ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

Journal of Computational Mathematica

Open Access

Bounds on the Connected Domination in Graphs

Vinolin J[1], Ramesh DST[2*], Athisayanathan S[3] and Anto Kinsley A[4]
[1,3,4] Department of Mathematics, St. Xaviers College, Palayamkottai,
Manonmaniam Sundaranar University, Tirunelveli - 627012, Tamil Nadu, India.
2 Department of Mathematics, Margchosis College, Nazareth,
Manonmaniam Sundaranar University, Tirunelveli - 627012, Tamil Nadu, India.

Abstract

A set $S \subseteq V$ of a connected graph G is a hop dominating set of G if for every vertex $v \in VS$ there exists a vertex $u \in S$ such that $d(u,v) = 2$. The cardinality of a minimum hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$. A hop dominating set D of a graph G is said to be a connected hop dominating set of G if the induced subgraph $< D >$ is connected. The cardinality of a minimum connected hop dominating set is called the connected hop domination number of G and it is denoted by $\gamma_h^c(G)$. In this paper some graphs G are characterized for which $\gamma_h(G) = 2$. Bounds based on diameter, girth and maximum degree for $\gamma_h(G)$ are developed. In addition the hop domination number of wounded spider is computed. The hop dominating sets are compared to the distance-2 dominating sets. An important result is proved that if $G_1, G_2, \ldots, G_s$ are the connected proper subgraphs of G with minimum connected hop dominating sets $D_1, D_2, \ldots, D_s$ as then $\gamma_h^c(G) \leq \gamma_h^c(G_i) + 2s$.

Key words: Graphs, distance, domination number, distance domination number, Hop domination number, Connected Hop domination number.

1. Introduction

Graph theory has immense application in the field of communication networks, interpersonal relations and several real life situations. Domination in graphs is one of the fastest growing areas in Graph theory. It was studied from 1950s onwards, but the rate of the research on domination significantly increased in the mid - 1970s. Several domination parameters can be found in the book [5] written by Haynes et al. For basic graph theoretic terminology we refer to [3]. By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of the shortest $x - y$ path in G. For any vertex u of G, the eccentricity of u is $e(u) = \{d(u, v) : v \in V\}$. The radius $rad(G)$ and the diameter $diam(G)$ are defined by $rad(G) = min\{e(v) : v \in V\}$ and

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

$diam\,(G) = max\{e\,(v) : v \in V\}$. The girth of a graph G is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles, its girth is defined to be infinity. For example, a 4-cycle has girth 4. S.K. Ayyaswamy et al. [1] have recently defined a new domination parameter called hop domination number of a graph. The definition is as follows: A set $S \subseteq V$ of a graph G is a hop dominating set of G if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $d(u, v) = 2$. The cardinality of a minimum hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$. In [1] S.K. Ayyaswamy et al. characterized the family of trees and unicyclic graphs for which $\gamma_h\,(G) = \gamma_t\,(G)$ and $\gamma_h\,(G) = \gamma_c\,(G)$ where $\gamma_t\,(G)$ and $\gamma_c\,(G)$ are the total domination and connected domination numbers of G respectively. Then they presented the strong equality of hop domination and hop independent domination number of trees. But in this paper we present some bounds for hop domination number based on diameter, girth and maximum degree and characterised graphs for which $\gamma_h\,(G) = 2$.

Example 1.1

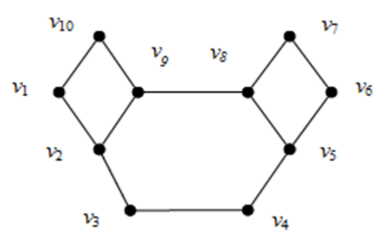

Figure 1: A Graph G

In the above graph G, the set $S = \{v_1, v_9, v_8\}$ is said to be the hop dominating set of G. Since each vertex in the set S hop dominates the vertices in $V - S$.Therefore $\gamma_h\,(G) = 3$.

In the above graph the usual minimum dominating set is $D = \{v_2, v_5, v_7, v_{10}\}$. But the minimum hop dominating set is $S = \{v_1, v_9, v_8\}$. Therefore, we can say that $\gamma_h\,(G) \leq \gamma\,(G)$ for some graphs G.

