Abstract

We study the problem: if \( \tilde{a}_i, i \in N \) are fuzzy numbers of triangular form, then what is the membership function of the infinite (or finite) sum \( \tilde{a}_1 + \tilde{a}_2 + \cdots \) (defined via the sup-product-norm convolution).

**Key words:** Triangular fuzzy number, Product-sum.

**AMS classification:** 05c72, 47s40.

1 Introduction

A fuzzy number is a convex fuzzy subset of the real line \( \mathbb{R} \) with a normalized membership function. A triangular fuzzy number \( \tilde{a} \) denoted by \( (a, \alpha, \beta) \) is defined as

\[
\tilde{a}(t) = \begin{cases} 
1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\
1 & \text{if } a \leq t \leq b \\
1 - \frac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\
0 & \text{otherwise}
\end{cases}
\]

where \( a \in \mathbb{R} \) is the center and \( \alpha > 0 \) is the left spread, \( \beta > 0 \) is the right spread of \( \tilde{a} \). If \( \alpha = \beta \), then the triangular fuzzy number is called symmetric triangular fuzzy number and denoted by \( (a, \alpha) \).

If \( \tilde{a} \) and \( \tilde{b} \) are fuzzy numbers, then their product-sum \( \tilde{a} + \tilde{b} \) is defined as,

\[
(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} \tilde{a}(x)\tilde{b}(y)
\]

The support \( \text{supp} \tilde{a} \) of a fuzzy number \( \tilde{a} \) is defined as

\[
\text{supp} \tilde{a} = \{ t \in \mathbb{R} \mid \tilde{a}(t) > 0 \}
\]
Preliminaries

Definition 2.1 Fuzzy set

Let us take a set \( \tilde{A} \), which is defined by \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0, 1]\} \).

If in the pair \( (x, \mu_{\tilde{A}}(x)) \), the first one, \( x \) belongs to the classical set \( A \) and the second one \( \mu_{\tilde{A}}(x) \) belongs to the interval \([0,1]\), then set \( \tilde{A} \) is called a fuzzy set. Here \( \mu_{\tilde{A}}(x) \) is called a Membership function.

Definition 2.2 Interval-valued fuzzy set (IVFS)

An IVFS \( \tilde{A} \) on \( \mathbb{R} \) is defined by

\[
\tilde{A}_n = \{(x, (\mu_{\tilde{A}}^U(x), \mu_{\tilde{A}}^L(x))) : x \in \mathbb{R}\}
\]

where \( x \in \mathbb{R} \) and \( \mu_{\tilde{A}}^U(x) \), maps \( \mathbb{R} \) into \([0, \lambda] \), \( \mu_{\tilde{A}}^L(x) \), maps \( \mathbb{R} \) into \([0, \omega] \) \( \forall \ x \in \mathbb{R}, \mu_{\tilde{A}}^L(x) \leq \mu_{\tilde{A}}^U(x) \). (\( \lambda \) and \( \omega \) are the maximum value of upper and lower membership function, respectively)

Definition 2.3 Non-linear interval-valued fuzzy number (IVFN)

An IVFN is denoted by

\[
\tilde{A}_n^{IVFN} = \{(a_1, b; \lambda), (a, b, c; \omega) ; n_1, n_2, n_3, n_4\}
\]

where \( 0 < \omega \leq \lambda \leq 1 \) and \( a_1 < a < b < c < c_1 \)

The upper and lower membership function of IVFN is defined by

\[
\begin{align*}
\mu_{\tilde{A}}^U(x) = \begin{cases} 
\lambda \left( \frac{x-a_1}{b-a_1} \right)^{n_1}, & a_1 \leq x \leq b \\
\lambda, & x = b \\
\lambda \left( \frac{c_1-x}{c_1-b} \right)^{n_2}, & b \leq x \leq c_1 \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\mu_{\tilde{A}}^L(x) = \begin{cases} 
\omega \left( \frac{x-a}{b-a} \right)^{n_3}, & a \leq x \leq b \\
\omega, & x = b \\
\omega \left( \frac{c-x}{c-b} \right)^{n_4}, & b \leq x \leq c \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

3 Product-sum of triangular fuzzy numbers

In this section we shall calculate the membership function of the product-sum \( \tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots \) where \( \tilde{a}_i, i \in \mathbb{N} \) are fuzzy numbers of triangular form.

The following theorem can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form.

