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A Study on Product-Sum of Triangular Fuzzy Numbers

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Abstract

We study the problem: if $\tilde{a}_i$, $i \in N$ are fuzzy numbers of triangular form, then what is the membership function of the infinite (or finite) sum $\tilde{a}_1 + \tilde{a}_2 + \cdots$ (defined via the sup-product-norm convolution)

Key words: Triangular fuzzy number, Product-sum.
AMS classification: 05c72, 47s40.

1 Introduction

A fuzzy number is a convex fuzzy subset of the real line $\mathbb{R}$ with a normalized membership function. A triangular fuzzy number $\tilde{a}$ denoted by (a, α, β) is defined as

$$\tilde{a}(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \frac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

where $a \in \mathbb{R}$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of $\tilde{a}$. If $\alpha = \beta$, then the triangular fuzzy number is called symmetric triangular fuzzy number and denoted by (a, α).
If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers, then their product-sum $\tilde{a} + \tilde{b}$ is defined as,

$$(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} \tilde{a}(x)\tilde{b}(y)$$

The support supp $\tilde{a}$ of a fuzzy number $\tilde{a}$ is defined as

$$\operatorname{supp} \tilde{a} = \{t \in \mathbb{R} \mid \tilde{a}(t) > 0\}$$

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2 Preliminaries

Definition 2.1 ***Fuzzy set***

Let us take a set $\tilde{A}$, which is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$. If in the pair $(x, \mu_{\tilde{A}}(x))$, the first one, x belongs to the classical set A and the second one $\mu_{\tilde{A}}(x)$ belongs to the interval [0,1], then set $\tilde{A}$ is called a fuzzy set. Here $\mu_{\tilde{A}}(x)$ is called a Membership function.

Definition 2.2 ***Interval-valued fuzzy set (IVFS)*** An IVFS $\tilde{A}$ on $\mathcal{R}$ is defined by

$$\tilde{A}_n = [\{x, (\mu_{\tilde{A}^U}(x), \mu_{\tilde{A}^L}(x))\} : x \in \mathcal{R}]$$

where $x \in \mathcal{R}$ and $\mu_{\tilde{A}_n^U}(x)$, maps $\mathcal{R}$ into $[0, \lambda]$, $\mu_{\tilde{A}_n^L}(x)$, maps $\mathcal{R}$ into $[0, \omega]$ $\forall\, x \in \mathcal{R}$, $\mu_{\tilde{A}_n^L}(x) \leq \mu_{\tilde{A}_n^U}(x)$. ($\lambda$ and ω are the maximum value of upper and lower membership function, respectively)

Definition 2.3 ***Non-linear interval-valued fuzzy number (IVFN)*** An IVFN is denoted by

$$\tilde{A}_{nLIVFN} = [\{(a_1, b, c_1; \lambda), (a, b, c; \omega)\}; n_1, n_2, n_3, n_4]$$

where $0 < \omega \leq \lambda \leq 1$ and $a_1 < a < b < c < c_1$

The upper and lower membership function of IVFN is defined by

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \lambda\left(\frac{x-a_1}{b-a_1}\right)^{n_1}, a_1 \leq x \leq b \\ \lambda, x = b \\ \lambda\left(\frac{c_1-x}{c_1-b}\right)^{n_2}, b \leq x \leq c_1 \\ 0, \text{ otherwise} \end{cases} \quad \text{and} \quad \mu_{\tilde{A}^L}(x) = \begin{cases} \omega\left(\frac{x-a}{b-a}\right)^{n_3}, a \leq x \leq b \\ \omega, x = b \\ \omega\left(\frac{c-x}{c-b}\right)^{n_4}, b \leq x \leq c \\ 0, \text{ otherwise} \end{cases}$$

3 Product-sum of triangular fuzzy numbers

In this section we shall calculate the membership function of the product-sum $\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots$ where $\tilde{a}_i, i \in \mathbf{N}$ are fuzzy numbers of triangular form.
The following theorem can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form.

Theorem 3.1 Let $\tilde{a}_i = (a_i, \alpha), i \in \mathrm{N}$. If

$$A := \sum_{i=1}^{\infty} a_i$$

exists and it is finite, then with the notations

$$\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n, \ \ A_n := a_1 + \cdots + a_n, n \in \mathbf{N}$$

we have

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \exp(-|A - z|/\alpha), z \in \mathbb{R}$$

proof: It will be sufficient to show that

$$\tilde{A}_n(z) = \begin{cases} \left[1 - \frac{|A_n - z|}{n\alpha}\right]^n & \text{if } |A_n - z| \leq n\alpha \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for each $n \geq 2$, because from (1) it follows that

