A Study on Compactness in Intuitionistic Fuzzy Topological Spaces

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Abstract

The purpose of this paper is to introduce and study the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness. Here we see that our notions satisfy hereditary and productive property.

Key words: Fuzzy set, Compactness, Intuitionistic Fuzzy, Topological Spaces

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1 Introduction

In this paper, we study the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness. Finally we observe that our notions preserve under one-one, onto and continuous mapping.

2 Compactness in intuitionistic fuzzy topological space

Definition 2.1 Let \((X, \tau)\) be an intuitionistic fuzzy topological space. A family \(\{(\mu_{G_j}, \nu_{G_j}) : j \in J\}\) of IFOS in \(X\) is called open cover of \(X\) if \(\bigcup \mu_{G_j} = 1\) and \(\bigcap \nu_{G_j} = 0\). If every open cover of \(X\) has a finite subcover then \(X\) is said to be intuitionistic fuzzy compact (IF-compact, in short).

Definition 2.2 A family \(\{(\mu_{G_j}, \nu_{G_j}) : j \in J\}\) of IFOS in \(X\) is called \((\alpha, \beta)\)-level open cover of \(X\) if \(\bigcup \mu_{G_j} \geq \alpha\) and \(\bigcap \nu_{G_j} \leq \beta\) with \(\alpha + \beta \leq 1\). If every \((\alpha, \beta)\)-level open cover of \(X\) has a finite subcover then \(X\) is said to be \((\alpha, \beta)\) -level IF-compact.

Theorem 2.3 Let \((X, T)\) be a topological space and \((X, \tau)\) be its corresponding IFTS, where \(\tau = \{(1_{A_j}, 1_{A_j}^c) : j \in J : A_j \in T\}\). Then \((X, T)\) is compact if \((X, \tau)\) is
IF-compact.

Proof Let \((X, T)\) be compact. Consider \(\{G_i|i \in J\}\) be the open cover of \(X\),

\[
i.e. \bigcup G_i = X \quad (1)
\]

Since \(X\) is compact then \(\exists G_{i_1}, G_{i_2}, \ldots, G_{i_n} \in T\) such that

\[
G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n} = X \quad (2)
\]

Now it is clear that \((1_{G_{i_1}}, 1_{(G_{i_1})^c}) \in T\) (by the definition).

Also we have,

\[
\bigcup (1_{G_i}, 1_{(G_i)^c}) = (\bigcup 1_{G_i}, \bigcap 1_{(G_i)^c})
= (1_{\bigcup G_i}, 1_{\bigcap (G_i)^c})
= (1_X, 1_{\bigcap (G_i)^c})
\]

But we have, \(1_X + 1_{\bigcap (G_i)^c} \leq 1\) then it must be \(1_{\bigcap (G_i)^c} = 0\). Therefore we get,

\[
\bigcup (1_{G_i}, 1_{(G_i)^c}) = (1_X, 0).
\]

Also by (ii) we get,

\[
(1_X, 0) = (1_{G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}}, 0)
= (\bigcup_{j=1}^n 1_{G_{i_j}}, 0)
= \bigcup (1_{G_{i_j}}, 0)
\]

Hence it is clear that the IFTS \((X, \tau)\) is IF-compact.

**Corollary 2.4** Let \((X, T)\) be a topological space and \((X, \tau)\) be its corresponding IFTS, where \(\tau = \{(1_{A_j}, 1_{(A_j)^c}), j \in J : A_j \in T\}\) Then \((X, T)\) is compact if \((X, \tau)\) is \((\alpha, \beta)\)-level IF-compact.

Proof: Here it is clear that for any \(\alpha, \beta \in I\) with \(\alpha + \beta \leq 1 \Rightarrow 1 \geq \alpha\) and \(\beta \geq 0\). So, \((X, \tau)\) is \((\alpha, \beta)\)-level IF-compact.

