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Oscillation of difference causal operator equations

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Abstract

New sufficient conditions are provided to guarantee that the (nontrivial) solutions of a discrete equation of the form $\Delta x(n) + F(n, x) = 0$, $n = 0, 1, 2, \cdots$, (where $F(n, \cdot)$ is a causal operator) either oscillate, or converge monotonically to zero.

Key words: Difference equations, oscillation

AMS classification: 39A21; 39A12

1. Introduction

Consider a Riemann-Stieljies delay integro-differential equation of the form

$$\dot{u}(t) + \int_{\alpha}^{\beta} g(t, \xi, u(\rho(t, \xi)))d_{\xi}\eta(t, \xi) = 0, \;\; t \geq 0, \tag{1}$$

where $g \in C(\mathbb{R}^+ \times [\alpha, \beta] \times \mathbb{R}, \mathbb{R})$, $\rho \in C(\mathbb{R}^+ \times [\alpha, \beta], \mathbb{R})$ and η is a function of bounded variation with respect to $\xi \in [\alpha, \beta]$, for all $t \geq 0$. Under the condition that $\rho(t, \xi) < t$, for all (t, ξ), on the delay, we say that the equation has an amnesia $[\sup_{\xi\in[\alpha,\beta]} \rho(t, \xi), t]$ and its general form is studied elsewhere, see, e.g. [12, 13], where the amnesia is bounded. Define $\mu := \inf\{\rho(t, \xi) : \;\; \xi \in [\alpha, \beta], \;\; t \geq 0\}(< 0)$ and let $\phi : [\mu, 0] \to \mathbb{R}$ be a initial function. It is easy to see that, under the previous conditions on g, ρ and η, the solution $u = u(\cdot; 0, \phi)$ (with initial value ϕ at 0) exists, it is unique and it can be extended on the whole interval $[\mu, +\infty)$. See, also, Lemma 2.2 in [13]).

Now suppose that we want to obtain an approximation of u. To do that we apply a discretization of (1) by a step $h > 0$. The smaller the h, the better approximation of u we have. This process leads to a relation of the form

$$\Delta x(n) + \int_{\alpha}^{\beta} f(n, \xi, x(\upsilon(n, \xi)))d_{\xi}\vartheta(n, \xi) = 0, \;\; n = 0, 1, 2, \cdots,$$

where $\Delta x(n) := x(n + 1) - x(n)$, $x(r) := u(hr)$, $r \geq 0$, $f(n, \xi, \cdot) := g(hn, \xi, \cdot)$, $\upsilon(n, \xi) := \frac{1}{h}\rho(hn, \xi)(\leq n)$ and $\vartheta(n, \xi) := h\eta(hn, \xi)$, for all $n = 0, 1, 2, \cdots$ and $\xi \in$

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$[\alpha, \beta]$. This equation belongs to the class of difference equations of the form

$$\Delta x(n) + F(n; x) = 0, \ \ n = 0, 1, 2, \cdots \tag{2}$$

where $F(n; \cdot)$ is defined on the space $C(\mathbb{R}, \mathbb{R})$ and it is causal, namely it satisfies the following condition:

(C1) For any index n there is a set $\mathcal{M}_n \subseteq (-\infty, n-1]$ such that if two functions x and y are such that $x(r) = y(r)$, for all $r \in \mathcal{M}_n$, then $F(n, x) = F(n, y)$.

We denote by $s(n), t(n)$ the infimum and the supremum, respectively, of the set $\mathcal{M}_n$ satisfying the condition (C1). The set $\mathcal{M}_n$ is the memory, while the interval $[t(n), n]$ is the amnesia of the equation. In discrete equations the set $\mathcal{M}_n$ consists of integer numbers. For technical reasons we shall consider the integer $\tau(n) := \min\{j \in \mathbb{N} : t(n) \leq j\}$. Also, we shall denote by I_n the interval $[s(n), \tau(n)]$ and let $k_n := n - \tau(n)$.

