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Equitable Coloring of Some Cycle Related Graphs

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Abstract

The graph is called equitably $k-colorable$ if the vertex set of the graph can be partitioned into k non empty independent sets $V_1, V_2, \cdots, V_k$ such that $||V_i - V_j|| \leq 1$, for every i and j. The smallest integer k for which the graph G is equitably $k-colorable$ is called the equitable chromatic number of graph G and is denoted by $\chi_e(G)$. If the connected graph G is neither a complete graph nor an odd cycle then the Equitable Coloring Conjecture (ECC) states that $\chi_c(G) \leq \Delta(G)$. In this work we investigate the equitable chromatic number of some cycle related graphs like middle graph, shadow graph and splitting graph of cycle.

Key words: Equitable coloring, Equitable chromatic number, Middle graph, Shadow graph, Splitting graph.

AMS Subject classification: 05C15, 05C76.

1. Introduction

We begin with finite, connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any undefined term in graph theory we refer to Bondy and Murty [6]. A *proper* $k-coloring$ of a graph G is a function $c: V(G) \rightarrow \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring. There are many variants of proper coloring like b-coloring, total coloring, dominator coloring, equitable coloring etc. The present work is intended to report some investigations on equitable coloring of graph.

The graph G is called *equitably* $k - colorable$ if the vertex set of G can be partitioned into k non empty independent sets $V_1, V_2, \cdots, V_k$ such that $||V_i| - |V_j|| \leq 1$, for every i and j. The smallest integer k for which G is equitably k- colorable is called the equitable chromatic number of G and is denoted by $\chi_e(G)$. If the connected

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graph G is neither a complete graph nor an odd cycle then the Equitable Coloring Conjecture (ECC) states that $\chi_e(G) \leq \Delta(G)$ [5, 3].

The notion of equitable coloring is first introduced by Meyer [5]. The equitable coloring of trees and bipartite graphs are studied by Chen et al [1] and Wu et al [4] respectively. The equitable coloring of graph products is discussed in [2]

Proposition 1.1 [8] If G and G^1 are simple graphs on the same set of vertices and $E(G) \subseteq E(G')$, then $\chi_e(G) \leq \chi_e(G')$.

Proposition 1.2 [7] If G contains a clique of order n then $\chi(G) \geq n$.

Proposition 1.3 [5] Since an equitable coloring is a proper coloring, $\chi_e(G) \geq \chi(G)$.

2. Main Results

Definition 2.1 The middle graph $M(G)$ of a graph G is the graph whose vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Theorem 2.2 $\chi_e(M(C_n)) = 3$, for all n.

Proof: Let $V(C_n) = v_1, v_2, \cdots, v_n$ and $E(C_n) = e_1, e_2, \cdots, e_n$ where $e_i = v_iv_{i+1}; 1 \leq i \leq n1$ and $e_n = v_nv_1$. By the definition of middle graph, $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = v_ie_i; 1 \leq i \leq n \cup e_iv_{i+1}; 1 \leq i \leq n1 \cup e_nv_1 \leq e_ie_{i+1}; 1 \leq i \leq n1 \cup e_ne_1$. Thus $|V(M(C_n))| = 2n$ and $|E(M(C_n))| = 3n$.

As $M(C_n)$ contains a clique of order 3, $\chi(M(C_n)) \geq 3$ according to Proposition 1.2 and so,

$$\chi_e(M(C_n)) \geq 3. \tag{1}$$

Case 1: when $n \equiv 0(mod3)$:
Consider the color function $c : V(M(C_n)) \to \mathcal{N}$ as
$c(v_3k) = c(e_{3k2}) = 1; c(v_{3k1}) = c(e_{3k}) = 3; c(v_{3k2}) = c(e_{3k1}) = 2$ where $k = 1, 2, \cdots, \frac{n}{3}$.
Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, ..., e_{n2}, v_3, v_6, v_9, ..., v_n\}$,
$V_2 = \{e_2, e_5, e_8, ..., e_{n1}, v_1, v_4, v_7, ..., v_{n2}\}$,
$V_3 = \{e_3, e_6, e_9, ..., e_n, v_2, v_5, v_8, ..., v_{n-1}\}$.

Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$. Also, $|V_1| = |V_2| = |V_3| = \frac{2n}{3}$.
Thus, $\chi_e(M(C_n)) \leq 3$.

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Case 2: when $n \equiv 1(mod3)$:
Consider the color function $c : V(M(C_n)) \to \mathcal{N}$ as
$c(v_n) = 1, c(e_n) = 2, c(v_1) = 3, c(v_{3k+1}) = 2; k = 1, 2, \cdots, \frac{n-4}{3}$.
$c(v_{3k}) = c(e_{3k2}) = 1; c(e_{3k1}) = 2; c(e_{3k}) = c(v_{3k1}) = 3; k = 1, 2, \cdots, \frac{n-1}{3}$.
Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, e_7, ..., e_n3, v_3, v_6, v_9, ..., v_{n1}, v_n\}$,

$V_2 = \{e_2, e_5, e_8, ..., e_{n2}, e_n, v_4, v_7, v_10, ..., v_{n3}\}$,

$V_3 = \{e_3, e_6, e_9, ..., e_{n1}, v_1, v_2, v_5, v_8, ..., v_{n2}\}$.

Now, $|V_1| = \frac{n-1}{3} + \frac{n-1}{3} + 1 = \frac{2n+1}{3}, |V_2| = \frac{n-1}{3} + 1 + \frac{n-4}{3} = \frac{2n-2}{3}$, and $|V_3| = \frac{n-1}{3} + 1 + \frac{n-1}{3} = \frac{2n+1}{3}$.
Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$.
Also, $|V_1| = |V_3|$ and $||V_1| - |V_2|| = ||V_2| - |V_3|| = |\frac{2n+1}{3}(\frac{2n2}{3})| = 1$. It holds the inequality $||V_i| - |V_j|| \leq 1$, for every i and j. Thus, $\chi_e(M(C_n)) \leq 3$.
Case 3: when $n \equiv 2(mod3)$:
Consider the color function $c : V(M(C_n)) \to \mathcal{N}$ as
$c(v_1) = 3; c(e_{3k2}) = 1; c(e_{3k1}) = 2; c(v_{3k1}) = 3$ where $k = 1, 2, ..., \frac{n+1}{3}$.
$c(v_{3k}) = 1; c(v_{3k+1}) = 2; c(e_{3k}) = 3$ where $k = 1, 2, ..., \frac{n2}{3}$.
Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, e_7, ..., e_{n1}, v_3, v_6, v_9, ..., v_{n2}\}$,

$V_2 = \{e_2, e_5, e_8, ..., e_n, v_4, v_7, v_10, ..., v_{n1}\}$,

$V_3 = \{e_3, e_6, e_9, ..., e_{n2}, v_1, v_2, v_5, v_8, ..., v_n\}$.

Now, $|V_1| = \frac{n+1}{3} + \frac{n-2}{3} + 1 = \frac{2n-1}{3}, |V_2| = \frac{n+1}{3} + \frac{n-2}{3} = \frac{2n-1}{3}$, and $|V_3| = \frac{n-2}{3} + 1 + \frac{n+1}{3} = \frac{2n+2}{3}$.
Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$.
Also, $|V_1| = |V_2|$ and $||V_1| - |V_3|| = ||V_2| - |V_3|| = |\frac{2n1}{3} - (\frac{2n+2}{3})| = 1$. It holds the inequality $||V_i||V_j|| \leq 1$, for every i and j. Thus, $\chi_e(M(C_n)) \leq 3$.
Thus,

$$\chi_e(M(C_n)) \leq 3, foralln. \tag{2}$$

Therefore from (1) and (2) we get, $\chi_e(M(C_n)) = 3$.

Definition 2.3 The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 2.4

$$\chi_e(D_2(C_n)) = \begin{cases} 2, & n \quad is \quad even \\ 3, & n \quad is \quad odd \end{cases}$$

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Proof: Let $V(C_n)) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n)) = \{e_1, e_2, \cdots, e_n\}$ where $e_i = v_i v_{i+1}$; $1 \leq i \leq n-1$ and $e_n = v_n v_1$. The shadow graph of $C_n, D_2(C_n)$, has two copies, say $C_n^{'}$ and $C_n^{''}$ with $V(D_2(C_n)) = \{v_i^i, v_i^{''}; 1 \leq i \leq n\}$. In $D_2(C_n), N(v_i^{'}) = N(v_i^{''})$ and v_i^i and $v_i^{''}$ are non adjacent vertices.