**Theorem 2.5** Let \((X, \mathcal{J})\) be an intuitionistic topological space and \((X, \tau)\) be its corresponding IFTS, where \(\tau = \{1_{A_j} = (1_{A_{j_1}}, 1_{A_{j_2}}), j \in J : A_j = (A_{j_1}, A_{j_2}) \in \mathcal{J}\}\). Then \((X, \mathcal{J})\) is intuitionistic compact if \((X, \tau)\) is IF-compact.

Proof Let \((X, \mathcal{J})\) be an intuitionistic compact space, we shall prove that \((X, \tau)\) is
IF-compact. Consider \{1_{Ak}\} be an open cover of \(\tau\), i.e. \(\cup 1_{Ak} = (1, 0)\), where (1,0) is intuitionistic fuzzy set. Now

\[
1_{Ak} = (1_{Ak_1}, 1_{Ak_2}) \Rightarrow \cup 1_{Ak} = \cup (1_{Ak_1}, 1_{Ak_2})
\]

\[
\Rightarrow \cup 1_{Ak} = (\cup 1_{Ak_1}, \cap 1_{Ak_2})
\]

\[
\Rightarrow 1_{\cup Ak} = (1_{\cup Ak_1}, 1_{\cap Ak_2})
\]

\[
\Rightarrow 1_{\cup Ak} = (1_X, 0)
\]

\[
\Rightarrow 1_{\cup Ak} = (1, 0)
\]

By the given definition \(\{A_k \in \mathcal{J}\}, k \in \land\) is the open cover of \(X\), since \(\cup A_k = (X, \phi)\). But we have \((X, \mathcal{J})\) is compact then \(\exists A_{k_1}, A_{k_2}, ..., A_{k_n} \in \mathcal{J}\) such that

\[
\cup_{j=1}^n A_{kj} = (X, \phi)
\]

\[
\Rightarrow \cup_{j=1}^n (A_{kij_1}, A_{kij_2}) = (X, \phi)
\]

\[
\Rightarrow \cup_{j=1}^n A_{kij_1}, \cap_{j=1}^n A_{kij_2} = (X, \phi)
\]

\[
\Rightarrow (\cup_{j=1}^n A_{kij_1}, 1_{\cap_{j=1}^n A_{kij_2}}) = (1_X, 1_\phi)
\]

\[
\Rightarrow (\cup_{j=1}^n A_{kij_1}, 1_{\cap_{j=1}^n A_{kij_2}}) = (1, 0)
\]

Hence \((X, \tau)\) is IF-compact.

**Theorem 2.6** Let \((X, \tau)\) and \((Y, \delta)\) be IFTSs and \(f : X \to Y\) is bijective, open and continuous. Then \((Y, \delta)\) is IF-compact \(\Rightarrow (X, \tau)\) is IF-compact.

Proof Let \(A_i = (\mu_i, \nu_i) \in \tau\) with \(\cup A_i = (1, 0)\). Now \(A_i \in \tau \Rightarrow f(A_i) \in \delta\) with \(\cup f(A_i) = (1, 0)\). For \(y \in Y\), \(f(A_i)(y) = (y, f(\mu_i))(y), f(\nu_i)(y))\), where \(f(\mu_i)(y) = \sup_{x \in f^{-1}(y)} \mu_i(x) = \mu_i(x)\). Similarly we get, \(f(\nu_i)(y) = \nu_i(x)\). Now \(\cup f(A_i) = \cup f(\mu_i), f(\nu_i) = (\cup f(\mu_i), \cap f(\nu_i))\), i.e. \(\cup f(\mu_i)(y) = \cup f(\nu_i)(y)\) is an open cover of \(Y\). Again \(Y\) is compact then there exist \(f(A_1), f(A_2), ..., f(A_n) \in \delta\) such that \(\cup_{j=1}^n f(A_{ji}) = (1, 0) \Rightarrow f(\cup_{j=1}^n A_{ji}) = (1, 0) \Rightarrow f^{-1}(f(\cup_{j=1}^n A_{ji})) = f^{-1}(1, 0) \Rightarrow f^{-1}(1, 0) \subseteq \cup_{j=1}^n A_{ji}\) (since from Chang \(\mu \subseteq f^{-1}(f(\mu))\)). Therefore \(\cup_{j=1}^n A_{ji} = (1, 0)\). Hence \((X, \tau)\) is IF-compact.