Now let us see how the machine (2) works. Given a initial function $\phi : [s(0), 0] \to \mathbb{R}$, we find $x(1)$ by the relation $x(1) = \phi(0) + F(0; \phi)$. Next, in order to obtain the value $x(2)$ we need to know the values of x on the set $I_1 \cup \{1\}$. Inductively we need obtain all values $x(n)$. But here there is a problem: In order to go from n to $n+1$ and it holds $[(-\infty, n) \setminus \mathbb{Z}] \cap [\cup_{j \leq n} I(j)] \neq \emptyset$, we have to know some values of x in the open intervals between the integers which are less than or equal to n. To overcome these uncertainties one could give any value of x on these intervals (and produce an uncountable set of solutions), but, for our purpose, we suggest that the values of x between the integers are defined as convex combination of the ends of the corresponding intervals. Namely, for our convenience, we accept x to be linear between the positive integers and so it must have the value $x((1-\lambda)(n-1) + \lambda n) := (1-\lambda)x(n-1) + \lambda x(n)$, for all $\lambda \in [0, 1]$ and $n = 1, 2, \cdots$ Apart of the fact that the most usual and traditional approximation of a C^1-arc is its chord, the suggestion to use "straight bridges" agrees with the following two facts: 1) Monotonicity of x on the interval $[0, +\infty)$ is equivalent with the same kind of monotonicity of the sequence $x(0), x(1), \cdots$. 2) Oscillation of x in the usual sense[2], is equivalent with the oscillation of the sequence $x(0), x(1), \cdots$

To shorten the text we shall say that a function has the property $(\mathbf{P})$ if it is either oscillating, or it converges monotonically to zero.

Our purpose in this work is to provide sufficient conditions for all solutions of equation (2) to have property $(\mathbf{P})$. To our knowledge, this general class of discrete equations (which, obviously, contains the discrete equations) is the first time introduced for further study. Difference and discrete equations were studied by a great number of authors. See, e.g., [1] - [27] and the references therein. There are,

[2] A function $x : [0, +\infty) \to \mathbb{R}$ is called oscillatory, if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

moreover, very good books on difference equations, as, for instance, [1, 3, 9, 11]. In particular the book [3] presents an extensive list of references on the subject.

Searching the literature we can find a great number of sufficient conditions which guarantee oscillation, concerning even higher order difference equations, see, e.g., [26, 27] and the references therein. Let us focus on conditions which guarantee the fact that all solutions of the linear difference equations

$$\Delta x(n) + p(n)x(n-k) = 0 \tag{3}$$

(which is a special case of equation (2)), are oscillating. Information about these conditions can be found elsewhere, see, e.g. [24]. Nevertheless, we focus our attention on the first sufficient condition for oscillation of all solutions of (3), which were given by Ladas et al. [15]:

$$\liminf_{n\to+\infty} \sum_{j=n-k}^{n-1} p(j) > \left(\frac{k}{k+1}\right)^{k+1}.$$

This condition was extended in [20] to

$$\liminf_{n\to+\infty} \sum_{j=h(n)}^{n-1} p(j) > \limsup_{n\to+\infty} \left(\frac{n-h(n)}{n-h(n)+1}\right)^{n-h(n)+1} \tag{4}$$

concerning the equation

$$\Delta x(n) + p(n)x(h(n)) = 0, \tag{5}$$

where $h(n)$ is a increasing integer-delay-sequence converging to $+\infty$. This condition was improved in [21].

Concerning the difference equation with several delays

$$\Delta x(n) + \sum_{j=1}^{m} p_j(n)x(\tau_j(n)) = 0, \ \ n \geq 1$$

(which is of the form (2)), the sufficient condition

$$\sum_{i=1}^{m} \left(\liminf_{n\to+\infty} p_i(n)\right) \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1$$

for oscillation of all solutions were given in [10]. (Here $k_i := n-\tau_i(n)$, $i = 1, 2, \cdots, m$.)

In this work we state and prove two theorems: In the first one we give a sufficient condition which resembles to (4) and in the second one we give a new sufficient

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condition for the property (**P**). In the last section enough applications are presented to show that these two conditions are independent and they do not imply each other.

