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Stability of Generalized Cubic Functional Equation in Matrix Normed Spaces

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Abstract

In this current work, we consider the following generalized cubic functional equation

$$f(x+ty)+f(x-ty)=2\left(2cos\left(\frac{t\pi}{2}\right)+t^2-1\right)f(x) - \frac{1}{2}\left(cos\left(\frac{t\pi}{2}\right)+t^2-1\right)f(2x)+t^2[f(x+y)+f(x-y)]$$

where t is an integer and $t \geq 2$. We prove the Hyers-Ulam stability of this cubic functional equation in matrix normed spaces by using the fixed point method.

Key words: Hyers-Ulam stability, fixed point, Generalized Cubic Functional Equation, Matrix Normed Spaces.

AMS Subject classification: 39B52, 39B82, 46L07, 47H10, 47L25.

1. Introduction

The first stability problem of functional equations created from a question of Ulam [22] relating to group homomorphisms and solved by Hyers [15]. The result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings. The result of Rassias has furnished a lot of influence during the past years in the development of the Hyers-Ulam cocepts. This new concept is called the Hyers-Ulam-Rassias stability. A generalization of Rassias's theorem was obtained by Gavruta [5] by replacing the difference cauchy equation by a general control function. In 2001, J.M. Rassias [21], introduced the cubic functional equation

$$f(x+2y)-3f(x-y)+3f(x)-f(x-y)=6f(y) \tag{1}$$

and established the solution of the Hyers-Ulam stability problem for these cubic mappings. It is easy to see that the function $h(x) = cx^3$ satisfies (1). Thus,

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every solution of the cubic functional equation (1) is said to be cubic function. The alternative cubic functional equations

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{2}$$

$$f(x+2y)+f(x-2y)=2f(x)-f(2x)+4f(x+y)+4f(x-y) \tag{3}$$

has been introduced by Jun and Kim in [7].

In [11], A. Najati introduced a generalization of the cubic functional equation (2) as follows

$$f(tx+y)+f(tx-y)=tf(x+y)+tf(x-y)+2t(t^2-1)f(x) \tag{4}$$

where t is an integer and $t \geq 2$. In [18], Bodaghi et al. introduced a generalization of the cubic functional equation (3) as follows:

$$\begin{aligned} f(x+ty)+f(x-ty) &= 2\left(2cos(\frac{t\pi}{2})+t^2-1\right)f(x) \\ &\quad -\frac{1}{2}\left(cos(\frac{t\pi}{2})+t^2-1\right)f(2x)t^2[f(x+y)+f(x-y)] \end{aligned} \tag{5}$$

where t is an integer and $t \geq 2$. They determined the general solution and proved the Hyers-Ulam stability problem for the equation (5).

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of matricially normed spaces [12] implies that quotients, mapping spaces and various tensor products of operator spaces may be treated as operator spaces. Owing this result, the theory of operator spaces is having a increasingly significant effect on operator algebra theory.

The proof given in [12] appealed to the theory of ordered operator spaces [3]. Effros and Ruan [14] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier and Haagerup (as modified in [4]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero ;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero. For $x \in M_n(X), y \in M_k(X)$.

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$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|.\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\| \|B\| \|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$ and $x = (x_{ij}) \in M_n(X)$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|.\|_n\})$ is a matrix normed space. A matrix normed space $(X, \{\|.\|_n\})$ is called an L^∞ -matrix normed space if $\|x \oplus y\|_{n+k} = max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by, $h_n([x_{ij}]) = [h(x_{ij})]$ for all $[x_{ij}] \in M_n(E)$.

In 1996, G. Isac and Th.M. Rassias [6] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using the fixed point method, the stability of several functional equations and inequalities in matrix normed spaces have been extensively investigated by number of authors [20, 19, 8, 16, 10, 13].

Throughout this paper, let $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed positive integer.

Here our purpose is to investigate the Hyers-Ulam stability for the equation (5) in matrix normed spaces by using the fixed point method.}

2. Stability of Generalized Cubic Functional Equation in Matrix Normed Spaces

In this section, we prove the Hyers-Ulam stability of the generalized cubic functional equation (5) in matrix normed spaces by using the fixed point method.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by,

$$\begin{aligned} Df(u+v) =& f(u+tv) + f(u-tv) - 2\left(2cos(\frac{t\pi}{2}) + t^2 - 1\right) f(u) \\ &+ \frac{1}{2}\left(cos(\frac{t\pi}{2}) + t^2 - 1\right) f(2u) - t^2[f(u+v) + f(u-v)] \end{aligned}$$

