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Oscillation Theory of q-Difference Equation

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Abstract

In this research article, the authors present the oscillation theory of the q-difference equation

$$k(t)y(qt) + k\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = r(t)y(t),$$

where $r(t) = k(t) + k\left(\frac{t}{q}\right) - q(t)$. In particular we prove that this q-difference equation is oscillatory or non-oscillatory for different conditions.

Key words: Oscillation, Non-oscillation, q-Difference equation.

AMS classification: 39A10, 39A21, 39A13, 26A33.

1 Introduction

The numerical and analytical solutions of q-difference operator has an important role in different fields such as science and engineering, whose solution has a better understanding of the physical features of the problem [6, 5]. The authors in [1] introduced a Δ_q operator and then derived many results using the generalized q-difference equation $\Delta_q^t v(k) = u(k),\ q \neq 1$ and for any real k.
The authors in [2, 3] developed the q-alpha multi-series formula for finite and higher order q-alpha formula for infinite series.

The branch of differential equation theory is widely used in oscillation theory [4, 9]. The existence or non-existence of oscillatory solutions to a given equation or system are contained in the basic problem of classical theory of oscillation [7, 8].Till recently no special importance was given to the study of oscillations using q-difference equation. Hence in this article, we are primarily interested in the oscillation theory of q-difference equation.

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2 Preliminaries

Here, we present some preliminaries which will be used for further discussion.

Definition 2.1 [1] Let $f(k)$ be the real valued funtion on $[0, \infty)$ and $q \neq 1$ be a fixed real number. Then the q-difference operator, denoted by Δ_q, on f(k) is defined as

$$\Delta_q f(k) = f(qk) - f(k), \;\; q \neq 1. \tag{1}$$

Definition 2.2 Let $1 \neq q$ and m be any positive integers and

$$y(qt) - y(t) + k(t) y\left(\frac{t}{q^m}\right) = 0, \tag{2}$$

where $k(t)$ is a sequence defined for $t \in Z^+$. Then $y(t)$ is called an oscillatory solution of (2) if $y(t)\, y(qt) \leq 0$. Otherwise it is called non oscillatory solution.

Result 2.3 If $\lim\limits_{t\to\infty} \inf p(t) = p > \frac{m^m}{(qm)^{qm}}$, then every solution of (2) oscillates.

2 $q-$self adjoint second order $q-$difference equation

Here, we developed the q-self adjoint second order q-difference equation

$$\Delta_q\left[k\left(\frac{t}{q}\right)\Delta_q y\left(\frac{t}{q}\right)\right] + q(t)y(t) = 0, \tag{3}$$

where $k(t) > 0$, $t \in Z^+$. By Definition 2.1, the above equation becomes

$$k(t)y(qt) + k\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = \left[k(t) + k\left(\frac{t}{q}\right) - q(t)\right] y(t),$$

which implies

$$k(t)y(qt) + k\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = r(t)y(t), \tag{4}$$

$$\text{where} \quad r(t) = k(t) + k\left(\frac{t}{q}\right) - q(t). \tag{5}$$

Hence any equation of the form

$$k_0 y(qt) + k_1(t)y(t) + k_2(t)y\left(\frac{t}{q}\right) = 0, \tag{6}$$

with $k_0(t) > 0$ and $k_2(t) > 0$, can be put in the q-self adjoint form (3) or (4). Now multiplying both sides of (6) by a positive sequence $h(t)$ yields

$$k_0(t)h(t)y(qt) + k_1(t)h(t)y(t) + k_2(t)h(t)y\left(\frac{t}{q}\right) = 0. \tag{7}$$

Comparing (4) and (7), we get $k(t) = k_0(t)h(t), k\left(\frac{t}{q}\right) = k_2(t)h(t)$

$$\text{and} \quad r(t) = -k_1(t)h(t). \tag{8}$$

Then $k_2(qt)h(qt) = k_0(t)h(t)$, which gives

$$h(t) = \frac{k_0\left(\frac{t}{q}\right) h\left(\frac{t}{q}\right)}{k_2(t)}.$$

Putting the value of $h(t/q)$ by replacing t by t/q in the above equation repeatedly, we obtain

$$h(t) = \left(\frac{k_0\left(\frac{t}{q}\right)}{k_2(t)}\right)\left(\frac{k_0\left(\frac{t}{q^2}\right)}{k_2\left(\frac{t}{q}\right)}\right)\left(\frac{k_0\left(\frac{t}{q^3}\right)}{k_2\left(\frac{t}{q^2}\right)}\right)\cdots\left(\frac{k_0\left(\frac{t}{q^{s+1}}\right)}{k_2\left(\frac{t}{q^s}\right)}\right) h\left(\frac{t}{q^{s+1}}\right),$$

and hence

$$h(t) = \prod_{j=0}^{s} \frac{k_0\left(\frac{t}{q^{j+1}}\right)}{k_2\left(\frac{t}{q^j}\right)} h\left(\frac{t}{q^{s+1}}\right)$$

is a solution of (6).

