definition 3.1 Let X be a limited universe of discourse, Interval-valued Pythagorean fuzzy set M in X is $M = \{hx, r_M(x) \cos(\theta_M(x)), r_M(x) \sin(\theta_M(x)) | x \in X\}$, the frame of discernment Θ_M of M and their basic probability assignments m_p are defined as:

$$\Theta_M = \{Y_M, N_M\},$$

- $m_p(Y_M) = r_M(x) \cos^2 \theta_M(x)$,

- $m_p(N_M) = r_M(x) \sin^2 \theta M(x)$,
- $m_p(Y_M N_M) = 1 - r_M(x)$,
- $m_p(\varphi) = 0$,

where $m_p(Y_M)$ represents the degree of evidence supporting membership of M. The $m_p(N_M)$ represents the degree of evidence supporting non-membership of M. The $m_p(Y_M N_M)$ represents the degree of evidence supporting membership and non-membership. Because the basic focal elements Y_M and N_M are totally exclusive, and the sum of them is equal to 1, they conform to the Dempster-Shafer evidence theory.

According to the meaning of $m_p(Y_M N_M)$ and the method of thinking of Deng entropy, it contains the degree of evidence supporting the elements of $\{Y_M\}$, $\{Y_N\}$ and $\{Y_M Y_N\}$, and Song redistributes the degree of supporting $\{Y_M N_M\}$ averagely among the three elements when calculating the distance of BPAs.

3.2. A New Divergence Measure of IVPFSs

Jensen–Shannon divergence is widely used in distance measure of probability distributions, and in this subsection, we propose an improved divergence measure of BPA based on Song’s divergence and Jensen–Shannon divergence. In addition, a new divergence measure of IVPFSs and its properties are proposed, which is capable of distinguishing IVPFSs better.

Definition 3.2 Let Θ be a frame of discernment $\Theta = \{A_1, A_2, \dots, A_n\}$ and the power set of Θ is $2^\Theta = \{\emptyset, \{A_1\}, \dots, \{A_n\}, \{A_1, A_2\}, \dots, \{A_1, A_n\}, \dots, \{A_1, \dots, A_n\}\} = \{\emptyset, F_1, F_2, \dots, F_{2^n-1}\}$. The Jensen-Shannon divergence measure $D_{JS}(m_1, m_2)$ of two BPAs m_1, m_2 is defined as:

$$D_{JS}(m_1, m_2) = \frac{1}{2} [D_{SD}(m_1, \frac{m_1+m_2}{2}) + D_{SD}(m_2, \frac{m_2+m_1}{2})]$$

$$= \frac{1}{2} [\sum_{F_i \in 2^\Theta} \frac{1}{2^{|F_i|-1}} m_1(F_i) \log(\frac{2m_1(F_i)}{m_1(F_i)m_2(F_i)}) + \sum_{F_i \in 2^\Theta} \frac{1}{2^{|F_i|-1}} m_2(F_i) \log(\frac{2m_2(F_i)}{m_1(F_i)m_2(F_i)})]$$

Where $|F_i|$ is the cardinal number of F_i . In addition, just in case there’s a zero in the denominator, 10^{-8} is used to replace zero in the calculation.

The improved method satisfies the symmetry and considers the number of elements in the power set. In addition, then, substituting the IVPFSs in the form of BPAs Which produces the new divergence measure of Interval-valued Pythagorean fuzzy sets.

Definition 3.3 Let X be a limited universe of discourse, two Interval-valued Pythagorean fuzzy sets M and N in X are $M = \{hx, r_M(x) \cos(\theta_M(x)), r_M(x) \sin(\theta_M(x)) | x \in X\}$, $N = \{hx, r_N(x) \cos(\theta_N(x)), r_N(x) \sin(\theta_N(x)) | x \in X\}$.