Theorem 3.1 Let \( \tilde{a}_i = (a_i, \alpha), i \in \mathbb{N} \). If

\[
A := \sum_{i=1}^{\infty} a_i
\]
exists and it is finite, then with the notations

\[ \bar{A}_n := \bar{a}_1 + \cdots + \bar{a}_n, \quad A_n := a_1 + \cdots + a_n, \quad n \in \mathbb{N} \]

we have

\[ \left( \lim_{n \to \infty} \bar{A}_n \right) (z) = \exp(-|A - z|/\alpha), \quad z \in \mathbb{R} \]

**proof:** It will be sufficient to show that

\[ \bar{A}_n(z) = \begin{cases} 
\left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n & \text{if } |A_n - z| \leq n\alpha \\
0 & \text{otherwise}
\end{cases} \tag{1} \]

for each \( n \geq 2 \), because from (1) it follows that

\[ \left( \lim_{n \to \infty} \bar{A}_n \right) (z) = \lim_{n \to \infty} \left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n = \exp \left( - \lim_{n \to \infty} A_n - z \right) /\alpha = \exp(-|A - z|/\alpha), \quad z \in \mathbb{R} \]

From the definition of product-sum of fuzzy numbers it follows that

\[ \text{supp } \bar{A}_n = \text{supp } (\bar{a}_1 + \cdots + \bar{a}_n) = \text{supp } \bar{a}_1 + \cdots + \text{supp } \bar{a}_n = \]

\[ [a_1 - \alpha, a_1 + \alpha] + \cdots + [a_n - \alpha, a_n + \alpha] = [A_n - n\alpha, A_n + n\alpha], \quad n \in \mathbb{N} \]

We prove (1) by making an induction argument on \( n \). Let \( n = 2 \). In order to determine \( A_2(z), \quad z \in [A_2 - 2\alpha, A_2 + 2\alpha] \) we need to solve the following mathematical programming problem:

\[ \left( 1 - \frac{|a_1 - x|}{\alpha} \right) \left( 1 - \frac{|a_2 - y|}{\alpha} \right) \to \max \]

subject to 

\[ |a_1 - x| \leq \alpha, \quad |a_2 - y| \leq \alpha, \quad x + y = z \]

By using Lagrange’s multipliers method and decomposition rule of fuzzy numbers into two separate parts, it is easy to see that \( A_2(z), \quad z \in [A_2 - 2\alpha, A_2 + 2\alpha] \) is equal to the optimal value of the following mathematical programming problem:

\[ \left( 1 - \frac{a_1 - x}{\alpha} \right) \left( 1 - \frac{a_2 - z + x}{\alpha} \right) \to \max \]

subject to 

\[ a_1 - \alpha \leq x \leq a_1, \quad a_2 - \alpha \leq z - x \leq a_2, \quad x + y = z \tag{2} \]
Using Lagrange’s multipliers method for the solution of \([2]\) we get that its optimal value is

\[
1 - \left| \frac{A_2 - z}{2\alpha} \right|^2
\]

and its unique solution is

\[
X = 1/2 (a_1 - a_2 + z)
\]

(where the derivative vanishes).

Indeed, it can be easily checked that the inequality

\[
1 - \left| \frac{A_2 - z}{2\alpha} \right|^2 \geq 1 - \frac{A_2 - z}{\alpha}
\]

holds for each \(z \in [A_2 - 2\alpha, A_2]\)

In order to determine \(\tilde{A}_2(z), z \in [A_2, A_2 + 2\alpha]\) we need to solve the following mathematical programming problem:

\[
(1 + \frac{a_1 - x}{\alpha}) \left(1 + \frac{a_2 - z + x}{\alpha}\right) \rightarrow \text{max}
\]

subject to \(a_1 \leq x \leq a_1 + \alpha, a_2 \leq z - x \leq a_2 + \alpha\)

In a similar manner we get that the optimal value of \([3]\) is

\[
1 - \left| \frac{z - A_2}{2\alpha} \right|^2
\]

Let us assume that \([1]\) holds for some \(n \in N\). By similar arguments we obtain

\[
\tilde{A}_{n+1}(z) = \left(\tilde{A}_n + \tilde{a}_{n+1}\right)(z) = \sup_{x+y=z} \tilde{A}_n(x) \cdot \tilde{a}_{n+1}(y) = \sup_{x+y=z} \left(1 - \frac{|A_n - x|}{n\alpha}\right) \left(1 - \frac{|a_{n+1} - y|}{\alpha}\right) = \left[1 - \frac{|A_{n+1} - z|}{(n+1)\alpha}\right]^{n+1}, z \in [A_{n+1} - (n + 1)\alpha, A_{n+1} + (n + 1)\alpha]
\]

and

\[
\tilde{A}_{n+1}(z) = 0, z \notin [A_{n+1} - (n + 1)\alpha, A_{n+1} + (n + 1)\alpha]
\]

This ends the proof.

**Theorem 3.2** Let \(\tilde{a}_i = (a_i, \alpha, \beta), i \in N\) be fuzzy numbers of triangular form. If
\[
A := \sum_{i=1}^{\infty} a_i \text{ exists and it is finite, then with the notations of Theorem 3.2 we have}
\]

\[
\left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \begin{cases} 
\exp \left( -\frac{|A-z|}{\alpha} \right) & \text{if } z \leq A \\
\exp \left( -\frac{|A-z|}{\beta} \right) & \text{if } z \geq A
\end{cases}
\]

4 Conclusion
In this paper, we studied about the Product-sum of triangular fuzzy numbers and also calculated the membership function of the product-sum \(\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots\)
where \(\tilde{a}_i, i \in N\) are fuzzy numbers of triangular form.

References