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A Study on Stochastic Differential Equation

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Abstract

In this paper we study solutions to stochastic differential equations (SDEs) with discontinuous drift. In this paper we discussed The Euler-Maruyama method and this shows that a candidate density function based on the Euler-Maruyama method. The point of departure for this work is a particular SDE with discontinuous drift.

Key words: Stochastic Differential Equation, Discontinuous Drift.

AMS classification: 60G10, 60H30.

1 Introduction

A general one dimensional SDE is given by

$$dx_t = b(t, x_t)dt + \sigma(t, x_t)dB_t, \quad x_0 = c, \tag{1}$$

where $x = x_t$ is an $\mathbb{R}$-valued stochastic process : $[0, T] \to \mathbb{R}$, $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are the drift and diffusion coefficient of x, $B = B_t$ is an $\mathbb{R}$-valued Wiener process, and c is a random variable independent of $B_t - B_0$ for $t \geq 0$. On $[0, T]$, existence and uniqueness of a solution x_t, continuous with probability 1, to (1) is guaranteed whenever the drift b and diffusion σ are measurable functions satisfying a Lipschitz condition together with a growth bound, both uniformly in t.
This paper focuses on the special case for (1), where $\sigma(t, x_t) = 1$ and $b(t, x_t) = -k$ sign(x_t) with $k > 0$ a control gain and the sign-function defined by

$$sign(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases} . \tag{2}$$

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Thus, we consider the SDE

$$dx_t = -k\text{sign}(x_t)dt + dB, \quad x_0 = c. \tag{3}$$

with c given.
In the next section, the Euler-Maruyama method is applied to approximate solutions to (3) and to investigate a theoretical methods to obtain candidate density functions for solutions to (3).

2 The Euler-Maruyama Method

The Euler-Maruyama method is a simple time discrete approximation technique which is used to approximate solutions to SDEs of the type given in (1), by discretizing the time interval $[0,T]$ in steps $0 < t_1 < \cdots < t_n < t_{n+1} \cdots < t_N$ with $N = \lceil \frac{T}{h} \rceil$, where $h = t_{n+1} - t_n$ is the step-length. Each recursive step is determined via the following method,

$$x_{n+1} = x_n + hb(t_n, x_n) + \sigma(t_n, x_n)W_n \ , \tag{4}$$

where $x_{t_i} = x_i$ and $W_n = B_{t_{n+1}} - B_{t_n}$ is i.i.d. normal with mean zero and variance h, which we denote by $W_n \sim N(0,h)$.
Given an initial condition $x_0 = c$, it is possible from (4) to approximate a solution to (1) by determination of $x_1, x_2, \ldots, x_N$. If the drift and diffusion coefficient in (1) are measurable, satisfy a Lipschitz condition and a growth bound, the Euler Maruyama method guarantee strong convergence to the solution of (1). Hence for SDEs with discontinuous drift we can, in general, not expect the Euler Maruyama method to produce meaningful results. Nevertheless, we will in the sequel apply this method to the special case (3) in order to obtain candidate solutions.

Analysis of the Deterministic Step

Application of the Euler-Maruyama method to the SDE in (3) gives the recursive step

$$x_{n+1} = x_n - hksign(x_n) + W_n \ . \tag{5}$$

Now, if $x_n > 0$, we have

$$x_{n+1} = x_n - hk + W_n.$$

Since $W_n \sim N(0,h)$, the expectation is that $x_{n+1} \in [x_n - h(k+1), x_n - h(k-1)]$ in most of the simulations. From this, we expect after a finite time $0 < t < \infty$ that there exists $N \in \mathbb{N}$ such that $x_{n+N} \leq 0$. Similar result is obtained if $x_m < 0$, then we expect that there exists $M \in \mathbb{N}$ such that $x_{m+M} \geq 0$. The influence from the control gain k determines how quick the evolution of the sequence $\{x_n\}_{n\geq 0}$ switches around zero. In other words, a big k minimizes the influence of the random variable W_n.
The Euler-Maruyama method is easy to implement in software, so following we have applied Matlab to simulate solutions to (3).