2. The main results

Consider the difference equation (2) and assume that the response operator F satisfies the following basic condition:

(C2) There is a sequence $(b(n))$ of positive real numbers, such that

$$F(n;x) \geq b(n) \inf_{r \in I(n)} x(r), \text{ whenever } \inf_{r \in I(n)} x(r) > 0 \tag{6}$$

and

$$F(n;x) \leq b(n) \sup_{r \in I(n)} x(r), \text{ whenever } \sup_{r \in I(n)} x(r) < 0. \tag{7}$$

It is clear that if $F(n;\cdot)$ is a positive linear operator then it satisfies condition (C2). Other examples of such operators will be given in the last section. There are cases where only one of these conditions is true. For example, the function $F(n;x) := n + x(n-2)e^{x(n-1)}$, satisfies (6), (with $b_n := 1$,) but not (7).

We observe that these two mathematical relations can be written in a unified form as

$$\min\{F(n;x), -F(n;-x)\} \geq b(n) \inf_{r \in I(n)} x(r), \text{ whenever } \min_{r \in I_n} x(r) > 0.$$

Therefore, if $F(n,\cdot)$ is odd, then the two conditions are equivalent.

The first result of this paper concerns an extension of the validity of condition (4) to this general setting.

Theorem 2.1 Consider equation (2), where the operator F satisfies conditions (C1) and (C2). Assume that it holds $\tau(n) \leq n-1$, for all n and the sequence (k_n) is bounded and moreover

$$\liminf_{n} s(n) = +\infty. \tag{8}$$

If the condition

$$\liminf_{n \to +\infty} \frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j > \limsup_{n \to +\infty} \frac{k_n^{k_n}}{(1+k_n)^{1+k_n}}, \tag{9}$$

is satisfied, then any solution has property (**P**).

Proof: Assume that x is a (nontrivial) non-oscillating solution of equation (2). If x is eventually negative, the second relation in (C2) is satisfied for all large n. Then

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the function $y := -x$ is eventually positive and it satisfies the difference equation $\Delta y(n) + F^*(n, y) = 0$, where the new function $F^*(n, u) := -F(n, -u)$, obviously, satisfies the first condition in (C2) for all large n. Thus, we can assume that x is eventually positive, i.e. there is some t_0 such that $x(r) \geq 0$, for all $r \geq t_0$. Due to (8), there is some t_1 such that $t_0 \leq s(t)$, for all $t \geq t_1$. Then, for all $n \geq n_1 := [t_1] + 1$ and $r \in I_n$, we have $x(r) > 0$, and therefore

$$\Delta x(n) = -F(n, x) < 0. \tag{10}$$

This relation implies that the solution x is an eventually monotonically decreasing function and therefore the following inequalities hold:

$$x(s(n)) \geq x(r) \geq x(\tau(n)) > x(n), \;\; n \geq n_1 \text{ and } r \in I_n. \tag{11}$$

From (2) and the first condition in (C2) we get

$$\frac{x(n+1)}{x(n)} - 1 = -\frac{1}{x(n)} F(n, x) \leq -\frac{x(\tau(n))}{x(n)} b_n < -b_n, \;\; n \geq n_1. \tag{12}$$

Summing up this inequality by parts we obtain

$$\sum_{j=\tau(n)}^{n-1} \frac{x(j+1)}{x(j)} < k_n - \sum_{j=\tau(n)}^{n-1} b_j, \tag{13}$$

for all $n \geq n_2$, where n_2 is a positive integer such that $n_2 \geq \min\{j : \;\; s(j) \geq n_1\}$.

By using the Arithmetic Mean-Geometric Mean inequality we obtain

$$\frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} \frac{x(j+1)}{x(j)} \geq \Big(\prod_{j=\tau(n)}^{n-1} \frac{x(j+1)}{x(j)} \Big)^{\frac{1}{k_n}} = \Big(\frac{x(n)}{x(\tau(n))} \Big)^{\frac{1}{k_n}}. \tag{14}$$

Then form (13) we get

$$0 < \Big(\frac{x(n)}{x(\tau(n))} \Big)^{\frac{1}{k_n}} < 1 - \frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j. \tag{15}$$

Therefore we have $\liminf_{n \to +\infty} \frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j \leq 1$.

