

Reductions and exact solutions of a cubic Schrödinger Partial Differential Equation

Phetogo Masemola T¹, Thilivhali Phidane AS²

Received: 10 September 2021/ Accepted: 02 January 2021/ Published online: 18 March 2022

©Sacred Heart Research Publications 2017

Abstract

Lie symmetry analysis is an established method for generating symmetries of differential equations. We apply this method together the generalized fundamental theorem of double reduction. In particular, Noether symmetries and some associated conservation laws are constructed in our investigation to find exact solutions of higher order partial differential equations and complex partial differential equations.

Key words: Wave equation, nonlinear Schrödinger equation, symmetries, multipliers, conservation laws, double reduction

AMS classification:35B06

1 Introduction

In this paper we use the method of double reduction to find the exact solutions of a Cubic nonlinear Schrödinger Partial Differential equation. Experiments on compressional dispersive Alven(CDA) waves have been done before on [10] and [11] where the relationship between CDA waves and the perturbations were analysed. Further in [11], the amplitude of the waves in a magnetic electron-positron plasma was discussed. In both experiments it was concluded that the system of equations under investigation can be written as the following equation:

$$u_{tt} - (3a^2 + c^2)u_{xx} - \delta^2 u_{xxx} - \delta^2 u_{xtt} = 0. \quad (1)$$

This equation was analysed in [13].

¹School of Mathematics, Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa.
Email:Phetogo.Masemola@wits.ac.za

²School of Mathematics, Faculty of Science, University of the Witwatersrand, Johannesburg, .
Email:Phidane@gmail.com

In order to get an envelope of CDA waves we need an interaction of a CDA pump and a quasi stationary compressional magnetic field. When the CDA envelope evolves we get a cubic nonlinear Schrödinger Partial Differential equation (NLSE) written as:

$$iq_t + \beta iq_x - \gamma q_{xx} + \delta q|q|^2 = 0, \quad (2)$$

where q is the complex valued dependent variable, and γ is the coefficient of the GVD, γ is the self-phase modulation (SPM) owing to /kerr law β is the inter-modal dispersion(IMD). We find that if we balance SPM and GVD, we get solitons.

To create a system of equation, we substitute $q = u + iv$ into (2) where $u = u(x, t)$, $v = v(x, t)$ and separate into real and imaginary parts to obtain:

$$\begin{aligned} G^1 &= u_t + \beta u_x - \gamma v_{xx} + \delta v(u^2 + v^2) = 0, \\ G^2 &= -v_t - \beta v_x - \gamma u_{xx} + \delta(u^2 + v^2) = 0. \end{aligned} \quad (3)$$

Conservation laws and conserved quantities of Cubic Schrödinger Partial Differential equation

To find multipliers and conservation laws, the condition below must hold.

$q^1(u_t + \beta u_x - \gamma v_{xx} + \delta v(u^2 + v^2)) + q^2(-v_t - \beta v_x - \gamma u_{xx} + \delta(u^2 + v^2)) = D_t T^t + D_x T^x$. where q^1, q^2 are multipliers of (2) and T^t, T^x conserved vectors of (2).

From total divergence we set

$\frac{\delta}{\delta u}[q^1(u_t + \beta u_x - \gamma v_{xx} + \delta v(u^2 + v^2)) + q^2(-v_t - \beta v_x - \gamma u_{xx} + \delta(u^2 + v^2))] = 0$ then use Maple and Mathematica to get the following results,

(i) $(q^1, q^2) = (v_x, u_x)$

$$\begin{aligned} T_1^x &= \frac{1}{4}(\delta u^4 + 2\delta u^2 v^2 + \delta v^4 + 2v u_t - 2u v_t - 2\gamma(u_x^2 + v_x^2)), \\ T_1^t &= \frac{1}{2}(-v u_x + u v_x). \end{aligned} \quad (4)$$

(ii) $(q^1, q^2) = (u, -v)$

$$\begin{aligned} T_2^x &= \frac{1}{2}(\beta u^2 + v(\beta v + 2\gamma u_x) - 2\gamma uv_x), \\ T_2^t &= \frac{1}{2}(u^2 + v^2). \end{aligned} \tag{5}$$