Numerical solutions to a SDE with Discontinuous Drift

We consider the recursive step in (5) and simulate the evolution of the stochastic process. For all simulations the initial condition is chosen to be $x_0 = 0$ and the considered time interval is $[0,T]$ where $T = 1$. The step-length is h such that the number of simulated steps is $N = \lceil \frac{T}{h} \rceil$. All simulations are repeated 500 times and histograms of the resulting values of x_T are presented.
In figure 1, one realization of a solution to the SDE in (3) is shown together with the average values of all the 500 simulations in the time interval $[0,T]$. The average of x_t is close to zero for all $0 \leq t \leq T$. Figure 2 shows the resulting histogram of x_T including 500 simulations. In order to investigate the influence of the step-size, figure

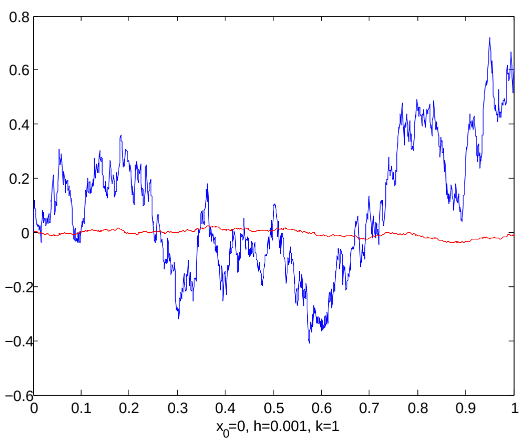

Figure 1: The blue graph shows one realization out of 500 simulations while the red graph represents the average of x_t for $0 \leq t \leq T$.

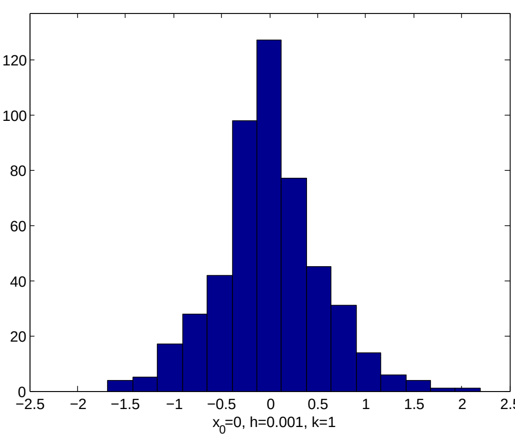

Figure 2: Resulting histogram of 500 simulations of x_T.

3 illustrates four different histograms of 500 simulations. Here h is $0.01, 0.001, 0.0001$ and 0.00001 respectively and $k = 1$. It can be seen right away that the result narrows

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slightly around zero when h becomes smaller, but changing in the step-size does not immediately give big effect.

In figure 4, the control gain is changing, $k = 1, 2, 3, 4$ and $h = 0.001$. Here it is clear that changing k as an influence on the result of x_T. The variance of x_T gets smaller when k increases. This is not surprising since the overall influence of the random variable W_n is decreased when k increases as mentioned in previous section.

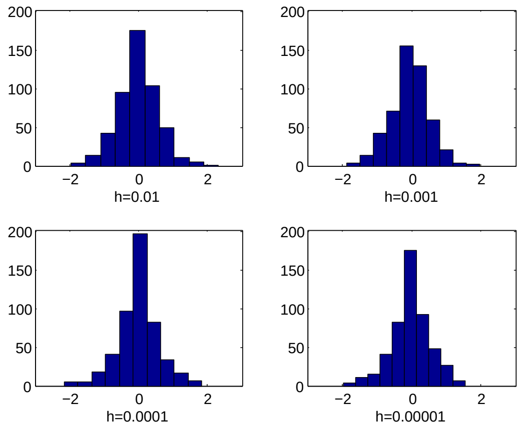

Figure 3: Histograms of 500 simulations of x_T with different step-size.

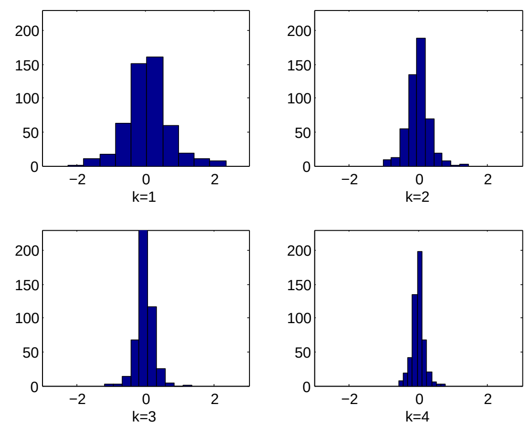

Figure 4: Histograms of 500 simulations of x_T with different control gains. $h = 0.001$.