Using the above equation, we get $k(t) = k_0(t) \prod_{j=0}^{s} \frac{k_0\left(\frac{t}{q^{j+1}}\right)}{k_2\left(\frac{t}{q^j}\right)} h\left(\frac{t}{q^{s+1}}\right)$.

Also from (5) and (8), we get $q(t) = k_1(t)h(t) + k(t) + k\left(\frac{t}{q}\right)$.

Result 3.1 Consider $z(t) = \dfrac{r(qt)y(qt)}{k(t)y(t)}$. Then (4) gives

$$\frac{k^2(t)}{r(t)r(qt)}z(t) + \frac{1}{z\left(\frac{t}{q}\right)} = 1,$$

which implies

$$c(t)z(t) + \frac{1}{z(\frac{t}{q})} = 1, \tag{9}$$

$$\text{where} \qquad c(t) = \frac{k^2(t)}{r(t)r(qt)}. \tag{10}$$

Result 3.2 If $c(t) \geq b(t) > 0$ for all $t > 0$ and $z(t) > 0$ is a solution of $c(t)z(t) + \dfrac{1}{z(\frac{t}{q})} = 1$, then the q-difference equation $b(t)x(t) + \dfrac{1}{x(\frac{t}{q})} = 1$, has a solution $x(t) \geq z(t) > 1$ for all $t \in \mathbb{Z}^+$.

3 Main Results

This section deals with the oscillatory and nonoscillatory solution of the q-difference equation $k(t)y(qt) + k\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = r(t)y(t)$, where $r(t) = k(t) + k\left(\frac{t}{q}\right) - q(t)$ based on the given conditions.

Lemma 4.1 If there exists a subsequence $r(t_m) \leq 0$ with $t_m \to \infty$ as $m \to \infty$, then every solution of (4) is oscillatory. proof On contrary, suppose there exists a non oscilltory solution $y(t) > 0$ for $t \geq N$ of (4). Then we obtain

$$k(t_m)y(qt_m) + k\left(\frac{t_m}{q}\right)y\left(\frac{t_m}{q}\right) - r(t_m)y(t_m) > 0 \quad for \quad t_m > N,$$

which is a contradiction and hence the proof.

Lemma 4.2 Suppose that $r(t) > 0$ for $t \in \mathbb{Z}^+$. Then every solution $y(t)$ of (4) is non oscillatory if and only if every solution $z(t)$ of (9) is positive for $t \geq N$, for some $N > 0$. proof Suppose that (4) has a non oscillatory solution $y(t)$, which yields $y(t)y(qt) > 0$ for $t \geq N$.
Also from (4), $z(t) > 0$ for $t > N$. Conversely assuming $z(t)$ is a positive solution of

(9).
Then we construct inductively $y(t)$ as $y(N) = 1$ and $y(qt) = \left(\frac{k(t)}{r(qt)}\right) z(t)y(t)$ with $t \geq N$, which gives $y(qN) = \left(\frac{k(N)}{r(qN)}\right) z(N)y(N) > 0$. Similarly, we can prove $y(qN) > 0$ for $n \geq 2$ and so $y(t) > 0$ for $n \geq N$. Thus $y(t)$ is a non oscillatory solution of equation (4).

Theorem 4.3 If $r(t)r(qt) \leq (4-\epsilon)\, k^2(t)$ for some $\epsilon > 0$ and for all $t \geq N$, then every solution of (4) oscillates. proof If $r(t)r\left(\frac{t}{q}\right) \leq (4-\epsilon)\, k^2(t)$ for some $\epsilon \geq 4$, then $r(t)r\left(\frac{t}{q}\right) \leq 0$.
By lemma 4.1,

every solution of (4) is oscillatory. Hence we may assume that $0 < \epsilon < 4$.
Now let us assume the contrary. Then by lemma 4.2, (9) has a positive solution $z(t)$ for $t \geq N$.

Using the assumption in (10) yields $c(t) \geq \frac{1}{(4-\epsilon)}$. Again using result 3.2, $s(t), s(t) \geq z(t) > 1$ for all $t \geq N$, is a solution of

$$\frac{1}{4-\epsilon} s(t) + \frac{1}{s\left(\frac{t}{q}\right)} = 1 \tag{11}$$

Now, we define a positive sequence $y(t)$ inductively as follows:

$$y(N) = 1, \quad y(qt) = \frac{1}{\sqrt{4-\epsilon}} s(t)y(t), \quad t \geq N.$$

$$Then \quad s(t) = \sqrt{4-\epsilon}\frac{y(qt)}{y(t)}.$$

Substituting $s(t)$ in (11), we get

$$\frac{1}{\sqrt{4-\epsilon}} \frac{y(qt)}{y(t)} + \frac{1}{\sqrt{4-\epsilon}} \frac{y(\frac{t}{q})}{y(t)} = 1,$$

which yields $y(qt) - \sqrt{4-\epsilon}y(t) + y(\frac{t}{q}) = 0,\ t \geq N,$

whose characteristic roots are $\dfrac{\sqrt{4-\epsilon} \pm i\sqrt{\epsilon}}{2}$.
Hence the solutions are non oscillatory, which is a contradiction.