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This relation and (9) permit us to choose a constant $\gamma \in (0,1)$ such that

$$\liminf_{n\to+\infty} \frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j > \gamma > \limsup_{n\to+\infty} \frac{k_n^{k_n}}{(1+k_n)^{k_n+1}}. \tag{16}$$

Thus there is some $n_3 \geq n_2$ such that for all $n \geq n_3$ it holds

$$\frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j > \gamma. \tag{17}$$

Assume that x does not converge to 0. Hence, due to its monotonicity, it must stay away from zero, i.e. there is some $\mu > 0$ such that $x(r) \geq \mu$, for all $r \geq n_3$. Letting $n_4 \geq n_3$ such that $s(n) \geq n_3$, for all $n \geq n_4$, we have $t(n) \geq n_3$, for all $n \geq n_4$. Then, due to (15), the quantity $\zeta := \liminf_{n\to+\infty} \frac{x(\tau(n))}{x(n)}$ satisfies

$$\zeta \geq \liminf_{n\to+\infty} \left(1 - \frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j\right)^{-k_n}.$$

Obviously we have $\frac{1}{1-\gamma} =: \delta > 1$ and from (17) it follows that

$$\zeta \geq \liminf_{n\to+\infty} \frac{1}{(1-\gamma)^{k_n}} = \liminf_{n\to+\infty} \delta^{k_n} \geq \delta > 1,$$

because, due to our assumptions, it holds $k_n \geq 1$. This relation implies that the number ζ belongs to the bounded interval $(1, \frac{x(\tau(n_3))}{\mu}]$.

Next, fix a small enough number $\varepsilon > 0$, such that

$$\varepsilon < \zeta - 1 \ \text{ and } \ \frac{\varepsilon}{1-\varepsilon^2} < \gamma. \tag{18}$$

Then, for some $n_5 \geq n_4$ and for all $n \geq n_5$, we have

$$\zeta - \varepsilon < \frac{x(\tau(n))}{x(n)}. \tag{19}$$

On the other hand, for some subsequence (n_m) converging to $+\infty$, it holds

$$\zeta + \varepsilon > \frac{x(\tau(n_m))}{x(n_m)}, \tag{20}$$

for all m. From (12) and (19), we get that

$$\frac{x(n+1)}{x(n)} - 1 < -(\zeta - \varepsilon)b_n, \ \ n \geq n_5.$$

Summing up by parts this inequality we obtain

$$\sum_{j=\tau(n)}^{n-1} \frac{x(j+1)}{x(j)} < k_n - (\zeta - \varepsilon) \sum_{j=\tau(n)}^{n-1} b_j, \tag{21}$$

for all $n \geq n_6$, where n_6 is a positive integer such that for all $n \geq n_6$ it holds $s(n) \geq n_5$.

By using (21), the Arithmetic Mean-Geometric Mean inequality (14) and relation (17), we obtain

$$\Big(\frac{x(n)}{x(\tau(n))}\Big)^{\frac{1}{k_n}} < 1 - (\zeta - \varepsilon)\frac{1}{k_n} \sum_{j=\tau(n)}^{n-1} b_j < 1 - (\zeta - \varepsilon)\gamma. \tag{22}$$

From (22) we conclude that $1 - \varepsilon < \zeta - \varepsilon < \frac{1}{\gamma}$. Then from (20)

$$\psi_m(\zeta) > 1, \tag{23}$$

for all large m, where

$$\psi_m(c) := (c + \varepsilon)(1 - (c - \varepsilon)\gamma)^{k_{n_m}}, \ \ m \geq n_6,$$

where $-\varepsilon \leq c \leq \frac{1}{\gamma} + \varepsilon$. The function ψ_m is positive on the interior of the interval of its definition and it vanishes at the end points. Notice that the number 1 belongs to this set. The first derivative of ψ_m vanishes only at the point

$$\theta_0 = \frac{1 - \varepsilon\gamma(k_{n_m} - 1)}{\gamma(k_{n_m} + 1)}.$$

Therefore ψ_m takes its maximum at this point. Assuming that $\theta_0 < 1$, the function ψ_m would be decreasing on the interval $[1, \frac{1}{\gamma} + \varepsilon]$. Then, it is clear that, its maximum on this interval is achieved at the point 1 and it must be equal to