In a complete graph every vertex is at a distance 1 from every other vertex. Take $S = V$ and so $V - S = \Phi$. Hence, $\gamma_h\,(K_n) = n$ [1].

Consider the star $K_{1,n\ 1}$, where u is the unique central vertex of G, and v any end vertex of G. Then $S = \{u, v\}$ is the hop dominating set of G and it is the minimum.

2*dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

Therefore, $\gamma_h(K_{1,n\ 1}) = 2$.

2. Characterization of graphs for which $\gamma_h(G) = 2$

Construction of a graph G obtained from C_4 by adding pendent vertices to at most two of the vertices of C_4.

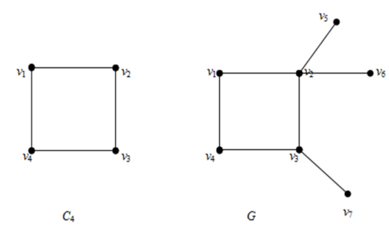

Figure 2: C_4 and the constructed Graph

Theorem 2.1 If G is a connected graph of order n containing a unique cycle C_n, then $\gamma_h(G) = 2$ if and only if G is obtained from $C_n\,(n = 4, 5, 6)$ by adding zero or more pendent vertices to at most two of the vertices of C_n.

Proof: If G is obtained from C_n, where $n = 4, 5, 6$, by adding zero or more pendent vertices to at most two of the vertices then it is easy to verify that $\gamma_h(G) = 2$. Conversely suppose that $\gamma_h(G) = 2$. Then G cannot have a cycle of length at least 7. By controversy if $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are the consecutive vertices on a cycle of length at least 7, then $\{v_1, v_6, v_7\}$ is the hop dominating set of G which is a contradiction to our assumption that $\gamma_h(G) = 2$. Suppose no pendent vertices are added to $C_n\,(n = 4, 5, 6)$ then the hop domination number of C_n is 2. Assume that G is a graph obtained from C_n by adding pendent vertices to at least three of the vertices of C_n. Then among the three vertices at which the pendent vertices are added, pendent vertices added at two of these vertices are dominated by the hop dominating vertices of C_n. And to dominate the pendent vertices added to the remaining vertex we need to choose another vertex from C_n. This increases the domination number by 1. This is a contradiction to our assumption that $\gamma_h(G) = 2$. Therefore the graph G obtained from $C_n\,(n = 4, 5, 6)$ by adding any number of pendent vertices to at most two of the vertices of C_n.

Now we construct a graph G obtained from C_3 by adding pendent vertices to at least two of the vertices of C_3.

Theorem 2.2 If a graph G is obtained from C_3 by adding any number of pendent vertices to at least two of the vertices of C_3, then $\gamma_h(G) = 2$.

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

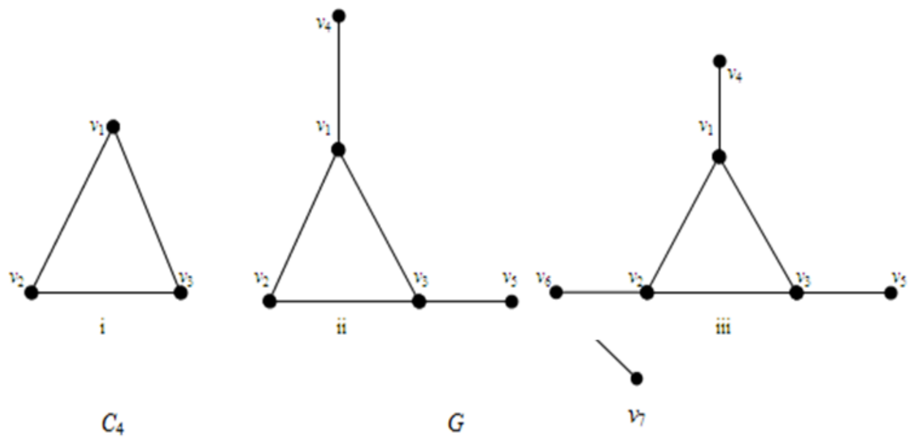

Figure 3: C_3 and reconstructed graphs

Proof: Let G be obtained from C_3 by adding more pendent vertices to at least any two of the vertices of C_3. See the above figure. Now we consider the following two cases:
Case 1: If the pendent vertices are added to any two of the vertices C_3. (See figure 3: (ii))

Without loss of generality we may assume that v_1 and v_3 be the two vertices of C_3 to which the pendent vertices are added. Let v_4 and v_5 be the pendent vertices added to v_1 and v_3 respectively. Then $\{v_4, v_5\}$ is the minimum hop dominating set of G. Then $\gamma_h(G) = 2$.
Case 2: If the pendent vertices are added to all the vertices of C_3.