(iii) $(q^1, q^2) = (u_t, v_t)$

$$\begin{aligned} T_3^x &= \frac{1}{2}(-\gamma(u_t u_x + v_x v_t) + u(\beta v_t + \gamma u_{xt}) + v(-\beta u_t + \gamma v_{xt})), \\ T_3^t &= \frac{1}{4}(\delta u^4 + 2\delta u^2 v^2 - 2u(\beta v_x + \gamma u_{xx}) + v(\delta v^3 + 2\beta u_x - 2\gamma v_{xx})). \end{aligned} \tag{6}$$

(iv) $(q^1, q^2) = (-\beta u + xu + 2\gamma tv_x, \beta tv - xv + 2\gamma tu_x)$

$$\begin{aligned} T_4^x &= \frac{1}{2}[t\gamma\delta u^4 + \beta(x - t\beta)v^2 + t\gamma\delta v^4 + u^2(\beta(x - t\beta) + 2t\gamma\delta v^2) \\ &\quad 2\gamma v(tu_t + (x - t\beta)u_x) - 2\gamma u(tv_t + (x - t\beta)v_x) - 2t\gamma^2(u_x^2 + v_x^2)], \\ T_4^t &= \frac{1}{2}((x - t\beta)u^2 + v((x - t\beta)v - 2t\gamma u_x) + 2t\gamma uv_x). \end{aligned} \tag{7}$$

Symmetries of Cubic Schrödinger Partial Differential equation

A one-parameter Lie group of transformations that leave (4) invariant will be written as a vector field

$$X = \tau(x, t, u, v)\partial_t + \xi(x, t, u, v)\partial_x + \eta^1(x, t, u, v)\partial_u + \eta^2(x, t, u, v)\partial_v \tag{8}$$

Equation (4) has the following variational symmetries:

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_x, \\ X_3 &= u\partial_v - v\partial_u, \\ X_4 &= 2\gamma t\partial_x + (\beta t - x)u\partial_v + (-\beta t + x)v\partial_u, \\ X_5 &= 2t\partial_t + (\beta t + x)\partial_x - u\partial_u - v\partial_v. \end{aligned}$$

Double reduction of Cubic Schrödinger Partial Differential equation

In order to perform a double reduction, we need to show association between the symmetries and the conservation laws calculated in the previous sections.

To check if a Lie-Bäcklund symmetry X of differential equation is associated with conservation law $T = (T^1, T^2, \dots, T^n)$ the following equation must hold

$$T^{*i} = X(T^i) + T^i D_j \xi^j - T^j D_j \xi^i = 0 \quad (9)$$

for $i = 1, 2, \dots, n$.

Double reduction of Cubic Schrödinger Partial Differential equation by symmetries X_1 and

X_3

In this section we use symmetries $\langle X_1, X_3 \rangle$ and conserved vectors given in (5) to reduce the Cubic nonlinear Schrödinger Partial Differential equation.

Firstly we show that Lie-Bäcklund symmetries X_1 and X_3 are associated with conservation law $T = (T^x, T^t)$ given in (5) using the following version of (9) for $i = 1, 2$:

$$T^* = X \begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t \xi^t & D_x \xi^t \\ D_t \xi^x & D_x \xi^x \end{pmatrix} \begin{pmatrix} T^t \\ T^x \end{pmatrix} + (D_t \xi^t + D_x \xi^x) \begin{pmatrix} T^t \\ T^x \end{pmatrix}$$

We obtain

$$\begin{aligned}
 \begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} &= X_1^{[1]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} \\
 &= X_1^{[1]} \begin{pmatrix} \frac{1}{2}(u^2 + v^2) \\ \frac{1}{2}(\beta u^2 + v(\beta v + 2\gamma u_x) - 2\gamma uv_x) \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 \begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} &= X_3^{[1]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} \\
 &= X_3^{[1]} \begin{pmatrix} \frac{1}{2}(u^2 + v^2) \\ \frac{1}{2}(\beta u^2 + v(\beta v + 2\gamma u_x) - 2\gamma uv_x) \end{pmatrix} \\
 &= \begin{pmatrix} u\frac{1}{2}2v - v\frac{1}{2}2u \\ u\frac{1}{2}(2\beta v + 2\gamma u_x) - v\frac{1}{2}(2\beta u + 2\gamma v_x) + u_x\frac{1}{2}(-2\gamma u) - v_x(\frac{1}{2}2\gamma v) \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
 \end{aligned}$$

where

$$\begin{aligned} X_1^{[1]} &= \partial_t \\ X_3^{[1]} &= u\partial_v - v\partial_u + u_t\partial_{v_t} + u_x\partial_{v_x} - v_t\partial_{u_t} - v_x\partial_{u_x} \end{aligned}$$