Case 1:

when n is even:

consider the color function $c : V(D_2(C_n)) \to \mathcal{N}$ as

$c(v_{2k-1}^{'}) = c(v_{2k-1}^{''}) = 1$ and $c(v_{2k}^{'}) = c(v_{2k}^{''}) = 2$

Now, partition the vertex set of $D_2(C_n)$ as

$V_1 = \{v_1^{'}, v_1^{''}, v_3^{'}, v_3^{''''}, \cdots, v_{n-1}^1, v_{n-1}^{''}\}$ and

$V_2 = \{v_2^{'}, v_2^{''}, v_4^{'}, v_4^{''}, \cdots, v_n^{'}, v_n^{''}\}$.

Clearly, V_1 and V_2 are independent sets of $D_2(C_n)$ and $||V_1| - |V_2|| = 1$. It holds the inequality $||V_i| - V_j|| \leq 1$ for every pair (i, j). Thus $\chi_e(D_2(C_n)) = 2$.

Case 2:

when n is odd:

As $D_2(C_n)$ contains an odd cycle, it is not a bipartite graph. So it is not possible to partition the vertex set of $D_2(C_n)$ into two independent sets. Therefore $\chi_e(D_2(C_m))\neg 2$. Thus we have the following three subcases:

Subcase 1:

$n \equiv 0(mod 3)$:

consider the color function $c : V(D_2(C_n)) \to \mathcal{N}$ as

$c(v_{3k-2}^{'}) = c(v_{3k-2}^{''}) = 1$

$c(v_{3k-1}^{'}) = c(v_{3k-1}^{''}) = 2$

$c(v_{3k}^{'}) = c(v_{3k}^{''}) = 3; k = 1, 2, \cdots, \frac{n}{3}$

Now, partition the vertex set $D_2(C_n)$ as

$V_1 = \{v_{3k-2}^{'}, v_{3k-2}^{''}\}$, $V_2 = \{v_{3k-1}^{'}, v_{3k-1}^{''}\}$ and $V_3 = \{v_{3k}^{'}, v_{3k}^{''}\}$; $k = 1, 2, \cdots, \frac{n}{3}$

Clearly, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$. Also,$|V_1| = |V_2| = |V_3| = \frac{2n}{3}$. Thus, $\chi_e(D_2(C_n)) = 2$.

Subcase 2:

$n \equiv 1(mod 3)$:

consider the color function $c : V(D_2(C_n)) \to \mathcal{N}$ as

$c(v_1^{'}) = c(v_{n-2}^{'}) = 1; c(v_{3k}^{'}) = c(v_{3k}^{''}) = 1; k = 1, 2, \cdots, \frac{n-4}{3}$.

$c(v_n^{'}) = 2, c(v_{3k-1}^{'}) = 2; k = 1, 2, \cdots, \frac{n-4}{3}$.

$c(v_n^{''}) = 2, c(v_{3k-1}^{''}) = 2; k = 1, 2, \cdots, \frac{n-1}{3}$.

$c(v_{n-1}^{'}) = 3, c(v_{3k+1}^{'}) = 3; k = 1, 2, \cdots, \frac{n-4}{3}$.

$c(v_{n-1}^{''}) = 3, c(v_{3k-2}^{''}) = 3; k = 1, 2, \cdots, \frac{n-1}{3}$.