Let v_4, v_5, v_6 be the pendent vertices added to v_1, v_3 and v_2 wrespectively as shown in the figure 3-(iii). Similarly we can prove that $\{v_2, v_6\}$ is the minimum hop dominating set of G. Thus $\gamma_h(G) = 2$

Note 2.3 It is not possible to construct a unicyclic graph from $C_n\,(n > 6)$ by adding zero or more pendent vertices to the vertices of C_n such that $\gamma_h(G) = 2$. Because in general for the graph $C_n\,(n > 6)$ itself $\gamma_h(G) = 3$. If we construct a graph from $C_n\,(n > 6)$ by adding zero or more pendent vertices to the vertices of C_n, the hop domination number should be greater than or equal to 3.
Next we develop some bounds having diameter, girth and maximum degree for $\gamma_h(G)$

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

3. Bounds based on diameter, girth and maximum degree Δ

Theorem 3.1 If G is a connected graph with diameter $d \geq 3$, then $\gamma_h(G) \geq \frac{d+1}{5}$.

Proof: Let $P : u_0, u_1, \cdots, u_d$ be a diametral path in G, joining the two peripheral vertices $u = u_0$ and $v = u_d$. Then P has length d. We show that every vertex of G hop dominates at most 5 vertices of P. Suppose to the contrary that there exists a vertex $w \in V(G)$ (may be in P) that hop dominates at least 6 vertices of P. Let Q be the set of vertices on the path P that are hop dominated by the vertex w in G. Then, by our supposition, $|Q| \geq 6$. Let i and j be the smallest and largest integers respectively, such that $u_i \in Q$ and $u_j \in Q$. Therefore, $Q \subseteq \{u_i, u_{i+1}, \cdots, u_j\}$. Thus, $6 \leq |Q.| \leq ji + 1$. Since P is the shortest (u, v) path in G and $Q \subseteq \{u_i, u_{i+1}, \cdots, u_j\}$, therefore $d_G(u_i, u_j) = d_P(u_i, u_j) \geq 5$. Let P_i be the shortest (u_i, w) path in G and P_j be (u_i, w) the shortest (w, u_j) path in G. Since the vertex w hop dominates both u_i and u_j in G, both paths P_i and P_j have length at most 2, therefore the path obtained by P_i proceeding P_j has length at most 4 implying that $d(u_i, u_j) \leq 4$, which is a contradiction. Therefore every vertex of G hop dominates at most 5 vertices of P. Let S be the minimum hop dominating set of G. Then $|S| = \gamma_h(G)$. Since each vertex of S hop dominates at most 5 vertices of P then S hop dominates $5(|S|)$ vertices of P. However S is a hop dominating set of G, every vertex of P is hop dominated by the set S and so S hop dominates $|V(P)| = d + 1$ vertices of P. Therefore $|S|(4 + 1) \geq d + 1$, or, equivalently, $\gamma_h(G) \geq (d + 1)/5$.
The following theorem founds a lower bound for $\gamma_h(G)$ having girth G of G.

Theorem 3.2 If G is a connected graph with girth g, then $\gamma_h(G) \geq \left\lceil \frac{g+1}{3} \right\rceil$

Proof: We prove this theorem by induction on the girth g of G. If G is a connected graph with girth 3, then it is easy to verify that $\gamma_h(G) \geq \left\lceil \frac{g+1}{3} \right\rceil$. Now we assume that the inequality holds for any connected graph G with girth $g - 1$. Now we prove for the case of the connected graph G with girth G. Let G be a connected graph G with girth $g - 1$. Then by our assumption the inequality holds for G. Let G be the graph obtained from G by subdividing any edge in the shortest cycle of G. Now the graph G has girth g. If the newly added vertex is dominated by any one of the vertices in the hop dominating set of G then the same set will be the hop dominating set of G also. So left hand side of the inequality remains the same and the right hand side of the inequality increases by one. Hence the equality holds. Therefore, $\gamma_h(G) = \left\lceil \frac{g+1}{3} \right\rceil$. If the newly added vertex is not dominated by the vertices in the hop dominating set of G' then we have to choose a vertex in order to dominate the newly added vertex. And so both the sides of the inequality increases by one or remains the same as previous case by the choice of the vertex selected to dominate the newly added