**Theorem 2.7** Let \((X, \tau)\) and \((A, \delta)\) be IFTSs and \(f : X \to Y\) is one-one, onto and continuous. Then \((X, \tau)\) is IF-compact \(\Rightarrow (A, \delta)\) is IF-compact.

Proof Let \(A_i = (\mu_i, \nu_i) \in \delta\) with \(\cup A_i = (1, 0)\). Since \(\delta\) is a topology so \(\cup A_i \in \delta \Rightarrow f^{-1}(\cup A_i) \in \tau\) with \(f^{-1}(\cup A_i) = (1, 0)\) (as \(f\) is continuous) \(\Rightarrow \cup f^{-1}(A_i) = (1, 0)\). But \(\cup f^{-1}(A_i) = \cup f^{-1}(\mu_i, \nu_i) = \cup (f^{-1}(\mu_i), f^{-1}(\nu_i)) \in \tau\) with
$\cup (f^{-1}(\mu_i), f^{-1}(\nu_i)) = (1, 0)$. Since $(X, \tau)$ is IF-compact then $\exists A_{i_1}, A_{i_2}, \ldots, A_{i_m} \in \delta$ where $(f^{-1}(\mu_{i_1}), f^{-1}(\nu_{i_1})), (f^{-1}(\mu_{i_2}), f^{-1}(\nu_{i_2})), \ldots, (f^{-1}(\mu_{i_m}), f^{-1}(\nu_{i_m})) \in \tau$ such that $(f^{-1}(\mu_{i_1}), f^{-1}(\nu_{i_1})) \cup (f^{-1}(\mu_{i_2}), f^{-1}(\nu_{i_2})) \cup \ldots \cup (f^{-1}(\mu_{i_m}), f^{-1}(\nu_{i_m})) = (1, 0)$

$\Rightarrow \cup_{j=1}^{m} (f^{-1}(\mu_{ij}), f^{-1}(\nu_{ij})) = (1, 0)$
$\Rightarrow (\cup_{j=1}^{m} f^{-1}(\mu_{ij}), \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = (1, 0)$
$\Rightarrow f(\cup_{j=1}^{m} f^{-1}(\mu_{ij}), \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = f(1, 0)$
$\Rightarrow (\cup_{j=1}^{m} f^{-1}(\mu_{ij}), \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = (1, 0)$,

since $f$ is one-one and onto, so $f(1, 0) = (1, 0)$. Therefore $(\cup_{j=1}^{m} \mu_{ij}, \cap_{j=1}^{m} \nu_{ij}) = (1, 0)$, i.e. $\cup_{j=1}^{m} (\mu_{ij}, \nu_{ij}) = (1, 0)$. Hence $(Y, \delta)$ is IF-compact.

**Theorem 2.8** Let $(X, \tau)$ be an IFTS and $(V, \tau_v)$ be a subspace of $(X, \tau)$ with $(X, \tau)$ is IF-compact. Let $f : (X, \tau) \rightarrow (V, \tau_v)$ be continuous, open and onto, then $(V, \tau_v)$ is IF compact.

**Proof** Let $\mathcal{M} = \{B_i : i \in J\}$ be an open cover of $(V, \tau_v)$ with $\cup B_i = (1_v, 0)$. By the definition of subspace topology, let $B_i = U_i|V$, where $U_i \in \tau$. Since $f$ is continuous then $f^{-1}(B_i) \in \tau$ implies that $f^{-1}(U_i|V) \in \tau$.

