Oscillation Theory of $q$-Difference Equation

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Abstract

In this research article, the authors present the oscillation theory of the $q$-difference equation

$$k(t)y(qt) + k\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = r(t)y(t),$$

where $r(t) = k(t) + k\left(\frac{t}{q}\right) - q(t)$. In particular we prove that this $q$-difference equation is oscillatory or non-oscillatory for different conditions.

Key words: Oscillation, Non-oscillation, $q$-Difference equation.


1 Introduction

The numerical and analytical solutions of $q$-difference operator has an important role in different fields such as science and engineering, whose solution has a better understanding of the physical features of the problem [6, 5]. The authors in [1] introduced a $\Delta_q$ operator and then derived many results using the generalized $q$-difference equation $\Delta_q^m v(k) = u(k), \quad q \neq 1$ and for any real $k$.

The authors in [2, 3] developed the $q$-alpha multi-series formula for finite and higher order $q$-alpha formula for infinite series.

The branch of differential equation theory is widely used in oscillation theory [4, 9]. The existence or non-existence of oscillatory solutions to a given equation or system are contained in the basic problem of classical theory of oscillation [7, 8]. Till recently no special importance was given to the study of oscillations using $q$-difference equation. Hence in this article, we are primarily interested in the oscillation theory of $q$-difference equation.

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2 Preliminaries

Here, we present some preliminaries which will be used for further discussion.

**Definition 2.1** Let \( f(k) \) be the real valued function on \([0, \infty)\) and \( q \neq 1 \) be a fixed real number. Then the \( q \)-difference operator, denoted by \( \Delta_q \), on \( f(k) \) is defined as

\[
\Delta_q f(k) = f(qk) - f(k), \quad q \neq 1.
\]

**Definition 2.2** Let \( 1 \neq q \) and \( m \) be any positive integers and

\[
y(qt) - y(t) + k(t)y \left( \frac{t}{qm} \right) = 0,
\]

where \( k(t) \) is a sequence defined for \( t \in \mathbb{Z}^+ \). Then \( y(t) \) is called an oscillatory solution of (2) if \( y(t) y(qt) \leq 0 \). Otherwise it is called non oscillatory solution.

**Result 2.3** If \( \lim_{t \to \infty} \inf p(t) = p > \frac{m^m}{(qm)^{qm}} \), then every solution of (2) oscillates.

2 \( q \)-self adjoint second order \( q \)-difference equation

Here, we developed the \( q \)-self adjoint second order \( q \)-difference equation

\[
\Delta_q \left[k \left( \frac{t}{q} \right) \Delta_q y \left( \frac{t}{q} \right) \right] + q(t)y(t) = 0,
\]

where \( k(t) > 0, \ t \in \mathbb{Z}^+ \). By **Definition 2.1** the above equation becomes

\[
k(t)y(qt) + k \left( \frac{t}{q} \right) y \left( \frac{t}{q} \right) = \left[k(t) + k \left( \frac{t}{q} \right) - q(t)\right] y(t),
\]

which implies

\[
k(t)y(qt) + k \left( \frac{t}{q} \right) y \left( \frac{t}{q} \right) = r(t)y(t),
\]

where \( r(t) = k(t) + k \left( \frac{t}{q} \right) - q(t) \).

Hence any equation of the form

\[
k_0 y(qt) + k_1(t)y(t) + k_2(t)y \left( \frac{t}{q} \right) = 0,
\]
with $k_0(t) > 0$ and $k_2(t) > 0$, can be put in the q-self adjoint form (3) or (4). Now multiplying both sides of (6) by a positive sequence $h(t)$ yields

$$k_0(t)h(t)y(qt) + k_1(t)h(t)y(t) + k_2(t)h(t)\left(\frac{t}{q}\right) = 0. \quad (7)$$

Comparing (4) and (7), we get $k(t) = k_0(t)h(t)$, $k\left(\frac{t}{q}\right) = k_2(t)h(t)$ and $r(t) = -k_1(t)h(t). \quad (8)$

Then $k_2(qt)h(qt) = k_0(t)h(t)$, which gives

$$h(t) = \frac{k_0\left(\frac{t}{q}\right)h\left(\frac{t}{q}\right)}{k_2(t)}.$$ 

Putting the value of $h(t/q)$ by replacing $t$ by $t/q$ in the above equation repeatedly, we obtain

$$h(t) = \left(\frac{k_0\left(\frac{t}{q}\right)}{k_2(t)}\right)\left(\frac{k_0\left(\frac{t}{q^2}\right)}{k_2\left(\frac{t}{q^2}\right)}\right)\cdots\left(\frac{k_0\left(\frac{t}{q^s}\right)}{k_2\left(\frac{t}{q^s}\right)}\right)h\left(\frac{t}{q^{s+1}}\right),$$

and hence

$$h(t) = \prod_{j=0}^{s} \frac{k_0\left(\frac{t}{q^{j+1}}\right)}{k_2\left(\frac{t}{q^j}\right)}h\left(\frac{t}{q^{s+1}}\right)$$

is a solution of (6).