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

vertex. Therefore in this case $\gamma_h(G) \geq \left\lceil \frac{g+1}{3} \right\rceil$. Therefore for any connected graph G with G $\gamma_h(G) \geq \left\lceil \frac{g+1}{3} \right\rceil$.
We prove a result in tree T that $\gamma_h(T) \leq \Delta(T)$ which is a relation between hop domination and maximum degree $\Delta(T)$ of T. But generally this relation $\gamma_h(T) \leq \Delta(T)$ is not true for all trees unless we add some conditions. Here we give one of such conditions that $nl \leq \Delta(T)$ where l is the number of pendent vertices in T. We prove this in the following theorem.

Theorem 3.3 If T is a tree of order $n \geq 3$ and $nl \leq \Delta(T)$ where l is the number of pendent vertices in T, $\Delta(T)$ maximum degree of T, then $\gamma_h(T) \leq \Delta(T)$.

Proof: Let T be a tree of order $n \geq 3$. Let l be the number of end vertices in the tree T. These end vertices are not included in the hop dominating set of T because it harms the minimality of the hop dominating set. Therefore, $\gamma_h(T) \leq nl$. Let $\Delta(T)$ be the maximum degree of the tree T. Since any tree T has at least $\Delta(T)$ end vertices and $nl \leq \Delta(T)$ we have $\gamma_h(T) \leq nl \leq \Delta(T)$. This implies that $\gamma_h(T) \leq \Delta(T)$.

Theorem 3.4 Let G be a connected graph that contains a cycle. Let C be the shortest cycle in G. If v is a vertex of G outside C that hop dominates two vertices (say) u and w of C, then there exists a shortest path (u, v) that does not contains w and a shortest (v, w) path that does not contain u.

Proof: Since, v is a vertex not on C, it has a distance at least one to every vertex of C. Let $Q = \{u, w\}$ be the set of vertices that are hop dominated by v. Thus, $Q \subseteq V(C)$ and $|Q| = 2$. Since, there are two vertices at a distance one from u on C and w is hop dominated by v, w is at a distance two from u on the cycle C. Therefore, $d_G(v, w) = d_G(u, v)$. Let P_u be a shortest (u, v) path and Let P_w be a shortest (v, w) path in G. If $w \in V(P_u)$, then $d_G(v, w) < d_G(u, v)$, contradicting the fact that $d_G(v, w) = d_G(u, v)$. Therefore, $w \notin V(P_u)$. Similarly, if $u \in V(P_v)$, then $d_G(u, v) < d_G(v, w)$, contradicting the fact that $d_G(v, w) = d_G(u, v)$. Therefore, $u \notin V(P_v)$.

Theorem 3.5 For any connected graph G with vertex cut R and if $|R| \geq 2$, then $\gamma_h(G) < |R| + k$, where k is the number of vertices in the open neighborhood of R.

Proof: Let G be a connected graph with vertex cut R such that $|R| \geq 2$. Let $N(R)$ be the open neighborhood of R and let $|N(R)| = k$. Since, R is a vertex cut of G,

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

the removal of R from the vertex set of G results in a disconnected graph. Therefore, each vertex in R is adjacent to at least one vertex in $v - R$. This implies that each vertex in R hop dominates the vertices which are adjacent to the vertices in $N(R)$. If the set R hop dominates the graph G, then R is our hop dominating set of G. If there are some vertices in G which are not hop dominated by the vertices in R, we can choose vertices from $N(R)$ to dominate those vertices. In general, we need to choose at most $|R| + \; |N(R)|$ vertices to hop dominate the graph G. Therefore, $\gamma_h(G) < |R| + k$.

Theorem 3.6 For any connected graph G with no vertex of degree one and clique number $\omega(G)$, $\gamma_h(G) \leq n - \omega(G) + 2$.