As, $(X, \tau)$ is IF-compact then $\cup_{i \in J} f^{-1}(U_i|V)(x) = (1_X, 0)$. Thus we see that, $\{f^{-1}(U_i|V) : i \in J\}$ is an open cover of $(X, \tau)$. Hence there exist

$$f^{-1}(U_{i_1}|V), f^{-1}(U_{i_2}|V), \ldots, f^{-1}(U_{i_n}|V) \in f^{-1}(U_i|V)$$

such that

$$\cap_{k=1}^{n} f^{-1}(U_{ik}|V) = (1_X, 0).$$

Put $B_{ik} = U_{ik}|V$, then it is clear that $B_{ik} \in \tau_v$ with $\cap_{k=1}^{n} f^{-1}(B_{ik}) = (1_X, 0)$

$\Rightarrow f(\cup_{k=1}^{n} f^{-1}(B_{ik})) = f(1_X, 0)$
$\Rightarrow \cup_{k=1}^{n} f(f^{-1}(B_{ik})) = (f(1_X), 0)$

$\Rightarrow \cup_{k=1}^{n} (B_{ik}) = (1_V, 0)$ as $f$ is open. Hence $(V, \tau_v)$ is IF-compact.

**Theorem 2.9** Show that the following statements are equivalent:

(i) $X$ is IF-compact,

(ii) For every $F_i$ where $F_i = (\nu_{F_i}, \mu_{F_i})$ of closed subset of $X$ with $\cap F_i = (0, 1)$ implies $\{F_i\}$ contains finite subclass $\{F_{i_1}, F_{i_2}, \ldots, F_{i_m}\}$ with $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m} = (0, 1)$.
Proof \((i) \Rightarrow (ii)\). Suppose \(\cap F_i = (0, 1)\) then by De Morgan’s law
\[
(\cap F_i)^c = (0, 1)^c
\]
\[\Rightarrow \bigcup F_i^c = (1, 0)\]
\[\Rightarrow \bigcup(\nu_{F_i}, \mu_{F_i})^c = (1, 0)\]
\[\Rightarrow \bigcup(\mu_{F_i}, \nu_{F_i}) = (1, 0)\]
\[\Rightarrow (\cup \mu_{F_i}, \cap \nu_{F_i}) = (1, 0)\].

So, \(\{F_i^c\}, (F_i^c = (\mu_{F_i}, \nu_{F_i}))\) is an open cover of \(X\). Since \(X\) is IF-compact hence \(\exists F_{i1}^c, F_{i2}^c, \ldots, F_{im}^c \in \{F_i^c\}\) such that \(F_{i1}^c \cup F_{i2}^c \cup \ldots \cup F_{im}^c = (1, 0)\). Then
\[
(0, 1) = (1, 0)^c = (F_{i1}^c \cup F_{i2}^c \cup \ldots \cup F_{im}^c)^c
\]
\[= (F_{i1}^c)^c \cap (F_{i2}^c)^c \cap \ldots \cap (F_{im}^c)^c \text{ By De Morgan's law} \]
\[= F_{i1}^c \cap F_{i2}^c \cap \ldots \cap F_{im}^c, \]
so we have shown that \((i) \Rightarrow (ii)\).

\((ii) \Rightarrow (i)\). Let \(\{G_i\}\) be an open cover of \(X\) where \(G_i = (\mu_{A_i}, \nu_{G_i})\), i.e. \(\cup_i G_i = (1, 0)\). By De Morgan’s law,
\[
(0, 1) = (1, 0)^c = (\cup_i G_i)^c = \cap_i G_i.
\]

Since each \(G_i\) is open, so \(\{G_i\}\) is a class of closed sets and by \((ii) \exists G_{i1}^c, G_{i2}^c, \ldots, G_{im}^c \in \{G_i^c\}\) such that
\[
G_{i1}^c \cap G_{i2}^c \cap \ldots \cap G_{im}^c = (0, 1).
\]

So by De Morgan’s law
\[
(1, 0) = (0, 1)^c = (G_{i1}^c \cap G_{i2}^c \cap \ldots \cap G_{im}^c) = G_{i1} \cup G_{i2} \cup \ldots \cup G_{im},
\]
hence \(X\) is IF-compact. So, we have shown that \((ii) \Rightarrow (i)\).