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$$Df_n([x_{ij}],[y_{ij}]) = f_n([x_{ij}]+t[y_{ij}]) + f_n([x_{ij}]-t[y_{ij}]) - 2\left(2cos(\frac{t\pi}{2})+t^2-1\right) f_n([x_{ij}])$$
$$+\frac{1}{2}\left(cos(\frac{t\pi}{2})+t^2-1\right) f_n(2[x_{ij}]) - t^2[f_n([x_{ij}]+[y_{ij}]) + f_n([x_{ij}]-[y_{ij}])]$$

for all $u, v \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2.1 Let $p = \pm 1$ be fixed, and let $\phi : X^2 \to [0, \infty)$ be a function such that there exists a $\eta < 1$ with

$$\phi(u, v) \leq 2^{3p}\eta\phi(\frac{u}{2^p}, \frac{v}{2^p}) \tag{6}$$

for all $u, v \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|Df_n([x_{ij}], [y_{ij}])\| \leq \sum_{i,j=1}^{n} \phi(x_{ij}, y_{ij}) \tag{7}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f_n([x_{ij}]) - C_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\eta^{\left(\frac{1-p}{2}\right)}}{4\left(cos(\frac{t\pi}{2})+t^2-1\right)(1-\eta)}\phi(x_{ij}, 0) \tag{8}$$

$\forall\ x = [x_{ij}] \in M_n(X)$.
Proof: For the cases $p = 1$ and $p = -1$, we consider $\eta < 1$. Put $n = 1$ in (7), we obtain

$$\|Df(u, v)\| \leq \phi(u, v) \tag{9}$$

$\forall\ u, v \in X$. Letting $v = 0$ in (9), we get

$$\left\|f(2u) - 2^3 f(u)\right\| \leq \frac{1}{4\left(cos(\frac{t\pi}{2})+t^2-1\right)}\phi(u, 0) \qquad \forall\ u \in X. \tag{10}$$

So
$$\left\|f(u) - \frac{1}{2^3p} f(2^p u)\right\| \leq \frac{\eta^{\left(\frac{1-p}{2}\right)}}{4\left(cos(\frac{t\pi}{2})+t^2-1\right)}\phi(u, 0), \qquad \forall\ u \in X. \tag{11}$$

Consider the set $N = \{f : X \to Y\}$ and introduce the generalized metric on N

$$\rho(f, g) = \inf\left\{\mu \in \mathbb{R}_+ : \|f(u) - g(u)\| \leq \mu\phi(u, 0), \forall u \in X\right\},$$

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where, as usual, $\inf \phi = +\infty$. It is easy to show that (N, ρ) is complete (see the proof of [9], Lemma 2.1).
Now we consider the linear mapping $L : N \to N$ such that

$$Lf(u) = \frac{1}{2^{3p}} f(2^p u), \qquad \forall\ u \in X.$$

Let $f, g \in N$ be given such that $d(f, g) = \nu$. This means that $\|f(u) - g(u)\| \leq \phi(u, 0)$ $\forall$ $u \in X$. Hence $\|Lf(u) - Lg(u)\| = \left\| \frac{1}{2^3 p} f(2^p u) - \frac{1}{2^3 p} g(2^p u) \right\| \leq \eta \phi(u, 0)$ $\forall$ $u \in X$.
It follows from (11) that $\rho(f, Lf) \leq \dfrac{\eta^{\left(\frac{1-p}{2}\right)}}{4\left(cos(\frac{t\pi}{2}) + t^2 - 1\right)}$.
By Theorem 2.2 in [2], there exists a mapping $C : X \to Y$ satisfying the following:
(1) C is a fixed point of L, i.e., $C(2u) = 2^3 C(u) \quad \forall\ u \in X$.
(2) $\rho(L^k f, C) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{2^{3k}} f(2^k u) = C(u) \qquad \forall\ u \in X.$$

(3) $\rho(f, C) \leq \dfrac{1}{1-\eta} \rho(f, Lf)$, which implies the inequality

$$\|f(u) - C(u)\| \leq \frac{\eta^{\left(\frac{1-p}{2}\right)}}{4\left(cos(\frac{t\pi}{2}) + t^2 - 1\right)(1-\eta)} \phi(u, 0) \tag{12}$$

$\forall$ $u \in X$.By (9), $\lim_{k\to\infty} \frac{1}{2^{3k}} \|Df(u, v)\| \leq \lim_{k\to\infty} \frac{1}{2^{3K}} \phi(2^k u, 2^k v) \to 0$ $\forall$ $u, v \in X$.Thus

$$C(u+tv) + C(u-tv) = 2\left(2cos(\frac{t\pi}{2}) + t^2 - 1\right) C(u) - \frac{1}{2}\left(cos(\frac{t\pi}{2}) + t^2 - 1\right) C(2u) + t^2[C(u+v) + C(u-v)].$$

So, the mapping $C : X \to Y$ is cubic. By Lemma 2.1 in [16] and utilizing (12), we can obtain (8). Thus $C : X \to Y$is a unique cubic mapping satisfying (8).