$$(1 + \varepsilon)(1 - (1 - \varepsilon)\gamma)^{k_{n_m}}.$$

But this quantity is smaller than or equal to

$$(1 + \varepsilon)(1 - (1 - \varepsilon)\gamma) = 1 - \gamma + \varepsilon(1 + \varepsilon\gamma),$$

which, due to the second condition in (18), is smaller than 1. Hence we must have $\theta_0 \geq 1$ and then

$$\sup \psi_m(c) = \frac{1}{\gamma} \frac{k_{n_m}^{k_{n_m}}}{(k_{n_m}+1)^{k_{n_m}+1}} (1+2\varepsilon\gamma)^{k_{n_m}+1}. \tag{24}$$

Since the sequence (k_n) is bounded we have

$$\limsup_{\varepsilon \to 0} \limsup_{m \to +\infty} (1+2\varepsilon\gamma)^{k_{n_m+1}} \leq \lim_{\varepsilon \to 0} (1+2\varepsilon\gamma)^{\sup k_n + 1} = 1. \tag{25}$$

Thus, from (23), (24) and (25) we obtain

$$\gamma \leq \limsup_{m \to +\infty} \frac{k_{n_m}^{k_{n_m}}}{(k_{n_m}+1)^{k_{n_m}+1}} \leq \limsup_{n \to +\infty} \frac{k_n^{k_n}}{(k_n+1)^{k_n+1}},$$

contrary to (16). Thus there is no positive solution nonconverging to zero. This completes the proof of the theorem.

Theorem 2.2 Assume that conditions (C1), (C2) and (8) hold as well as

$$\limsup_{n \to +\infty} \sum_{j=\tau(n)}^{n} b_j < +\infty. \tag{26}$$

If there is a term B_λ of the sequence defined by

$$B_0 := 1 \text{ and } B_{i+1} := \liminf_{n \to +\infty} \prod_{j=\tau(n)}^{n} [1 + B_i b_j], \quad i = 0, 1, 2, \cdots$$

satisfying the relation

$$B_{\lambda+1} - B_\lambda \geq 1, \tag{E_λ}$$

then property (**P**) keeps in force.

Proof: Assume that x is a nonoscillating solution. Then, again, as in Theorem (2.1), we can assume that x is an eventually positive function satisfying relation (11). Let us define

$$z := \liminf_{n \to \infty} \frac{x(\tau(n))}{x(n+1)},$$

which, in case the solution does not converge to zero, is bounded above. Obviously, $z \geq 1$. Let $0 < \varepsilon << 1$, be small enough. Corresponding to this ε, there is some $n(\varepsilon)_0$

such that
$$\frac{x(\tau(n))}{x(n+1)} \geq z - \varepsilon,$$
for all $n \geq n_0(\varepsilon)$. Then due to (2), (11) and (6) we have
$$\frac{x(n)}{x(n+1)} = 1 + \frac{1}{x(n+1)}F(n,x) \geq 1 + \frac{x(\tau(n))}{x(n+1)}b_n \geq 1 + (z-\varepsilon)b_n,$$
for all $n \geq n_0(\varepsilon)$. So, there is some $n_1(\varepsilon)$, such that for all $n \geq n_1(\varepsilon)$ it holds $s(n) \geq n_0(\varepsilon)$. Then, for all $n \geq n_1(\varepsilon)$, we get
$$\frac{x(\tau(n))}{x(n+1)} = \prod_{j=\tau(n)}^{n} \frac{x(j)}{x(j+1)} \geq \prod_{j=\tau(n)}^{n} (1 + (z-\varepsilon)b_j).$$
Therefore it holds
$$z \geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} (1 + (z-\varepsilon)b_j), \tag{27}$$
and so
$$\ln z \geq \liminf_{n\to+\infty} \sum_{j=\tau(n)}^{n} \ln(1 + (z-\varepsilon)b_j).$$
By the monotonicity of the logarithm and the fact that $z \geq 1$, we obtain
$$\ln z \geq \liminf_{n\to+\infty} \sum_{j=\tau(n)}^{n} \ln(1 + (1-\varepsilon)b_j). \tag{28}$$