Thus X_1 and X_3 are both associated with T_2 . We then consider a linear combination of X_1 and X_3 , of the form $X = X_1 + cX_3$ and transform this generator to its canonical form $Y = \frac{\partial}{\partial_s}$, where we assume that this generator is of the form

$$Y = 0\frac{\partial}{\partial_r} + \frac{\partial}{\partial_s} + 0\frac{\partial}{\partial_w} + 0\frac{\partial}{\partial_p}.$$

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$, and $X(p) = 0$, we obtain

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{-cv} = \frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0} = \frac{dp}{0} = \frac{dv}{cu}. \quad (10)$$

We then solve (10) using the method of invariants, and the results are summarized in the table below.

Invariants of $X = \frac{\partial}{\partial t} + c(u\partial_v - v\partial_u)$	
$\frac{dt}{1} = \frac{ds}{1}$	$b_1 = s - t$
$\frac{du}{-cv} = \frac{dv}{ku}$	$b_2 = u^2 + v^2$
$\frac{dv}{cu} = \frac{dt}{1}$	$b_3 = \arctan\left(\frac{v}{u}\right) - ct$
$\frac{dr}{0}$	$b_4 = r$
$\frac{dp}{0}$	$b_5 = p$
$\frac{dw}{0}$	$b_6 = w$
$\frac{dx}{0}$	$b_7 = x$

Table 1

By choosing $b_1 = 0$, $b_4 = b_7$, $b_6 = \sqrt{b_2}$, and $b_3 = b_5$ we get

$$\begin{aligned}
 r &= x \\
 s &= t \\
 w &= \sqrt{u^2 + v^2} \\
 p &= \arctan\left(\frac{v}{u}\right) - ct
 \end{aligned}$$

The matrices A and A^{-1} can be computed using the canonical coordinates above

$$A = \begin{pmatrix} D_s t & D_s x \\ D_r t & D_r x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (A^{-1})^T \quad (11)$$

and $J = \det(A) = 1$.

The inverse canonical coordinates are presented below

$$\begin{aligned} x &= r, \\ t &= s, \\ u &= w \cos(p + cs), \\ v &= w \sin(p + cs). \end{aligned}$$

The first and second partial derivatives of u and v in terms of new dependent variables w and p are,

$$\begin{aligned} u_x &= w_r \cos(p + cs) - wp_r \sin(p + cs), \\ u_{xx} &= w_{rr} \cos(p + cs) - 2w_r p_r \sin(p + cs) - wp_r^2 \cos(p + cs) - wp_{rr} \sin(p + cs), \\ u_t &= -cw \sin(p + cs), \\ v_x &= w_r \sin(p + cs) + wp_r \cos(p + cs), \\ v_{xx} &= w_{rr} \sin(p + cs) + 2w_r p_r \cos(p + cs) + wp_{rr} \cos(p + cs) - wp_r^2 \sin(p + cs), \\ v_t &= cw \cos(p + cs). \end{aligned}$$

The reduced conserved form will be

$$\begin{aligned} \begin{pmatrix} T_2^s \\ T_2^r \end{pmatrix} &= J(A^{-1})^T \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(u^2 + v^2) \\ \frac{1}{2}(\beta u^2 + v(\beta v + 2\gamma u_x) - 2\gamma uv_x) \end{pmatrix}. \end{aligned} \quad (12)$$