**Theorem 2.10** Let the IFTS’s \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be IF-compact. Then the product IFT \(\tau_1 \times \tau_2\) on \(X_1 \times X_2\) is IF-compact.

Proof Consider, \((X_1, \tau_1)\) and \((X_2, \tau_2)\) is IF-compact. Let \(A_i = (\mu_{A_i}, \nu_{A_i}) \in \tau_1\) with \(\cup A_i = (1, 0)\) and \(B_i = (\mu_{B_i}, \nu_{B_i}) \in \tau_2\) with \(\cup B_i = (1, 0)\). Now
\[
A_i \times B_i = (\mu_{A_i}, \nu_{A_i}) \times (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_i} \times \mu_{B_i}, \nu_{A_i} \times \nu_{B_i})
\]
where
\[
(\mu_{A_i} \times \mu_{B_i})(x, y) = \min(\mu_{A_i}(x), \mu_{B_i}(y)), \text{ where } x \in X_1, y \in X_2 \\
= \min(1, 1) \\
= 1.
\]
Similarly,
\[
(\nu_{A_i} \times \nu_{B_i})(x, y) = \max(\nu_{A_i}(x), \nu_{B_i}(y)), \\
= \max(0, 0) = 0.
\]
So, \(A_i \times B_i = (1, 0)\).

But by the definition of product topology, \(A_i \times B_i \in \tau_1 \times \tau_2\), i.e. \(\{A_i \times B_i\}\) is a family of intuitionistic fuzzy open set in \(X_1 \times X_2\).

Choose \(\cup (A_i \times B_i) = (1, 0)\). Since \((X_1, \tau_1)\) is IF-compact, then \(\{A_i\}\) has finite subclass \(\{A_{ij}\}\) such that \(\cup_{j=1}^{n} A_{ij} = (1, 0)\). Similarly, since \((X_2, \tau_2)\) is IF-compact, then \(\{B_i\}\) has finite subclass \(\{B_{ik}\}\) such that \(\cup_{k=1}^{m} B_{ik} = (1, 0)\). Therefore
\[
\cup_{j=1}^{n} A_{ij} \times \cup_{k=1}^{m} B_{ik} = (1, 0) \\
\Rightarrow \cup_{j=1}^{n} (\mu_{A_{ij}}, \nu_{A_{ij}}) \times \cup_{k=1}^{m} (\mu_{B_{ik}}, \nu_{B_{ik}}) = (1, 0) \\
\Rightarrow (\cup_{j=1}^{n} \mu_{A_{ij}}, \cap_{j=1}^{n} \nu_{A_{ij}}) \times (\cup_{k=1}^{m} \mu_{B_{ik}}, \cap_{k=1}^{m} \nu_{B_{ik}}) \\
= (1, 0).
\]

Hence there exist four cases:

**Case-I:** If \(\cup_{j=1}^{n} \mu_{A_{ij}} = 1, \cup_{k=1}^{m} \mu_{B_{ik}} = 1\)

**Case-II:** If \(\cup_{j=1}^{n} \mu_{A_{ij}} = 1, \cap_{k=1}^{m} \mu_{B_{ik}} = 0\)

**Case-III:** If \(\cap_{j=1}^{n} \nu_{A_{ij}} = 0, \cup_{k=1}^{m} \mu_{B_{ik}} = 1\)

**Case-IV:** If \(\cap_{j=1}^{n} \nu_{A_{ij}} = 0, \cap_{k=1}^{m} \nu_{B_{ik}} = 0\)

Here from four cases, we see that the product topology \((X_1 \times X_2, \tau_1 \times \tau_2)\) is IF-compact.

### 3 Conclusion
In this paper, we discussed about the compactness in intuitionistic fuzzy topological spaces. Here we define two new notions of intuitionistic fuzzy compactness in intuitionistic fuzzy topological space and find their relation. Also we find the relationship between intuitionistic general compactness and intuitionistic fuzzy compactness.

### References