Next we are going to eliminate the parameter ε from (27) and (28). (For any future reference to this fact we call the procedure **M**ain Step.) To do that, define the sequences
$$a_{\varepsilon,n} := \sum_{j=\tau(n)}^{n} \ln(1 + (1-\varepsilon)b_j) \text{ and } a_n := \sum_{j=\tau(n)}^{n} \ln(1 + b_j)$$
and observe that
$$0 \leq |a_{\varepsilon,n} - a_n| = a_n - a_{\varepsilon,n} = \sum_{j=\tau(n_n)}^{n} \Big(\ln(1 + b_j) - \ln(1 + (1-\varepsilon)b_j) \Big)$$

$$= \sum_{j=\tau(n)}^{n} \ln\Big(1 + \frac{\varepsilon b_j}{1+(1-\varepsilon)b_j}\Big) \leq \sum_{j=\tau(n)}^{n} \frac{\varepsilon b_j}{1+(1-\varepsilon)b_j} \leq \varepsilon \sum_{j=\tau(n)}^{n} b_j.$$

This fact together with (26) implies that $\lim_{\varepsilon\to 0}(a_{\varepsilon,n} - a_n) = 0$, uniformly for all large n. Therefore, given $\delta > 0$ there is ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$ it holds $0 \leq a_n - a_{\varepsilon,n} \leq \delta$, uniformly for all large n and so, for all such ε, it holds $0 \leq \liminf_{n\to+\infty}[a_n - a_{\varepsilon,n}] \leq \delta$. This relation means that

$$\lim_{\varepsilon\to 0} \liminf_{n\to+\infty}[a_n - a_{\varepsilon,n}] = 0 \tag{29}$$

and so, from (28), we can get

$$\ln z \geq \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} a_{\varepsilon,n} \geq \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty}[a_{\varepsilon,n} - a_n] + \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} a_n.$$

Therefore, because of (29), we conclude that $\ln z \geq \liminf_{n\to+\infty} a_n$, which implies that

$$z \geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} [1 + b_j] =: B_1 \tag{30}$$

and the Basic Step is done.

Now, we observe that relations (30) and (27) give

$$\ln z \geq \liminf_{n\to+\infty} \sum_{j=\tau(n)}^{n} \ln(1 + (B_1 - \varepsilon)b_j), \tag{31}$$

which, by following the Basic Step, as previously, implies that

$$z \geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} [1 + B_1 b_j] =: B_2.$$

Continue in this way and obtain a increasing sequence (B_i), with $B_0 = 1$, such that

$$z \geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} [1 + B_{(i-1)} b_j] =: B_i,$$

for any $i = 0, 1, 2, \cdots$ Now, fix a $\lambda \in \{0, 1, \cdots\}$. By using the fact that $z \geq B_\lambda$, from

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(27), we obtain

$$
\begin{aligned}
z &\geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} (1+(z-\varepsilon)b_j) \\
&= 1 + \liminf_{n\to+\infty} \Big((z-\varepsilon) \sum_{j=\tau(n)}^{n} b_j + \cdots + (z-\varepsilon)^{k_n+1} \prod_{j=\tau(n)}^{n} b_j \Big) \\
&> \liminf_{n\to+\infty} \Big((z-\varepsilon) \sum_{j=\tau(n)}^{n} b_j + \cdots + (z-\varepsilon)^{k_n+1} \prod_{j=\tau(n)}^{n} b_j \Big).
\end{aligned}
$$

Due to the strict inequality, we can take a constant number $c \in (0,1)$ such that

$$
cz > \liminf_{n\to+\infty} \Big((z-\varepsilon) \sum_{j=\tau(n)}^{n} b_j + \cdots + (z-\varepsilon)^{k_n+1} \prod_{j=\tau(n)}^{n} b_j \Big).
$$