Proof: Let G be a connected graph G with no vertex of degree one and clique number $\omega(G)$. Let H be the maximal clique in the graph G. Then to hop dominate the vertices in the maximal clique, we need to choose any two vertices of G outside H. Then to hop dominate the remaining vertices outside the maximal clique we need to choose at most $n - \omega(G)$ vertices. In general we need to choose at most $n - \omega(G) + 2$ vertices to hop dominate the graph G.
Therefore, $\gamma_h(G) \leq n - \omega(G) + 2$.

Theorem 3.7 For any connected graph G with l $(l > 2)$ cut vertices,$\gamma_h(G) \leq l$.

Proof: Let G be a connected graph. Let U be the set of all cut vertices in the graph G such that $|U| = l$ and $l > 2$. Since, all the vertices in U are cut vertices of G removal of U from the vertex set V of G results in l different components of G. Hence each vertex in U is adjacent to at least one vertex in at least one component of G. Therefore, vertices in U hop dominates all the vertices of G. Therefore, we need to choose at most l vertices (since,$|U| = l$) to hop dominate the graph G. Therefore, $\gamma_h(G) \leq |U| = l$.

Theorem 3.8 For the graph $K_n \odot P_2$, $\gamma_h(K_n \odot P_2) = n$.

Proof: Let $G = K_n \odot P_2$. Since, P_2 is attached to each vertex of K_n, eccentricity of all the vertices of K_n is 3, eccentricity of all the pendent vertices is 5 and the eccentricity of all the support vertices is 4. Here, the eccentricities of the vertices of K_n is less than the eccentricities of the vertices of P_2. Therefore, choosing all the vertices of K_n will hop dominate the graph $K_n \odot P_2$. Therefore, $\gamma_h(K_n \odot P_2) = n$.

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

4. Hop Domination Number of a Wounded Spider

Definition 4.1 A subdivision of an edge uv is obtained by replacing the edge uv with the edges uw and vw with a new vertex w. A spider is a tree on $2n+1$ vertices obtained by subdividing each edge of a star. One or more (but not all) of the edges from this subdivision exempted results a wounded spider.

Theorem 4.2 For any wounded spider $K_{1,t}^{*}$, where $t \geq 2$, $\gamma_h\left(K_{1,t}^{*}\right) = 2$.

Proof: Consider the following wounded spider $K_{1,t}^{*}$, where k edges are subdivided and $k < t$. Let u be the central vertex. Referring the above graph we say that the

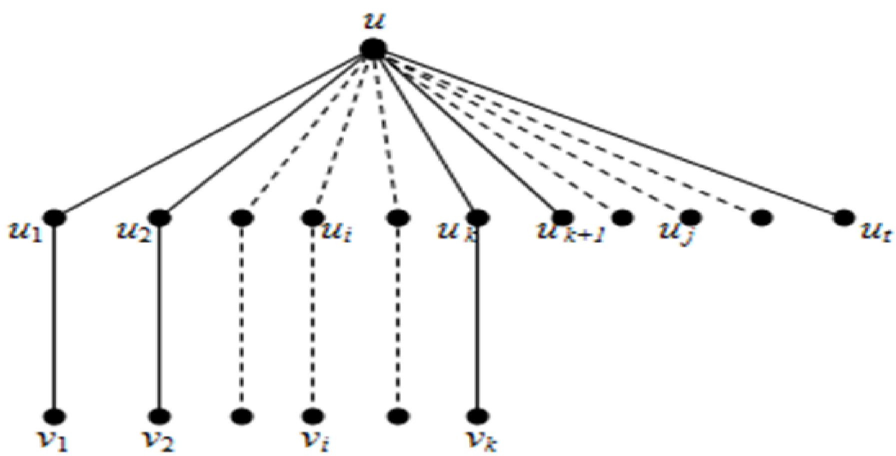

Figure 4: A Wonder spider $K_{1,t}^{*}$

vertices $v_1, v_2, v_3, \ldots\ldots, v_k$ are hop dominated by the vertex u. Take any one of the vertex from the set of vertices $\{u_{k+1}, u_{k+2}, \ldots\ldots, u_t\}$ without loss of generality take that vertex to be u_t. Then u_t hop dominates all other remaining vertices. Therefore $\{u, u_t\}$ is the minimum hop dominating set of $K_{1,t}^{*}$. Therefore $\gamma_h\left(K_{1,t}^{*}\right) = 2$.