Now, partition the vertex set of $D_2(C_n)$ as

$V_1 = \{v_1^{'}, v_{n-2}^{'}, v_{3k}^{'}, v_{3k}^{''}; k = 1, 2, \cdots, \frac{n-4}{3}\}$,

$V_2 = \{v_{n-1}^{'}, v_n^{''}, v_{3k-1}(k = 1, 2, \cdots, \frac{n-4}{3}), v_{3k-1}^{''}(k = 1, 2, \cdots, \frac{n-1}{3})\}$, and

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$V_3 = \{v^{'}_{n-1}, v^{''}_{n-1}, v_{3k+1}(k = 1, 2, \cdots, \frac{n-4}{3}), v^{''}_{3k-2}(k = 1, 2, \cdots, \frac{n-1}{3})\}$.
Clearly, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$
Also, $|V_1| = \frac{2n-2}{3}, |V_2| = \frac{2n+1}{3}$ and $|V_3| = \frac{2n+1}{3}$. Now, $|V_2| = |V_3|$ and $||V_1| - |V_2|| = ||V_1| - |V_3|| = |\frac{2n-2}{3} - (\frac{2n+1}{3})| = 1$. It holds the inequality $||V_i| - |V_j|| \leq 1$, for every pair (v_i, v_j). Thus $\chi_e(D_2(C_n)) = 3$.

Subcase 3:
$n \equiv 2(mod3)$:
consider the color function $c : V(D_2(C_n)) \rightarrow \mathcal{N}$ as
$c(v^{'}_{3k-2}) = 1; k = 1, 2, \cdots, \frac{n+1}{3}$.
$c(v^{''}_{3k+1}) = 1; k = 1, 2, \cdots, \frac{n-2}{3}$.
$c(v^{'}_{3k-1}) = c(v^{''}_{3k-1}) = 2; k = 1, 2, \cdots, \frac{n+1}{3}$.
$c(v^{''}_{1}) = 3; c(v^{'}_{3k}) = c(v^{''}_{3k}) = 3; k = 1, 2, \cdots, \frac{n-2}{3}$.
Now, partition the vertex set of $D_2(C_n)$ as
$V_1 = \{v^{'}_1, v^{'}_4, v^{'}_7, \cdots, v^{'}_{n1}, v^{''}_4, v^{''}_7, \cdots, v^{''}_{n1}\}$,
$V_2 = \{v^{'}_2, v^{'}_5, v^{'}_8, \cdots, v^{1}_{n}, v^{''}_2, v^{''}_5, \cdots, v^{''}_n\}$,
$V_3 = \{v^{'}_3, v^{'}_6, \cdots, v^{'}_{n2}, v^{''}_1, v^{''}_3, v^{''}_6, \cdots, v^{''}_{n2}\}$.
Now, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$
Also, $|V_1| = \frac{n+1}{3} + \frac{n-2}{3} = \frac{2n-1}{3}, |V_2| = \frac{2(n+1)}{3}$ and $|V_3| = \frac{2(n-2)}{3} + 1 = \frac{2n-1}{3}$.
Now, $|V_1| = |V_3|$ and $||V_2| - |V_3|| = ||V_2| - |V_1|| = |\frac{2n+2}{3} - (\frac{2n-1}{3})| = 1$. It holds the inequality $||V_i| - |V_j|| \leq 1$, for every pair (v_i, v_j). Thus $\chi_e(D_2(C_n)) = 3$.

Definition 2.5 The splitting graph $S^{'}(G)$ of a connected graph G is obtained by adding new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$ where $N(v)$ and $N(v^{'})$ are the neighborhood sets of v and $v^{'}$ respectively.

The following theorem can be proved by the arguments analogous to the previous Theorem 2.4

Theorem 2.6

$$\chi_e(S^{'}(C_n)) = \begin{cases} 2, & n \quad is \quad even \\ 3, & n \quad is \quad odd \end{cases} \tag{3}$$

Proof: We have $V(S^{'}(C_n)) = V(D_2(C_n))$ and $E(S^{'}(C_n)) \subset E(D_2(C_n))$. Then by proposition 1.1, $\chi_e(S^{'}(C_n)) \leq \chi_e(D_2(C_n))$. Now, we can assign the same coloring as in $D_2(C_n)$ and hence, $\chi_e(S^{'}(C_n)) = \chi_e(D_2(C_n))$.

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3. Conclusion

The equitable chromatic number of some standarad graphs like cycle, path, wheel, etc., are known. We have investigated the equitable chromatic number for the larger graphs such as $M(C_n), D_2(C_n)$ and $S^{'}(C_n)$ obtained from cycle C_n.

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