The divergence measure denoted as $D_{IVPFS}(M, N)$ is defined as follows:

$$D_{IVPFS}(M, N) = \frac{1}{2} \left[r_M(x) \cos^2 \theta_M(x) \log \left(\frac{2r_M(x) \cos^2 \theta_M(x)}{r_M(x) \cos^2 \theta_M(x) + r_N(x) \cos^2 \theta_N(x)} \right) \right. \\
+ r_N(x) \cos^2 \theta_N(x) \log \left(\frac{2r_N(x) \cos^2 \theta_N(x)}{r_M(x) \cos^2 \theta_M(x) + r_N(x) \cos^2 \theta_N(x)} \right) \\
r_M(x) \sin^2 \theta_M(x) \log \left(\frac{2r_M(x) \sin^2 \theta_M(x)}{r_M(x) \sin^2 \theta_M(x) + r_N(x) \sin^2 \theta_N(x)} \right) \\
+ r_N(x) \sin^2 \theta_N(x) \log \left(\frac{2r_N(x) \sin^2 \theta_N(x)}{r_M(x) \sin^2 \theta_M(x) + r_N(x) \sin^2 \theta_N(x)} \right) \\
+ \frac{1}{3}(1 - r_M(x)) \log \left(\frac{2(1 - r_M(x))}{(1 - r_M(x)) + (1 - r_N(x))} \right) \\
\left. + \frac{1}{3}(1 - r_N(x)) \log \left(\frac{2(1 - r_N(x))}{(1 - r_M(x)) + (1 - r_N(x))} \right) \right]$$

In order to obtain higher resolution when making distance measurement, the divergence measure of IVPFSs, IVPFSDM distance, denoted as $D_{mp}(M, N)$, is defined by $D_{mp}(M, N) = \sqrt{D_{IVPFS}(M, N)}$.

According to the properties of Jensen–Shannon divergence [82], the larger IVPFSDM distance, the more different IVPFSs, and the smaller IVPFSDM, the more similar IVPFSs. The properties of the PFSDM distance are displayed as follows:

Definition 3.4 Let M and N be two IVPFSs in a limited universe of $X = \{x_1, x_2, \dots, x_n\}$, where $M = \{(x_i, r_M(x_i) \cos(\theta_M(x_i)), r_M(x_i) \sin(\theta_M(x_i)))\}$, $N = \{(x_i, r_N(x_i) \cos(\theta_N(x_i)), r_N(x_i) \sin(\theta_N(x_i)))\}$.

The normalized IVPFSDM distance, $D_{mp}(M, N)$ is defined as follows:

$$D_{mp}(M, N) \\
= \frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{2} \left[r_M(x) \cos^2 \theta_M(x) \log \left(\frac{2r_M(x) \cos^2 \theta_M(x)}{r_M(x) \cos^2 \theta_M(x) + r_N(x) \cos^2 \theta_N(x)} \right) \right. \right.$$

$$\begin{aligned}
 &+r_N(x) \cos^2 \theta_N(x) \log \left(\frac{2r_N(x) \cos^2 \theta_N(x)}{r_M(x) \cos^2 \theta_M(x) + r_N(x) \cos^2 \theta_N(x)} \right) \\
 &r_M(x) \sin^2 \theta_M(x) \log \left(\frac{2r_M(x) \sin^2 \theta_M(x)}{r_M(x) \sin^2 \theta_M(x) + r_N(x) \sin^2 \theta_N(x)} \right) \\
 &+r_N(x) \sin^2 \theta_N(x) \log \left(\frac{2r_N(x) \sin^2 \theta_N(x)}{r_M(x) \sin^2 \theta_M(x) + r_N(x) \sin^2 \theta_N(x)} \right) \\
 &+\frac{1}{3}(1 - r_M(x)) \log \left(\frac{2(1 - r_M(x))}{(1 - r_M(x)) + (1 - r_N(x))} \right) \\
 &+\frac{1}{3}(1 - r_N(x)) \log \left(\frac{2(1 - r_N(x))}{(1 - r_M(x)) + (1 - r_N(x))} \right) \Bigg\}^{\frac{1}{2}}
 \end{aligned}$$

Property 3.5 Let M, N, and O be three arbitrary IVPFSs in the limited universe of discourse X, then

- (P1) $D_{mp}(M, N) = 0$ if $M = N$, for $M, N \in X$.
- (P2) $D_{mp}(M, N) + D_{mp}(N, O) \geq D_{mp}(M, O)$, for $M, N, O \in X$.
- (P3) $D_{mp}(M, N) \in [0, 1]$ for $M, N \in X$.
- (P4) $D_{mp}(M, N) = D_{mp}(N, M)$, for $M, N \in X$.