Theorem 4.3 For any connected graph G, $\gamma_2(G) \leq \gamma_h(G)$.

Proof: Let G be any connected graph G. Let D be the minimum distance 2 dominating set of G. Let D' be the minimum hop dominating set of G. Every vertex in D dominates every vertex in VD at a distance less than or equal to 2 but every vertex in D' dominates every vertex in $V - D'$ at a distance exactly 2. Then, $D \subseteq D'$. Hence, $|D| \leq |D'|$ which implies that $\gamma_2(G) \leq \gamma_h(G)$.

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

5. Connected Hop Domination Number

Definition 5.1 A hop dominating set D of a graph G is said to be a connected hop dominating set of G if the induced subgraph $< D >$ is connected. The cardinality of a minimum connected hop dominating set is called the connected hop domination number of G and it is denoted by $\gamma_h^c(G)$.

Example 5.2 Consider the following graph G:
Consider the minimum hop dominating set $D = \{c, e, f\}$.
The induced sub graph of D is not connected. Now consider another minimum

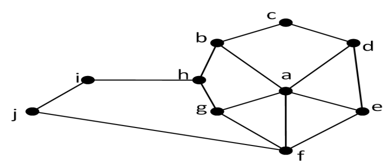

Figure 5: A graph G

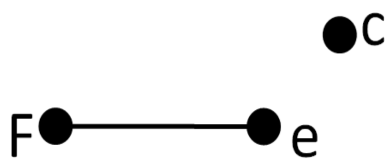

Figure 6: Induced subgraph of D

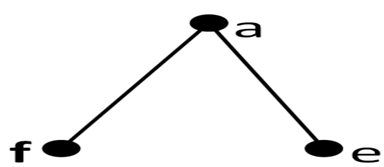

Figure 7: Induced subgraph of D_1

hop dominating set $D_1 = \{a, e, f\}$. The induced sub graph of D_1 is is connected. Therefore, $D_1 = \{a, e, f\}$ is the minimum connected hop dominating set of G. Therefore, $\gamma_h^c(G) = 3$.
Next we study the connected hop domination number for some standard graphs.

Theorem 5.3 For any complete graph K_n, $\gamma_h^c(K_n) = n$.

Proof: Let K_n be any complete graph. Since in a complete graph each and every vertex is adjacent to the other $n1$ vertices so it is not easy to find a hop dominating set unless all the vertices are included in the hop dominating set D. Since it is a complete graph and all the vertices are included in the hop dominating set D the induced subgraph of D is connected. Therefore, $\gamma_h^c(K_n) = n$.

Theorem 5.4 For any complete bipartite graph $K_{m,n}$, $\gamma_h^c(K_{m,n}) = 2$.

Proof: Let $K_{m,n}$ be any complete bipartite graph with partite set $V = V_1 \cup V_2$. In the set V_1 distance between any two vertices is 2. Therefore choosing one vertex in the set V_1 say v hop dominates the remaining $m1$ vertices. Similarly in the set V_2 distance between any two vertices is 2. Therefore choosing one vertex in the set V_2 say w hop dominates the remaining $n1$ vertices. Therefore, the minimum hop dominating set of $K_{m,n}$ is $D = \{v, w\}$. Since it is a complete bipartite graph the induced subgraph of D will be connected. Since $D = \{v, w\}$ is the minimum connected hop dominating set of $K_{m,n}$. Therefore, $\gamma_h^c(K_{m,n}) = 2$.

Theorem 5.5 For any star graph $K_{1,n}$, $\gamma_h^c(K_{1,n}) = 2$.

Proof: Let $K_{1,n}$ be any star graph with $n+1$ vertices. Let the vertex set of $K_{1,n}$ be $V = \{u, v_{1,}v_2, .., v_n\}$. Here u is the only central vertex with eccentricity 1 and the remaining n pendant vertices have eccentricity 2. Any one of the pendant vertices say v_i hop dominates the remaining $n1$ vertices. Since u is the only vertex with eccentricity 1 it dominates itself. Therefore the minimum hop dominating set of $K_{1,n}$ is $D = \{u, v_i\}$. Since, it is a star graph the induced sub graph of D will be connected. Therefore, $D = \{u, v_i\}$ $D =$ u, viis the minimum connected hop dominating set of $K_{1,n}$. Therefore, $\gamma_h^c(K_{1,n}) = 2$.