Therefore it holds

$$
\begin{aligned}
cz &> \liminf_{n\to+\infty} [(z-\varepsilon) \sum_{j=\tau(n)}^{n} b_j + \cdots + (z-\varepsilon)^{k_n+1} \prod_{j=\tau(n_m)}^{n_m} b_j] \\
&= (z-\varepsilon) \liminf_{n\to+\infty} \Big[\sum_{j=\tau(n)}^{n} b_j + \cdots + (z-\varepsilon)^{k_n} \prod_{j=\tau(n_m)}^{n_m} b_j \Big] \\
&\geq (z-\varepsilon) \liminf_{n\to+\infty} \Big[\sum_{j=\tau(n)}^{n} b_j + \cdots + (B_\lambda-\varepsilon)^{k_n} \prod_{j=\tau(n)}^{n} b_j \Big] \\
&= \frac{z-\varepsilon}{B_\lambda-\varepsilon} \liminf_{n\to+\infty} \Big[(B_\lambda-\varepsilon) \sum_{j=\tau(n)}^{n} b_j + \cdots + (B_\lambda-\varepsilon)^{k_n+1} \prod_{j=\tau(n)}^{n} b_j \Big] \\
&= -\frac{z-\varepsilon}{B_\lambda-\varepsilon} \\
&+ \frac{z-\varepsilon}{B_\lambda-\varepsilon} \liminf_{n\to+\infty} \Big[1 + (B_\lambda-\varepsilon) \sum_{j=\tau(n)}^{n} + \cdots + (B_\lambda-\varepsilon)^{k_n+1} \prod_{j=\tau(n)}^{n} b_j \Big],
\end{aligned}
$$

namely

$$
cz > -\frac{z-\varepsilon}{B_\lambda-\varepsilon} + \frac{z-\varepsilon}{B_\lambda-\varepsilon} \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} [1+(B_\lambda-\varepsilon)b_j].
$$

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Therefore we have

$$cB_\lambda + 1 + c\varepsilon \frac{B_\lambda - z}{z - \varepsilon} > \liminf_{n\to+\infty} [\prod_{j=\tau(n)}^{n} [1 + (B_\lambda - \varepsilon) b_j].$$

Getting the limits in both parts as ε tends to zero and following the previous Basic Step we can eliminate ε, and finally, obtain

$$B_\lambda + 1 > cB_\lambda + 1 \geq \liminf_{n\to+\infty} \prod_{j=\tau(n)}^{n} (1 + B_\lambda b_j) = B_{\lambda+1},$$

contrary to (E_λ). The proof is complete.

3. Applications

First we give four examples where both theorems are applicable.

Application 3.1 Motivated from [7] we consider the equation

$$\Delta x(n) + \frac{1}{3e} x(n-1) + \frac{2}{3e} x(n-2) = 0.$$

Now we see that $\tau(n) = n - 1$, $k_n = k = 1$ and $b_n = \frac{1}{e}$. Thus condition (9) is satisfied and so all solutions have property $(\mathbf{P})$. On the other hand condition (2.3) of [7] is not true and therefore Theorem 2.1 of [7] is not applicable. Notice that condition (E_λ) is also satisfied for at least $\lambda = 1$.

Application 3.2 Consider the difference equation

$$\Delta x(n) + \frac{l}{an} x(n-1) + \frac{n-l}{an} x(n - 6 + (-1)^n) = 0, \ \ n = 6, 7, 8, \cdots$$

Here we have $l > 0$, $\tau(n) = n - 1$, $b_n = \frac{1}{a}$ and $k(n) = k = 1$. Observe that condition (9) takes the form $\frac{1}{a} > \frac{1}{4}$ and it is true for $a < 4$. Also condition (E_λ), for $\lambda = 0$, holds for $a < 1 + \sqrt{2}$. We can observe that when we increase the parameter a but keeping it smaller than 4, condition (E_λ) holds for large values of λ. For example[3], if $a = 3,84$ we obtain $B_{22} = 7,30826$ and $B_{23} = 8,42853$, thus condition (E_{22}) is true. Therefore, for all such values of a both theorems are applied and so all solutions have the property $(\mathbf{P})$.