Proof (P1)

Suppose two Interval-valued Pythagorean fuzzy sets M and N in the limited universe of discourse X. In addition, the IVPFSs of them are given as follows:

$$M = \{hx, r_M(x) \cos(\theta_M(x)), r_M(x) \sin(\theta_M(x)) | x \in X\},$$

$$N = \{hx, r_N(x) \cos(\theta_N(x)), r_N(x) \sin(\theta_N(x)) | x \in X\}.$$

- If $M=N$, we can get $r_M(x) = r_N(x)$ and $\theta_N(x) = \theta_M(x)$. we get a result that $D_{mp}(M, N) = 0$.
- If $D_{mp}(M, N) = 0$, we get $r_M(x) = r_N(x)$ and $\theta_N(x) = \theta_M(x)$

Hence, it also be found that $M = N$.

Thus, the property (P1) in the Property 2 is proven.

Proof. (P2)

Suppose two Interval-valued Pythagorean fuzzy sets M and N in the limited

universe of discourse X. In addition, the IVPFSs of them are given as follows:

$$M = \{hx, r_M(x) \cos(\theta_M(x)), r_M(x) \sin(\theta_M(x)) | x \in X\},$$

$$N = \{hx, r_N(x) \cos(\theta_N(x)), r_N(x) \sin(\theta_N(x)) | x \in X\},$$

$$O = \{hx, r_O(x) \cos(\theta_O(x)), r_O(x) \sin(\theta_O(x)) | x \in X\}.$$

Given four assumptions:

1. (A1) $r_M(x) \cos^2(\theta_M(x)) \leq r_N(x) \cos^2(\theta_N(x)) \leq r_O(x) \cos^2(\theta_O(x))$.
2. (A2) $r_O(x) \cos^2(\theta_O(x)) \leq r_N(x) \cos^2(\theta_N(x)) \leq r_M(x) \cos^2(\theta_M(x))$.
3. (A3) $r_N(x) \cos^2(\theta_N(x)) \leq \min\{r_M(x) \cos^2(\theta_M(x)), r_O(x) \cos^2(\theta_O(x))\}$.
4. (A4) $r_N(x) \cos^2(\theta_N(x)) \geq \max\{r_M(x) \cos^2(\theta_M(x)), r_O(x) \cos^2(\theta_O(x))\}$.

Let $A = r_M(x) \cos^2(\theta_M(x))$, $B = r_N(x) \cos^2(\theta_N(x))$ and $C = r_O(x) \cos^2(\theta_O(x))$.

According to the above, it is obvious that $|A - C| = |A - B| + |B - C|$ is satisfied under the (A1) and (A2). We can easily find $A - B = 0$ and $C - B = 0$ in terms of A3 and A4. Therefore, we have:

$$\begin{aligned} |A - B| + |B - C| - |A - C| &= A - B + C - B - A + C, \text{ if } A = C, \\ &= A - B + C - B + A - C, \text{ if } A = C, \\ &= 2(B - \max\{A, C\}) = 0. \end{aligned}$$

Hence, the inequality $|A - C| = |A - B| + |B - C|$ is valid under A3 and A4. In the same way, the $r(x) \sin(\theta(x))$ and $1-r$ also satisfy the $|A - C| = |A - B| + |B - C|$. Therefore, the inequality in the Property 2 (P2), $D_{mp}(M, N) + D_{mp}(N, O) = D_{mp}(M, O)$, has been proven.

Proof. (P3 & P4)

Given two IVPFSs $M = hx, \alpha, \beta i$ and $N = hx, \beta, \alpha i$ in the limited universe of discourse X. The values α and β represent membership degree and non-membership degree in two IVPFSs.

$$M = \{r_M(x) \cos(\theta_M(x)), r_M(x) \sin(\theta_M(x))\},$$

$$N = \{r_N(x) \cos(\theta_N(x)), r_N(x) \sin(\theta_N(x))\},$$

where $r_M(x) = r_N(x) = \sqrt{\alpha + \beta}$; $\theta_M(x) = \text{Arctan}(\beta/\alpha)$; $\theta_N(x) = \text{Arctan}(\alpha/\beta)$. The α and β satisfy the $X \rightarrow [0, 1]$ and $0 \leq \alpha^2 + \beta^2 = 1$.

4.Numerical Examples

Example 4.1 Assume there are three patients, Ram, Mari, Somu denoted as $P = \{P_1, P_2, P_3\}$. Three Symptoms ,Temperature , Headache , Cough are observed as denoted as

$S = \{S_1, S_2, S_3\}$. Additionally three diagnoses , Viral fever, Malaria, Typhoid are represented as $D = \{D_1, D_2, D_3\}$, then the Interval-valued Pythagorean Fuzzy realtions $P \rightarrow S$ and $D \rightarrow S$ are displayed as follows .