Theorem 5.6 For any wheel graph W_n, $\gamma_h^c(W_n) = 3$.

Proof: Let W_n be any wheel graph with n vertices. Let the vertex set of W_n be $V = \{u, v_{1,}v_2, \cdots, v_{n-1}\}$. Here u is the only central vertex with eccentricity 1 and the remaining $n1$ vertices have eccentricity 2. Any one of the vertices having eccentricity 2 say v_i hop dominates the other vertices except the vertices adjacent to it. Therefore to hop dominate the adjacent vertices of v_i choose the antipodal vertex say v_j of v_i. Since, u is the only central vertex with eccentricity 1 it dominates itself. Therefore, the minimum hop dominating set of W_n is $D = \{u, v_i, v_j\}$. Since, it is a wheel graph the induced sub graph of D will be connected. Therefore, $D = \{u, v_i, v_j\}$ is the minimum connected hop dominating set of W_n. Therefore, $\gamma_h^c(W_n) = 3$.
The following theorem states the existence of a connected hop dominating set in an arbitrary graph.

[2*]dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

Theorem 5.7 Let $G = (V, E)$ be a connected graph. Let $G_1, G_2, \ldots, G_s (s \geq 2)$ be connected proper subgraphs of G with connected hop dominating sets $D_1, D_2, \ldots, D_s$ respectively. If $\bigcup_{i=1}^{s} V(G_i) = V$ then there exists a connected hop dominating set D of G such that $D \subseteq \bigcup_{i=1}^{s} D_i$ and $|D| \leq \sum_{i=1}^{s} |D_i| + 2s$.

Proof: We proceed the proof by induction on $s \geq 2$. First we assume that $s = 2$. By the assumption there exists $x \in N[D_1]$ and $y \in N[D_2]$ with $xy \in E(G)$ since G is connected. Thus by adding two vertices in v together with $\{x, y\}$ to $D_1 \bigcup D_2$ we get a connected dominating set D of G with $D \subseteq D_1 \bigcup D_2$ such that $|D| \leq |D_1| + |D_2| + 2$. Assume now that the result is true for $s = 2, 3, \cdots, k$ and we prove the result for the case $s = k + 1$. We construct a new graph with vertices of G_i, $1 \leq i \leq k + 1$. We say that G_i and G_j are adjacent if $V(G_i) \cap V(G_j) \neq \Phi$ or there exists $u \in V(G_i) = N[D_i]$ and $v \in V(G_j) = N[D_j]$ v such that $uv \in E(G)$. It is obvious that κ is a connected graph and the fact that $\bigcup_{i=1}^{k+1} V(G_i)$. By deleting a non-cut vertex of κ, say G_{k+1}, we get a new connected graph $\kappa' = G\left[\bigcup_{i=1}^{k+1} V(G_i) - V(G_{K+1})\right]$. By the induction hypothesis there exists a connected hop dominating set D' of κ' such that $D' \subseteq \bigcup_{i=1}^{k} D_i$ and $|D'| \leq \sum_{i=1}^{k} |D_i| + 2k$. By the same argument in the case $s = 2$, there exists a connected hop dominating set D of κ with $D \subseteq D' \bigcup D_{k+1} \subseteq \bigcup_{i=1}^{k} D_i \bigcup D_{k+1} = \bigcup_{i=1}^{k+1} D_i$ which implies that $|D| \leq |D'| + |D_{k+1}| + 2$. Hence, $|D| \leq \sum_{i=1}^{k} |D_i| + 2k + |D_{k+1}| + 2$. And so, $|D| \leq \sum_{i=1}^{k+1} |D_i| + 2(k+1) = \sum_{i=1}^{s} |D_i| + 2s$. Thus it is true for all $s = k+1$ and it completes the induction hypothesis.

Corollary 5.8 In the theorem 5.7, if $D_1, D_2, \ldots, D_s$ are minimum connected hop dominating sets of $G_1, G_2, \ldots, G_s$, then $\gamma_h^c(G) \leq \sum_{i=1}^{s} \gamma_h^c(G_i) + 2s$.