Substituting the first and second partial derivatives of u and v into (12) we get

$$\begin{pmatrix} T_2^s \\ T_2^r \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w(r)^2 \\ \frac{1}{2}(2\beta w(r)^2 + \gamma w(r)w(r)_r \sin(2(p + ks)) + 2\gamma w(r)^2 \cos(2(p + ks))) \end{pmatrix}$$

where the reduced conserved form is also given by

$$D_s T_2^s = 0. \quad (13)$$

The second step of double reduction can also be given as

$$w(r)^2 = \epsilon \quad (14)$$

ϵ is constant, or equivalently

$$w(r) = \sqrt{\epsilon} \quad (15)$$

Differentiating (15) implicitly with respect to r we get

$$\frac{d}{dr} w(r) = 0. \quad (16)$$

Our original system is equivalent to

$$sys_1 = \begin{cases} g_1^1 Q^1 + g_1^2 Q^2 = 0, \\ g_1^1 Q^1 - g_1^2 Q^2 = 0. \end{cases} \quad (17)$$

This system can be rewritten as

$$\begin{aligned} D_t T_2^t + D_x T_2^x &= 0, \\ G_1^1 Q^1 - G_1^2 Q^2 &= 0. \end{aligned} \quad (18)$$

The second equation of sys_1 from is given by

$$u(u_t + \beta u_x - \gamma v_{xx} + \delta v(u^2 + v^2)) - (-v)(-v_t - \beta v_x - \gamma u_{xx} + \delta(u^2 + v^2)) = 0 \quad (19)$$

Substituting (15),(16),(19) and the first and second partial derivatives of u and v in terms of $w(r)$ and $p(r)$ we get

$$\begin{aligned} -c\epsilon \sin(2p(r) + 2cs) - \beta\epsilon p'(r) \sin(2p(r) + 2cs) - \gamma\epsilon p''(r) \cos(2p(r) + 2cs) \\ + \gamma\epsilon p'(r)^2 \sin(2p(r) + 2cs) + \delta\epsilon^2 \sin(2p(r) + 2cs) = 0 \end{aligned} \quad (20)$$

When computing the final solution to equation (20) we get tedious solution. We then compute the solution of (20) for the following cases.

Case 1: $c = \gamma = 0$

$$p(r) = 0, \tag{21}$$

$$p(r) = -\frac{\pi}{2}, \tag{22}$$

$$p(r) = \frac{\pi}{2}, \tag{23}$$

$$p(r) = \frac{r\delta\epsilon}{\beta} + c_1. \tag{24}$$

From (21), $p(r) = 0$:

$$u = \sqrt{\epsilon},$$

$$v = 0,$$

$$q = \sqrt{\epsilon}.$$

From (22), $p(r) = -\frac{\pi}{2}$

$$u = 0,$$

$$v = -\sqrt{\epsilon},$$

$$q = -i\sqrt{\epsilon}.$$

From (23), $p(r) = \frac{\pi}{2}$

$$u = 0,$$

$$v = \sqrt{\epsilon},$$

$$q = i\sqrt{\epsilon}.$$

From (24), $p(r) = \frac{r\delta\epsilon}{\beta} + c_1$

$$u = \sqrt{\epsilon} \cos\left(\frac{x\delta\epsilon}{\beta} + c_1\right),$$
$$v = \sqrt{\epsilon} \sin\left(\frac{x\delta\epsilon}{\beta} + c_1\right).$$

From canonical coordinates

$$q = \sqrt{\epsilon} e^{i\left(\frac{x\delta\epsilon}{\beta} + c_1\right)} \quad (25)$$

Case 2 $c = \beta = 0$:

$$p(r) = 0, \quad (26)$$

$$p(r) = -\frac{\pi}{2}, \quad (27)$$

$$p(r) = \frac{\pi}{2}, \quad (28)$$

$$p(r) = c_1. \quad (29)$$

From (29) $p(r) = c_1$

$$u = \sqrt{\epsilon} \cos(c_1),$$

$$v = \sqrt{\epsilon} \sin(c_1),$$

$$q = \sqrt{\epsilon} e^{ic_1}.$$

Case 3 $\delta = \gamma = 0$:

$$p(r) = -cs, \quad (30)$$

$$p(r) = -\frac{\pi}{2} - cs, \quad (31)$$

$$p(r) = \frac{\pi}{2} - cs, \quad (32)$$

$$p(r) = -\frac{cr}{\beta} + c_1. \quad (33)$$

From (30) $p(r) = -\frac{cr}{\beta} + c_1$

$$u = \sqrt{\epsilon} \cos\left(\frac{c(\beta s - r)}{\beta} + c_1\right)$$

$$v = \sqrt{\epsilon} \sin\left(\frac{c(\beta s - r)}{\beta} + c_1\right)$$

From canonical coordinates

$$q = \sqrt{\epsilon} e^{i\left(\frac{c(\beta t - x)}{\beta} + c_1\right)} \quad (34)$$