Theorem 4.4 Let $r(t)r(qt) \geq 4k^2(t)$ for $t \geq N$. Then every solution of (4) is nonoscillatory. proof From the given assumption, $\dfrac{1}{r(t)r(qt)} \leq 4k^2(t)$.
Hence (10) gives $c(t) \leq \frac{1}{4}$.
We shall construct indutively a solution $z(t)$ of (9) as follows: Put $z(t) = 2$ and $z(t) = \dfrac{1}{c(t)}\left[1 - \dfrac{1}{z(\frac{t}{q})}\right]$.
Now,

$$z(qN) = \frac{1}{c(qN)}\left[1 - \frac{1}{z(N)}\right] \geq 4(1 - \frac{1}{2}) = 2.$$

Similarly, we can prove that $z(t) \geq 2$ for $t \geq N$.
Hence by lemma 4.2, every solution of (4) is nonoscillatory.

Example 4.5 Let m be a positive integer. Then $\left(\dfrac{1}{q}\right)^{t-1}$, $t > 1$ is a nonoscillatory solution of $y(qt) - y(t) + \dfrac{m^m}{(qm)^{qm}} y\left(\frac{t}{q^m}\right) = 0$.

Theorem 4.6 If $k(t) \geq 0$ and $\sup k(t) < \dfrac{m^m}{(qm)^{qm}}$, $q > 1$, then (2) has a non oscillatory solution. proof We have

$$y(qt) - y(t) + k(t)y\left(\frac{t}{q^m}\right) = 0 \tag{12}$$

Dividing both sides by $y(t)$, we get

$$\frac{y(qt)}{y(t)} = 1 - \frac{k(t)y\left(\frac{t}{q^m}\right)}{y(t)}. \tag{13}$$

Let $z(t) = \dfrac{y(t)}{y(qt)}$. Then we have

$$\frac{y\left(\frac{t}{q^m}\right)}{y(t)} = z\left(\frac{t}{q^m}\right) z\left(\frac{t}{q^{m-1}}\right) \cdots z\left(\frac{t}{q}\right).$$

So (13) becomes

$$\frac{1}{z(t)} = 1 - k(t) z\left(\frac{t}{q^m}\right) z\left(\frac{t}{q^{m-1}}\right) \cdots z\left(\frac{t}{q}\right). \tag{14}$$

To complete the proof, it is enough to show that equation (14) has a positive solution. Now, we define

$$z\left(\frac{1}{q^m}\right) = z\left(\frac{1}{q^{m-1}}\right) = \cdots = z\left(\frac{1}{q}\right) = b = \frac{mq}{m}(= q) > 1, \tag{15}$$

and from (13), we get

$$z(1) = \left[1 - k(1) z\left(\frac{1}{q^m}\right) z\left(\frac{1}{q^{m-1}}\right) \cdots z\left(\frac{1}{q}\right)\right]^{-1}, \tag{16}$$

which implies $z(1) > 1$.
Now we claim that $z(1) < b$. From (16), we write

$$\frac{z(1)}{b} = \frac{1}{b}\left[\frac{1}{1 - k(1) z\left(\frac{1}{q^m}\right) z\left(\frac{1}{q^{m-1}}\right) \cdots z\left(\frac{1}{q}\right)}\right].$$

Using the given assumption and (15), the above equation becomes

$$\frac{z(1)}{b} \le \frac{1}{q}\left[1 - \frac{m^m}{(qm)^{qm}} q^m\right]^{-1},$$

which yields $z(1) < b$. Thus $1 < z(1) < b$. Hence by induction, $1 < z(t) < b$, $t = 1, 2, 3, \cdots$
Also $z(t)$ is a solution of (14) and $y(qt) = \dfrac{y(t)}{z(t)}$.

Hence $y(t)$ is a nonoscillatory solution of (12).

Theorem 4.7 Every solution of $y(qt) - y(t) + ky\left(\frac{t}{q^m}\right) = 0$, where m is a positive integer and k is a non negative real number, is oscillatory if and only if $k > \frac{m^m}{(qm)^{qm}}$.
proof The proof completes by using the result 2.3, 4.5 and the Theorem 4.6.

4 Conclusion

n our research work, we discussed about the oscillation theory of the q-difference equation $k(t)y(qt)+k\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = r(t)y(t)$, where $r(t) = k(t)+k\left(\frac{t}{q}\right) - q(t)$. Some theorems are also proved using the concept of oscillation theory.

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