Proof: Proof is immediate from the above theorem by taking $D_1, D_2, \ldots, D_s$ as minimum connected hop dominating sets of $G_1, G_2, \ldots, G_s$.

Corollary 5.9 In the theorem 5.7, if $V(G) - \bigcup_{i=1}^{s} V(G_i) = X$, then there exists a connected hop dominating set D of G such that $D \subseteq \bigcup_{i=1}^{s} D_i$ and $D \subseteq \sum_{i=1}^{s} |D_i| + 2s + |X|$.

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

Proof: Let the components of $G[X]$ be $X_1, X_2, \ldots, X_r, 1 \leq r \leq |X|$. Then for each X_j; $1 \leq j \leq r$, there exists $u_j \in X_j$ and $v_j \in V(G_j)$ such that u_j is adjacent to v_j in G_j. Denote this existence as X_j is adjacent to G_j. Joining X_j to one of its adjacent sub graphs say G_j to get a new graph $G_j^{'}$ and obtain a connected hop dominating set of $D_j^{'}$ of $G_j^{'}$ by adding at most $|X_j|$ vertices to the connected dominating set D_j of G_j. By the previous theorem if $|X_j| \geq 2$ then ignore at least one non cut vertex of X_j at the time of addition of vertices to the hop dominating set of G_j. By repeating the above process for all X_j; $1 \leq j \leq r$, we obtain a subgraph of G denoted as $G_i^{'}$, $1 \leq i \leq s$. Then $V(G) = \bigcup\limits_{i=1}^{s} V\left(G_j^{'}\right) = \bigcup\limits_{i=1}^{s} N\left[G_j^{'}\right]$ and $G[G_{i'}]$ is connected. Also we have $\sum\limits_{i=1}^{s} |D_{_i}^{'}| \leq \sum\limits_{i=1}^{s} |D_i| + \sum\limits_{j=1}^{r} |X_j| = \sum\limits_{i=1}^{s} |D_i| + |X|$. It follows from the above theorem that G has a connected dominating set D with $D \subseteq \bigcup\limits_{i=1}^{s} D_{_i}^{'}$ and so $|D| \leq \sum\limits_{i=1}^{k} |D_{_i}^{'}| + 2s = \sum\limits_{i=1}^{s} |D_i| + 2s + |X|$. It completes the proof.

6. Conclusion

In this paper, we characterized graphs for which $\gamma_h(G) = 2$ by constructing new graphs from the cycles C_3, C_4, C_5, C_6 by adding pendent vertices. The bounds for the hop domination number based on diameter, girth and maximum degree were found. The hop domination number of a wounded spider was determined. The relation between the hop domination number and the distance-2 domination number was discussed. The connected hop domination number of special graphs like Corona graphs, splitting graphs and subdivided graphs can be studied. This study can be extended as hop domination based central structures in graphs.

References

[1] Ayyaswamy SK and Natarajan C, Hop Domination in Graphs - II, An, St. Univ. Ovidius Constanta,23(2), 2015, 187 -199.

[2] Fang Tian, A note on distance domination number of graphs, Australian Journal of Combinatorics, 43, 2009, 181 - 190.

[3] Vaidya SK and Niran GJ Kothari, Some New Results on Distance k - Domination in Graphs, International Journal of Combinatorics, Article ID 795401, 4 pages, 2013.

[4] Chartrand G and Lesniak L, Graphs and Digraphs, Chapman and Hall, CRC, 4th edition, (2005).

2*dstramesh@gmail.com

ISSN: 2456-8686, Volume 2, Issue 1, 2018:1-13
DOI : http://doi.org/10.26524/cm22

[5] Chartrand G and Zhan G P, Introduction to Graph Theory, McGraw Hill Education (India) Private Limited, New Delhi, (2006).

[6] Chellali M, Favaron O, Hansberg A and Volkmann L, k-domination and k-independence in graphs: A survey, Graphs and Combinatorics, (2012).

[7] Haynes TW, Hedetniemi ST and Slater PJ, Fundamentals of Domination in graphs, Marcel Dekker, New York, (1998).

[2*]dstramesh@gmail.com