3 Conclusion

In this paper we have calculated multipliers and conservation laws of the respective Cubic Schrödinger partial differential equation. Further, we showed that if a symmetry is associated with a conservation law we can apply the fundamental theorem of double reduction to obtain an ordinary differential equation. Consequently we used Maple and Mathematica to calculate exact solutions for special cases.

References

- [1] A. Doleman, B. Sandstede, A. Scheel and G. Schneider, "The Propagation of hexagonal patterns near onset", *Euro. Jnl of Applied Mathematics*, **14** (2003), 85110.
- [2] A. H. Kara and F. M. Mahomed, "A basis of conservation laws for partial differential equations", *J. Nonlinear Math Phys*, **9** (2002), 60-72.

- [3] A. H. Bokhari, A. Al-Dweik, F. D. Zaman, A. H. Kara and F. M. Mahomed, “Generalization of the double reduction theory, Nonlinear analysis: Real World Applications”, **11** (2010), 3763-3769.
- [4] J. B. van den Berg, A. Deschenes, J. P. Lessard and J. D. Mireles James, “Stationary Coexistence of Hexagons and Rolls vis Rigorous Computations”, *SIAM J. App. Dyn. Sys.*, **14**(2) (2015), 942-979.
- [5] J. B. van den Gerg and J. P. Lessard, “Rigorous Numerics in Dynamics”, G. W. Bluman and J. D. Cole, *Notices of the AMS*, **62**(2015), 1057-1061.
- [6] J. B. Swift and P. C. Hohenberg, “Hydrodynamic fluctuations at the convective instability”, *Phys. Rev. A*, **15** (1977), 319328.
- [7] M. Bahiana, M and Y. Oono, “Cell dynamical system approach to block copolymers”, *Physical Review A*, **41**(12) (1990), 6763.
- [8] Masemola,P.,Phidane,T. (2019). “Reductions and exact solutions of a cubic Schrodinger Partial Differential Equation”.Arxiv <https://arxiv.org/pdf/1912.05816.pdf>
- [9] N. Q. Le, “On the convergence of the Ohta-Kawasaki equation to motion by nonlocal Mullins-Sekerka law”, *SIAM Journal on Mathematical Analysis*, **42**(4) (2010), 1602-1638.
- [10] P. K. Shukla, B. Eliasson and L. Stenflo, “Dark and grey compressional dispersive Alven solutions in plasmas”, *Physics of Plasmas of Mathematical Mechanics*, **18**(6) (2011), 065411.
- [11] P. K. Shukla, B. Eliasson and L. Stenflo, “Electromagnetic solitary pulses in a magnetized electro-positron plasma”, *Physical Review E. Statistical, Nonlinear and Soft Matter Physics*, **84**(3) (2011), 037401.
- [12] R. Choksi and X. Ren, “On the derivation of a density functional theory for microphase separation of diblock copolymers”, *Journal of Statistical Physics*, **113**(1-2) (2003), 151-176.
- [13] R. Morris, P. Masemola, A. H.Kara and A. Biswas, “On symmetries, reductions, conservation laws and conserved quantities of optical solitons with inter-modal dispersion”, *Optik*, **124**(21) (2013), 5116-5123.

- [14] T. Ohta and K. Kawasaki, "Equilibrium morphology of block copolymer melts", *Macromolecules*, **19**(10) (1986), 2621-2632.
- [15] T. Phidane, A. H. Kara and G. Wang, "Analysis of fourth-order partial differential equations 'patterns' and 'diblock' polymers", Wiley Online Library, *Mathematical Methods in the Applied Sciences*, (2017).
- [16] W. Sarlet, "Comment on Conservation laws of higher order nonlinear partial differential equations and the variational conservation laws in the class with mixed derivatives", *J.Physics. A: Math. Theor.*, **43** (2010), 458001.