Using the above equation, we get $k(t) = k_0(t)\prod_{j=0}^{s} \frac{k_0\left(\frac{t}{q^{j+1}}\right)}{k_2\left(\frac{t}{q^j}\right)}h\left(\frac{t}{q^{s+1}}\right).$

Also from (5) and (8), we get $q(t) = k_1(t)h(t) + k(t) + k\left(\frac{t}{q}\right).$
Result 3.1 Consider \( z(t) = \frac{r(qt)y(qt)}{k(t)y(t)} \). Then (4) gives
\[
\frac{k^2(t)}{r(t)r(qt)}z(t) + \frac{1}{z\left(\frac{t}{q}\right)} = 1,
\]
which implies
\[
c(t)z(t) + \frac{1}{z\left(\frac{t}{q}\right)} = 1, \tag{9}
\]
where
\[
c(t) = \frac{k^2(t)}{r(t)r(qt)}. \tag{10}
\]

Result 3.2 If \( c(t) \geq b(t) > 0 \) for all \( t > 0 \) and \( z(t) > 0 \) is a solution of \( c(t)z(t) + \frac{1}{z\left(\frac{t}{q}\right)} = 1 \), then the q-difference equation \( b(t)x(t) + \frac{1}{x\left(\frac{t}{q}\right)} = 1 \), has a solution \( x(t) \geq z(t) > 1 \) for all \( t \in \mathbb{Z}^+ \).

3 Main Results

This section deals with the oscillatory and nonoscillatory solution of the q-difference equation \( k(t)y(qt) + k \left(\frac{t}{q}\right) y \left(\frac{t}{q}\right) = r(t)y(t) \), where \( r(t) = k(t) + k \left(\frac{t}{q}\right) - q(t) \) based on the given conditions.

Lemma 4.1 If there exists a subsequence \( r(t_m) \leq 0 \) with \( t_m \to \infty \) as \( m \to \infty \), then every solution of (4) is oscillatory. proof On contrary, suppose there exists a non oscillatory solution \( y(t) > 0 \) for \( t \geq N \) of (4). Then we obtain
\[
k(t_m)y(qt_m) + k \left(\frac{t_m}{q}\right) y \left(\frac{t_m}{q}\right) - r(t_m)y(t_m) > 0 \text{ for } t_m > N,
\]
which is a contradiction and hence the proof.

Lemma 4.2 Suppose that \( r(t) > 0 \) for \( t \in \mathbb{Z}^+ \). Then every solution \( y(t) \) of (4) is non oscillatory if and only if every solution \( z(t) \) of (9) is positive for \( t \geq N \), for some \( N > 0 \). proof Suppose that (4) has a non oscillatory solution \( y(t) \), which yields \( y(t)y(qt) > 0 \) for \( t \geq N \).
Also from (4), \( z(t) > 0 \) for \( t > N \). Conversely assuming \( z(t) \) is a positive solution of
Then we construct inductively \( y(t) \) as 

\[ y(N) = 1 \quad \text{and} \quad y(qt) = \left( \frac{k(t)}{r(qt)} \right) z(t)y(t) \]

with \( t \geq N \), which gives 

\[ y(qN) = \left( \frac{k(N)}{r(qN)} \right) z(N)y(N) > 0. \]

Similarly, we can prove \( y(qN) > 0 \) for \( n \geq 2 \) and so \( y(t) > 0 \) for \( n \geq N \). Thus \( y(t) \) is a non oscillatory solution of equation (4).

**Theorem 4.3** If 

\[ r(t)r(qt) \leq (4 - \epsilon) k^2(t) \]

for some \( \epsilon > 0 \) and for all \( t \geq N \), then every solution of (4) oscillates.proof If 

\[ r(t)r \left( \frac{t}{q} \right) \leq (4 - \epsilon) k^2(t) \]

for some \( \epsilon \geq 4 \), then 

\[ r(t)r \left( \frac{t}{q} \right) \leq 0. \]

By lemma 4.1, every solution of (4) is oscillatory. Hence we may assume that \( 0 < \epsilon < 4 \).

Now let us assume the contrary. Then by lemma 4.2, (9) has a positive solution \( z(t) \) for \( t \geq N \).

Using the assumption in (10) yields \( c(t) \geq \frac{1}{4 - \epsilon} \). Again using result 3.2, \( s(t), s(t) \geq z(t) > 1 \) for all \( t \geq N \), is a solution of

\[ \frac{1}{4 - \epsilon} s(t) + \frac{1}{s \left( \frac{t}{q} \right)} = 1 \quad (11) \]

Now, we define a positive sequence \( y(t) \) inductively as follows:

\[ y(N) = 1, \quad y(qt) = \frac{1}{\sqrt{4 - \epsilon}} s(t)y(t), \quad t \geq N. \]

Then 

\[ s(t) = \sqrt{4 - \epsilon} \frac{y(qt)}{y(t)}. \]

Substituting \( s(t) \) in (11), we get

\[ \frac{1}{\sqrt{4 - \epsilon}} \frac{y(qt)}{y(t)} + \frac{1}{\sqrt{4 - \epsilon}} \frac{y \left( \frac{t}{q} \right)}{y(t)} = 1, \]

which yields 

\[ y(qt) - \sqrt{4 - \epsilon} y(t) + y \left( \frac{t}{q} \right) = 0, \quad t \geq N, \]
whose characteristic roots are \( \frac{\sqrt{4 - \epsilon} \pm i \sqrt{\epsilon}}{2} \).
Hence the solutions are non oscillatory, which is a contradiction.