Theoretical Distribution of x_n

Consider the recursive determination of x_{n+1} in (5). Define an intermediate variable $z_n = x_n + hksign(x_n)$ and let f_{z_n} and f_n denote the density functions of z_n and x_n, respectively. Moreover, let $N(0, h)$ denote the density function for W_n. From probability theory we get

$$f_{n+1}(x) = f_{z_n} * N(0, h) \ , \tag{6}$$

Hence, we proceed by studying the density function $f_{z_n}(z)$. Let the distribution function of z_n be denoted by F_{z_n} such that $F_{z_n}(z) = P(z_n \leq z) = P(x_n - hksign(x_n) \leq z)$, which can be expressed by

$$\begin{aligned} P(z_n \leq z) &= P(x_n - hk \leq z, x_n > 0) + P(x_n + hk \leq z, x_n < 0) + P(x_n \leq z, x_n = 0) \\ &= P(x_n \leq z + hk, x_n > 0) + P(x_n \leq z - hk, x_n < 0) \ . \end{aligned}$$

For different values of z, the probability $P(z_n \leq z)$ can be expressed differently. If $z < -hk$

$$P(z_n \leq z) \;=\; P(x_n \leq z - hk, x_n < 0) = P(x_n \leq z - hk) = F_n(z - hk)\;.$$

If $z > hk$

$$P(z_n \leq z) \;=\; P(x_n \leq z + hk, x_n > 0) + P(x_n \leq z - hk) = P(x_n \leq z + hk) = F_n(z + hk)\;,$$

and if $-hk \leq z \leq hk$.

$$\begin{aligned} P(z_n \leq z) &= P(x_n \leq z + hk) - P(x_n < 0) + P(x_n \leq z - hk) \\ &= F_n(z + hk) - F_n(0) + F_n(z - hk)\;. \end{aligned}$$

By introducing the indicator function $\mathbb{I}$, the expression of the distribution function of z_n is

$$\begin{aligned} P(z_n \leq z) &= F_n(z - hk)\mathbb{I}_{(-\infty,-hk)}(z) + (F_n(z + hk) - F_n(0) \\ &\qquad + F_n(z - hk))\mathbb{I}_{[-hk,hk]}(z) + F_n(z + hk)\mathbb{I}_{(hk,\infty)}(z) \\ &= F_n(z - hk)\mathbb{I}_{(-\infty,hk]}(z) + F_n(z + hk)\mathbb{I}_{[-hk,\infty)}(z) - F_n(0)\mathbb{I}_{[-hk,hk]}(z)\;. \end{aligned}$$

By differentiating with respect to z, the density function of z_n is

$$\begin{aligned} \frac{\partial}{\partial z}F_{z_n} &= f_n(z - hk)\mathbb{I}_{(-\infty,hk]}(z) - F_n(z - hk)\delta(z - hk) + f_n(z + hk)\mathbb{I}_{[-hk,\infty)}(z) \\ &\qquad + F_n(z + hk)\delta(z + hk) - F_n(0)(\delta(z + hk) - \delta(z - hk)) \\ &= f_n(z - hk)\mathbb{I}_{(-\infty,hk]}(z) + f_n(z + hk)\mathbb{I}_{(-hk,\infty)}(z) + \delta(z + hk)(F_n(z + hk) - F_n(0)) \\ &\qquad + \delta(z - hk)(F_n(0) - F_n(z - hk)) \\ &= f_n(z - hk)\mathbb{I}_{(-\infty,hk]}(z) + f_n(z + hk)\mathbb{I}_{[-hk,\infty)}(z)\;. \end{aligned}$$

Therefore

$$f_{z_n}(z) = f_n(z - hk)\mathbb{I}_{(-\infty,hk]}(z) + f_n(z + hk)\mathbb{I}_{[-hk,\infty)}(z)\;. \tag{7}$$

By substituting the above into (6), the density function f_{n+1} is found from the density function f_n. In the following section, the solution to (7) is investigated numerically.