<i>patient</i>	S_1	S_2	S_3
p_1	$\{s_1, [0.10, 0.80][0.10, 0.30]\}$	$\{s_2, [0.10, 0.80][0.10, 0.30]\}$	$\{s_3, [0.10, 0.80][0.10, 0.30]\}$
p_2	$\{s_1, [0.10, 0.80][0.10, 0.30]\}$	$\{s_2, [0.10, 0.80][0.10, 0.30]\}$	$\{s_3, [0.10, 0.80][0.10, 0.30]\}$
p_3	$\{s_1, [0.10, 0.80][0.10, 0.30]\}$	$\{s_2, [0.10, 0.80][0.10, 0.30]\}$	$\{s_3, [0.10, 0.80][0.10, 0.30]\}$

<i>Diagnose</i>	S_1	S_2	S_3
D_1	$\{s_1, [0.20, 0.40][0.10, 0.30]\}$	$\{s_2, [0.10, 0.80][0.40, 0.60]\}$	$\{s_3, [0.20, 0.40][0.70, 0.80]\}$
D_2	$\{s_1, [0.10, 0.30][0.50, 0.70]\}$	$\{s_2, [0.10, 0.30][0.50, 0.80]\}$	$\{s_3, [0.10, 0.20][0.10, 0.90]\}$
D_3	$\{s_1, [0.10, 0.20][0.10, 0.30]\}$	$\{s_2, [0.40, 0.60][0.10, 0.30]\}$	$\{s_3, [0.10, 0.40][0.60, 0.80]\}$

D_1 -represent the diagnoses of viral fever;

D_2 represent the diagnoses of Malaria ;

D_3 represent the diagnoses of Typhoid

CALCULATION FOR THE EXISTING METHOD :

The results generated by Exsisting method

<i>patient</i>	D_1	D_2	D_3	<i>Classification</i>
p_1	0.225	0.535	0.215	D_3 -Typhoid
p_2	0.305	0.380	0.593	D_1 – Viral fever
p_3	0.452	0.242	0.275	D_2 – Malaria

CALCUATION FOR THE PROPOSED METHODS:

<i>patient</i>	D_1	D_2	D_3	<i>Classification</i>
p_1	0.4268	0.4389	0.4064	D_3 -Typhoid
p_2	0.6156	0.6351	0.7005	D_1 – Viral fever
p_3	0.4491	0.4197	0.4967	D_2 – Malaria

Calculation for diagnoses of Viral Fever :

$$M = \{(x_i, M_Y(x_i), M_N(x_i)) \mid x_i \in X\} ; N = \{(x_i, N_Y(x_i), N_N(x_i)) \mid x_i \in X\}$$

$$i) P_1 \{ s_1, [0.10 \ 0.80] [0.10 \ 0.30] \} ; D_1 \{ s_1, [0.20 \ 0.40] [0.10 \ 0.30] \}$$

$$\tilde{D}_{HM}(M, N) = \frac{1}{2N} \sum_{i=1}^n (|M_\gamma^2(x) - N_\gamma^2(x)| + |M_N^2(x) - N_N^2(x)| + |M_H^2(x) - N_H^2(x)| + |M_N^2(x) - N_N^2(x)|)$$

$$= \frac{1}{2} \{ |0.10^2 - 0.20^2| + |0.80^2 - 0.40^2| + |0.10^2 - 0.10^2| + |0.30^2 - 0.30^2| \}$$

$$= \frac{1}{2} \{ 0.03 + 0.48 \} = 0.255$$

$$ii) P_2 \{ s_1, [0.30 \ 0.60] [0.60 \ 0.80] \} ; D_2 \{ s_1, [0.10 \ 0.30] [0.50 \ 0.70] \}$$

$$= \frac{1}{2} \{ |0.30^2 - 0.10^2| + |0.60^2 - 0.10^2| + |0.60^2 - 0.50^2| + |0.50^2 - 0.70^2| \} = 0.305$$

$$iii) P_3 \{ s_1, [0.60 \ 0.80] [0.10 \ 0.20] \} ; D_3 \{ s_1, [0.10 \ 0.20] [0.10 \ 0.30] \}$$

$$= \frac{1}{2} \{ |0.60^2 - 0.10^2| + |0.80^2 - 0.20^2| + |0.10^2 - 0.10^2| + |0.20^2 - 0.30^2| \} = 0.452$$