**Theorem 4.4** Let \( r(t)r(qt) \geq 4k^2(t) \) for \( t \geq N \). Then every solution of (4) is nonoscillatory. proof From the given assumption, \( \frac{1}{r(t)r(qt)} \leq 4k^2(t) \).
Hence (10) gives \( c(t) \leq \frac{1}{4} \).
We shall construct inductively a solution \( z(t) \) of (9) as follows: Put \( z(t) = 2 \) and \( z(t) = \frac{1}{c(t)} \left[ 1 - \frac{1}{z(t/q)} \right] \).
Now, \( z(qN) = \frac{1}{c(qN)} \left[ 1 - \frac{1}{z(qN)} \right] \geq 4 \left( 1 - \frac{1}{2} \right) = 2 \).
Similarly, we can prove that \( z(t) \geq 2 \) for \( t \geq N \).
Hence by lemma 4.2, every solution of (4) is nonoscillatory.

**Example 4.5** Let \( m \) be a positive integer. Then \( \left( \frac{1}{q} \right)^{t-1}, t > 1 \) is a nonoscillatory solution of \( y(qt) - y(t) + \frac{m^m}{(qm)^m} y \left( \frac{t}{q^m} \right) = 0 \).

**Theorem 4.6** If \( k(t) \geq 0 \) and \( \sup k(t) < \frac{m^m}{(qm)^m}, q > 1 \), then (2) has a non oscillatory solution. proof We have
\[
y(qt) - y(t) + k(t) y \left( \frac{t}{q^m} \right) = 0
\]
(12)
Dividing both sides by \( y(t) \), we get
\[
\frac{y(qt)}{y(t)} = 1 - \frac{k(t) y \left( \frac{t}{q^m} \right)}{y(t)}.
\]
(13)
Let \( z(t) = \frac{y(t)}{y(qt)} \). Then we have

\[
\frac{y\left(\frac{t}{q^m}\right)}{y(t)} = z\left(\frac{t}{q^m}\right) z\left(\frac{t}{q^{m-1}}\right) \cdots z\left(\frac{t}{q}\right).
\]

So (13) becomes

\[
\frac{1}{z(t)} = 1 - k(t) z\left(\frac{t}{q^m}\right) z\left(\frac{t}{q^{m-1}}\right) \cdots z\left(\frac{t}{q}\right) \quad (14)
\]

To complete the proof, it is enough to show that equation (14) has a positive solution. Now, we define

\[
z\left(\frac{1}{q^m}\right) = z\left(\frac{1}{q^{m-1}}\right) = \cdots = z\left(\frac{1}{q}\right) = b = \frac{mq}{m} (= q) > 1, \quad (15)
\]

and from (13), we get

\[
z(1) = \left[1 - k(1) z\left(\frac{1}{q^m}\right) z\left(\frac{1}{q^{m-1}}\right) \cdots z\left(\frac{1}{q}\right)\right]^{-1}, \quad (16)
\]

which implies \( z(1) > 1 \).

Now we claim that \( z(1) < b \). From (16), we write

\[
\frac{z(1)}{b} = \frac{1}{b} \left[1 - k(1) z\left(\frac{1}{q^m}\right) z\left(\frac{1}{q^{m-1}}\right) \cdots z\left(\frac{1}{q}\right)\right].
\]

Using the given assumption and (15), the above equation becomes

\[
\frac{z(1)}{b} \leq \frac{1}{q} \left[1 - \frac{mm}{(qm)qm}q^m\right]^{-1},
\]

which yields \( z(1) < b \). Thus \( 1 < z(1) < b \). Hence by induction, \( 1 < z(t) < b \), \( t = 1, 2, 3, \cdots \).

Also \( z(t) \) is a solution of (14) and \( y(qt) = \frac{y(t)}{z(t)} \).
Hence \( y(t) \) is a nonoscillatory solution of (12).

**Theorem 4.7** Every solution of \( y(qt) - y(t) + ky\left(\frac{t}{q^m}\right) = 0 \), where \( m \) is a positive integer and \( k \) is a non negative real number, is oscillatory if and only if \( k > \frac{m^m}{(qm)^{qm}} \).

**proof** The proof completes by using the result 2.3, 4.5 and the Theorem 4.6.

### 4 Conclusion

In our research work, we discussed about the oscillation theory of the \( q \)-difference equation \( k(t)y(qt) + k\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = r(t)y(t) \), where \( r(t) = k(t)+k\left(\frac{t}{q}\right) - q(t) \). Some theorems are also proved using the concept of oscillation theory.

### References