Corollary 2.2 Let $p = \pm 1$ be fixed, and let q, σ be non-negative real numbers with

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$q \neq 3$. Let $f : X \to Y$ be a mapping such that

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \sigma(\|x_{ij}\|^q + \|y_{ij}\|^q) \tag{13}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f_n([x_{ij}]) - C_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\sigma}{\left(cos(\frac{t\pi}{2}) + t^2 - 1\right)|8 - 2^q|} \|x_{ij}\|^q \tag{14}$$

for all $x = [x_{ij}] \in M_n(X)$.
Proof: The proof follows from Theorem (2.1) by taking $\phi(u, v) = \sigma(\|u\|^q + \|v\|^q)$ for all $u, v \in X$. Then we can choose $\eta = 2^{p(q-3)}$.

Corollary 2.3 Let $p = \pm 1$ be fixed, and let q, σ be non-negative real numbers with $q = r + s \neq 3$. Let $f : X \to Y$ be a mapping such that

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \sigma(\|x_{ij}\|^r . \|y_{ij}\|^s + \|x_{ij}\|^q + \|y_{ij}\|^q) \tag{15}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that (14).
Proof: The proof follows from Theorem 2.1 by taking
$\phi(u, v) = \sigma(\|x_{ij}\|^r . \|y_{ij}\|^s + \|x_{ij}\|^q + \|y_{ij}\|^q)$ for all $u, v \in X$. Then we can choose $\eta = 2^{p(q-3)}$.

References

[1] Aoki T, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2, 1950, 64-66.

[2] Cadariu L and Radu V, Fixed points and the stability of Jensen's functional equation., J. Inequal. Pure Appl. Math., 4(1), 2003, 1-7.

[3] Choi MD, Effros E, Injectivity and operator spaces, J. Funct. Anal., 24, 1977, 156-209.

[4] Effros E, Ruan ZJ, On matricially normed spaces, Pac. J. Math., 132, 1988, 243-264.

[1]*shcrmurali@yahoo.co.in

[5] Gavruta P, A generalization of the Hyers-Ulam Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184, 1994, 431-436.

[6] Isac G and Th M Rassias, Stability of φ -additive mappings: Applications to nonlinear analysis, J. Funct. Anal., 19, 1996, 219-228.

[7] Jun KW and Kim HM, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, J Math Anal Appl., 274(2), 2002, 267-278.

[8] Lee J, Shin D and Park C, Hyers-Ulam Stability of Functional Equations in Matrix Normed Spaces, Journal of Inequalities and Applications, 22, 2013, 1-11.

[9] Mihet D and Radu V, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl., 343, 2008, 567-572.

[10] Murali R and Vithya V, Hyers-Ulam-Rassias stability of functional equations in matrix normed spaces: A fixed point approach, Assian Journal of Mathematics and Computer Research, 4(3), 2015, 155-163.

[11] Najati A, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, Turkish J. Math., 31, 2007, 395-408.

[12] Ruan ZJ, Subspaces of C*-algebras, J. Funct. Anal., 76, 1988, 217-230.

[13] Wang Z and Sahoo PK, Stability of an ACQ- Functional Equation in Various Matrix Normed Spaces, J. Nonlinear Sci. Appl., 8, 64-85, (2015).

[14] Effros E, Ruan ZJ, On the abstract characterization of operator spaces, Proc. Am. Math. Soc., 119, 579-584, (1993)

[15] Hyers DH, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA, 27, 222-224,(1941)

[16] Lee J, Park C and Shin D, Functional Equations in Matrix Normed Spaces, Proc. Indian Acad. sci., 125(3), 399-412, (2015).

[17] TH M Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc., 72, 297-300, (1978).

[18] Badaghi A, Moosavi SM and Rahimi H, The generalized cubic functional equation and stability of cubic jordan*-derivations, Ann Univ Ferrara, 5, 235-250, (2013).

[19] J. Lee, D. Shin and C. Park, An AQCQ- Functional Equation in Matrix Normed Spaces, Result. Math., 64, 146, 1-15, (2013).

1*shcrmurali@yahoo.co.in

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[20] Lee JR, Shin DY and Park C, An additive functional inequality in matrix normed spaces, Mathematical inequalities and applications, 16, no. 4 , 1009-1022, (2013).

[21] Rassias JM, Solution of the Ulam stability problem for cubic mappings, Glasnik Mathematicki Series III, 36(1), 63-72, (2001).

[22] Ulam SM, Problems in Modern Mathematics, Science Editions, Wiley, NewYork, (1964).

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