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Recursive Developing of the Density Function in Matlab

The recursive density function is given by

$$f_{n+1}(x) \quad = \quad \left(f_n(x+kh)\mathbb{I}_{(-\infty,hk]} + f_n(x-hk)\mathbb{I}_{[-hk,\infty)}\right) * N(0,h)\ . \tag{8}$$

Following, we apply Matlab to investigate the evolution of the function $f_{n+1}(x)$ for n increasing. Assume that the density function for the initial condition $x_0 = c$ is normal distributed with mean zero and variance h. The Euler-Maruyama method is expected to converge to a stochastic process (or a distribution of a stochastic process) when $h \to 0$. (Under certain regularity conditions, so actually we cannot expect it here but only conjecture.) We hope that the developing of the recursive density functions in (8) will reach stationary condition for $n \to \infty$. For this reason, the number n of simulations is chosen to depend on the step size, such that $n = \left\lceil \frac{1}{h^{1+\alpha}} \right\rceil$, where $\alpha > 0$. This ensures that both convergence criteria are fulfilled.
Equation (8) is simulated in Matlab for $h = 0.01, \alpha = 0.5, k = 1$ such that $n = 1000$, the result is shown in figure 5. At the end of section, there is a comparison between the convergence of the recursive density function and the result obtained there is presented.

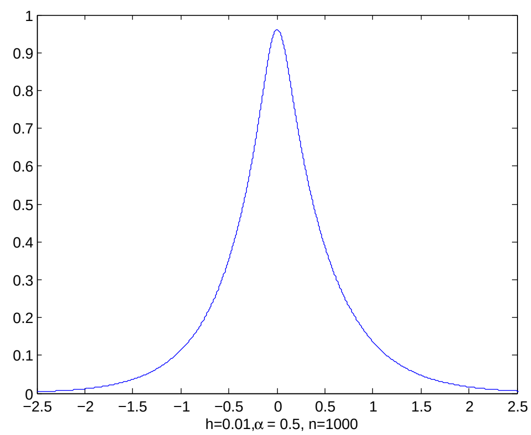

Figure 5: Recursive density function, f_{1000}.

In the following, we continue the study of (7) under stationary assumptions.

Stationary State

In the sequel we continue the study of the recursive density function under the assumptions that it is possible to reach stationary state in (8) for $n \to \infty$, such that

$$f_{z_n}(z) = f(z-hk)\mathbb{I}_{(-\infty,hk]}(z) + f(z+hk)\mathbb{I}_{(-hk,\infty)}(z) \; ,$$

where $f(x)$ is the stationary density function of the recursive development of $f_n(x)$. In this case we define an operator H_h taking f_n to f_{n+1} by

$$H_h[f](x) \;\; = \;\; \int_{\mathbb{R}} \left(f(z-hk)\mathbb{I}_{(-\infty,hk]}(z) + f(z+hk)\mathbb{I}_{(-hk,\infty)}(z)\right) \frac{1}{\sqrt{2\pi h}} \exp\left(\frac{-(x-z)^2}{2h}\right) dz \; .$$

By taking the derivative with respect to h and then the limit $h \to 0$, we obtain,

$$\lim_{h\to 0} \frac{\partial}{\partial h} H_h[f](x) = \begin{cases} -kf'(x) + \frac{1}{2}f''(x) & \text{for} \quad x < 0 \\ \psi(x) & \text{for} \quad x = 0 \\ kf'(x) + \frac{1}{2}f''(x) & \text{for} \quad x > 0 \end{cases} . \tag{9}$$

The case $x = 0$ is not important for the sequel developing, hence we leave $\psi(x)$ unspecified. Note that the right hand side of the first and last case in (9) have similarity with the stationary Fokker-Planck equation.

An approximation of the operator H_h is

$$\begin{aligned} H_h[f](x) &\approx f(x) + h(ksign(x)f'(x) + \frac{1}{2}f''(x)) \\ &= f(x) + hG[f](x) \; . \end{aligned} \tag{10}$$

For a fixed h define f_h to be the stationary density function and assume that $\lim_{h\to 0} f_h$ exists, say f_0 and that $H_h(f_h) = f_h$. Hence, if we disregard the approximation in (10), we look for a function f_h such that $G[f_h] = 0$, which by (9) means that f_h is a stationary solution to the Fokker-Planck equation. We conjecture that this heuristic will constitute the main ideas in the proof that the stationary distribution f_0 $(= \lim_{h\to 0} f_h)$ of the Euler-Maruyama simulation converges to the Fokker-Planck equation.

3 Conclusion

In this paper, we have discussed The Euler-Maruyama method and this shows that a candidate density function based on the Euler-Maruyama method. The point of departure for this work is a particular SDE with discontinuous drift.

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