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \lim_{n\to\infty}\left[1 - \frac{|A_n - z|}{n\alpha}\right]^n = \exp\left(-\left|\lim_{n\to\infty} A_n - z\right|/\alpha\right) =$$
$$\exp(-|A - z|/\alpha), z \in \mathbb{R}$$

From the definition of product-sum of fuzzy numbers it follows that

$$\operatorname{supp} \tilde{A}_n = \operatorname{supp}(\tilde{a}_1 + \cdots + \tilde{a}_n) = \operatorname{supp} \tilde{a}_1 + \cdots + \operatorname{supp} \tilde{a}_n =$$

$$[a_1 - \alpha, a_1 + \alpha] + \cdots + [a_n - \alpha, a_n + \alpha] = [A_n - n\alpha, A_n + n\alpha], n \in \mathbf{N}$$

We prove (1) by making an induction argument on n. Let $n = 2$. In order to determine $\tilde{A}_2(z), z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ we need to solve the following mathematical programming problem:

$$\left(1 - \frac{|a_1 - x|}{\alpha}\right)\left(1 - \frac{|a_2 - y|}{\alpha}\right) \to \max$$
$$\text{subject to } \ |a_1 - x| \leq \alpha,$$
$$|a_2 - y| \leq \alpha, \ x + y = z$$

By using Lagrange's multipliers method and decomposition rule of fuzzy numbers into two separate parts, it is easy to see that $\tilde{A}_2(z), z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ is equal to the optimal value of the following mathematical programming problem:

$$\left(1 - \frac{a_1 - x}{\alpha}\right)\left(1 - \frac{a_2 - z + x}{\alpha}\right) \to \max$$
$$\text{subject to } a_1 - \alpha \leq x \leq a_1 \tag{2}$$
$$a_2 - \alpha \leq z - x \leq a_2, \ x + y = z$$

Using Lagrange's multipliers method for the solution of (2) we get that its optimal value is

$$\left[1-\frac{|A_2-z|}{2\alpha}\right]^2$$

and its unique solution is

$$X=1/2\,(a_1-a_2+z)$$

(where the derivative vanishes).
Indeed, it can be easily checked that the inequality

$$\left[1-\frac{|A_2-z|}{2\alpha}\right]^2 \geq 1-\frac{A_2-z}{\alpha}$$

holds for each $z\in[A_2-2\alpha,A_2]$
In order to determine $\tilde{A}_2(z), z \in [A_2, A_2+2\alpha]$ we need to solve the following mathematical programming problem:

$$\begin{gathered}\left(1+\tfrac{a_1-x}{\alpha}\right)\left(1+\tfrac{a_2-z+x|}{\alpha}\right)\to \max\\ \text{subject to } a_1\leq x\leq a_1+\alpha, a_2\leq z-x\leq a_2+\alpha\end{gathered} \tag{3}$$

In a similar manner we get that the optimal value of (3) is

$$\left[1-\frac{|z-A_2|}{2\alpha}\right]^2$$

Let us assume that (1) holds for some $n\in N$. By similar arguments we obtain

$$\begin{gathered}\tilde{A}_{n+1}(z)=\left(\tilde{A}_n+\tilde{a}_{n+1}\right)(z)=\\ \sup_{x+y=z}\tilde{A}_n(x)\cdot\tilde{a}_{n+1}(y)=\sup_{x+y=z}\left(1-\frac{|A_n-x|}{n\alpha}\right)\left(1-\frac{|a_{n+1}-y|}{\alpha}\right)=\\ \left[1-\tfrac{|A_{n+1}-z|}{(n+1)\alpha}\right]^{n+1}, z\in[A_{n+1}-(n+1)\alpha, A_{n+1}+(n+1)\alpha]\end{gathered}$$

and

$$\tilde{A}_{n+1}(z)=0, z\notin[A_{n+1}-(n+1)\alpha, A_{n+1}+(n+1)\alpha]$$

This ends the proof.

Theorem 3.2 Let $\tilde{a}_i=(a_i,\alpha,\beta), i\in\mathbf{N}$ be fuzzy numbers of triangular form. If

$A := \sum_{i=1}^{\infty} a_i$ exists and it is finite, then with the notations of Theorem 3.2 we have

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \begin{cases} \exp\left(-\frac{|A-z|}{\alpha}\right) & \text{if } z \leq A \\ \exp\left(-\frac{|A-z|}{\beta}\right) & \text{if } z \geq A \end{cases}$$

4 Conclusion

In this paper, we studied about the Product-sum of triangular fuzzy numbers and also calculated the membership function of the product-sum $\tilde{a}_1 + \tilde{a}_2 + \cdots \tilde{a}_n + \cdots$ where $\tilde{a}_i, i \in N$ are fuzzy numbers of triangular form.

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