Calculation for diagnoses of Viral Fever :

$$M = \{ (x_i, r_M(x_i)\cos(\theta_M(x_i)), r_M(x_i)\sin(\theta_M(x_i))) \}$$

$$N = \{ (x_i, r_N(x_i)\cos(\theta_N(x_i)), r_N(x_i)\sin(\theta_N(x_i))) \}$$

$$i) P_1 \{ s_1, [0.10 \ 0.80] [0.10 \ 0.30] \} ; D_1 \{ s_1, [0.20 \ 0.40] [0.10 \ 0.30] \}$$

$$= \frac{1}{2} [(0.0039794) + (0.0039794) + (0.90127) + (0.008164799) + (0.014082) + (0.68333)]$$

$$= 0.4074024$$

$$D_{mpt}^i(D_t^i P_t^i) = \sqrt{0.4074024} = 0.6382$$

Next calculate weight of every symptom w_t

$$w_t = \frac{D_{mpt}^i(D_t^i, P_t^i)}{\sum_{i=1}^n D_{mp}(D_i, P_i)} \quad (1 \leq t \leq n)$$

$$= \frac{D_{mpt}^i(D_t^i, P_t^i)}{\sum_{i=1}^4 D_{mp}(D_i, P_i)} = \frac{0.6382}{0.039794 + 0.90127 + 0.816479 + 0.014082} = 0.5685$$

Calculated the weighted average IVPFDM distance $\check{D}_{mp}(D_i, P_i) = D_{mpt}^i(D_t^i, P_t^i) w_t$

$$= 0.6382 * 0.5685 = \mathbf{0.4628}$$

As shown in the table obviously the proposed method generates the same results as Existing Method .By observing the table we can find that the proposed method has better resolution than Existing method .

Example 4.2 Suppose a limited universe of discourse $X = \{x_1, x_2\}$, and the PFSs M_i and N_i in the discourse under the Case i ($i \in \{1, 2, 3, \dots, 6\}$), which are given in Table 1

Table1 :Two IVPFSs M_i and N_i Under Different cases

<i>IVPSs</i>	<i>CASE1</i>	<i>CASE2</i>
1	$\{x_1[0.4][0.6]; [0.3 \ 0.5]\}, \{x_2[0.5][0.7]; [0.4 \ 0.6]\}$	$\{x_1[0.4][0.6]; [0.3 \ 0.5]\}, \{x_2[0.5][0.7]; [0.4 \ 0.6]\}$
2	$\{x_1[0.2][0.4]; [0.4 \ 0.6]\}, \{x_2[0.4][0.6]; [0.4 \ 0.6]\}$	$\{x_1[0.3][0.5]; [0.4 \ 0.6]\}, \{x_2[0.4][0.6]; [0.5 \ 0.7]\}$

<i>Methods</i>	<i>Case 1</i>	<i>Case 2</i>
D_{HD}	0.1823	0.2032
D_E	0.2013	0.1964
D_{IVPFS}	0.1569	0.2041

5. Conclusions

In this paper, we propose a new divergence measure, called IVPFSDM distance, based on belief function, and modify the algorithm based on Xiao's method. The proposed method can produce intuitive results and its feasibility is proven by comparing with the existing method. In addition to this, the new method has more even change trend and better performance when the IVPFSs have larger hesitancy.

In addition, we then apply the new algorithm to medical diagnosis and get the desired effect. The new algorithm has better resolution, which is helpful to ruling thresholds in practical applications. Consequently, the main contributions of this article are as follows:

- A method to express the IVPFS in the form of BPA is proposed, which is the first time to establish a link between them.
- A new distance measure between IVPFSs, called the IVPFSDM distance, based on Jensen–Shannon divergence and belief function, is proposed. Combining the characters of divergence and BPA contributes to more powerful resolution and even more of a change trend than existing methods.
- The new divergence measure and the modified algorithm both have satisfying performance in the applications of pattern recognition and medical diagnosis.

It is well known that medical diagnosis is not a 100% accurate procedure and uncertainty is always present in cases. In future research, we will try to explore more IVPFSs' properties by using the belief function in evidence theory further and apply them to more situations such as the multi-criteria decision-making and pattern recognition.

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