[3]by using The Graphing Calculator 3.5, of Pacific Tech. 1993-2004

Application 3.3 Consider the integral difference equation

$$\Delta x(n) + n\int_{n-6}^{n-5} \frac{x(s)}{s} ds = 0, \quad n = 7, 8, \cdots$$

Here we have $\tau(n) = n - 5$, $k_n = k = 5$, and therefore

$$\lim_{n\to+\infty} \frac{1}{5} \sum_{j=n-5}^{n-1} j \ln \frac{j-5}{j-6} = \frac{1}{5} > \frac{5^5}{6^6} = \lim_{n\to+\infty} \frac{k^k}{(k+1)^{k+1}}.$$

Thus Theorem 2.1 applies. Moreover Theorem 2.2 (with $\lambda = 0$,) can also be applied since we have $b_n := n\ln(\frac{n-5}{n-6})$, and

$$2 < 2^6 = \lim_{n\to+\infty} \prod_{j=n-5}^{n} \Big(1 + j\ln\frac{j-5}{j-6}\Big).$$

Therefore property (**P**) is true for all solutions.

Application 3.4 For the difference equation

$$\Delta x(n) + \frac{1}{3}x(n-1) + \int_e^3 \frac{x(n-s)ds}{s^2}, \quad n = 0, 1, \cdots \tag{32}$$

we have $\tau(n) := n - 1$, $k_n := 1$ and

$$b_n := \frac{1}{3} + \int_e^3 \frac{ds}{s^2} = \frac{1}{e}.$$

For this equation property (P) is satisfied. Indeed, observe that condition (9) becomes $\frac{1}{e} > \frac{1}{4}$, which is true. Also the first three terms of the sequence B_λ are defined by $B_0 = 1$, $B_1 = 1,87109$, $B_2 = 2,8504$ and $B_3 = 4,1969$. We see that condition (E_λ), for $\lambda = 2$, becomes $4,1969 - 2,8504 = 1,3465 > 1$, which is a true relation.

Next we give a difference equation where Theorem 2.2 is applicable, but not Theorem 2.1.

Application 3.5 Consider the difference equation

$$\Delta x(n) + \frac{n}{5}e^{x^2(n-7)} \int_{n-6}^{n-5} \frac{x(s)}{s} ds = 0, \quad n = 7, 8, \cdots$$

Here, again, as in the previous application, we have $\tau(n) = n - 5$, $k_n = k = 5$ and

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$b_n := \frac{n}{5}\ln(\frac{n-5}{n-6})$. Therefore

$$\lim_{n\to+\infty}\frac{1}{5}\sum_{j=n-5}^{n-1}\frac{j}{5}\ln\frac{j-5}{j-6}=\frac{1}{5^2}<\frac{5^5}{6^6}=\lim_{n\to+\infty}\frac{k_n^{k_n}}{(k_n+1)^{k_n+1}}.$$

Thus Theorem 2.1 does not apply. On the other hand Theorem 2.2 applies, since, for $\lambda = 0$, we have

$$B_1 - B_0 = \lim_{n\to+\infty}\prod_{j=n-5}^{n}\Big(1+\frac{j}{5}\ln\frac{j-5}{j-6}\Big)-1$$

$$=(1+\frac{1}{5})^6-1=\frac{6^6-5^6}{5^6}=\frac{46656-15625}{15625}\approx 1,9859>1.$$

So property (**P**) is satisfied.

Application 3.6 Finally we shall discuss the simple linear difference equation $x_{n+1} - x_n + px_{n-1} = 0$, where $p > 0$. Here we have $\tau(n) = n - 1$, $k_n = 1$ and $b_n = p$. Let us compute the first terms of the sequence (B_l). Indeed, we obtain that relation (E_i), for $i = 0, 1, 2, \cdots, 35$ holds as equality for the values of the parameter p described in the following tabular:

$E_0 : p \geq 0,414214$	$E_9 : p \geq 0,277820$	$E_{18} : p \geq 0,262560$	$E_{27} : p \geq 0,257165$
$E_1 : p \geq 0,370810$	$E_{10} : p \geq 0,274973$	$E_{19} : p \geq 0,261700$	$E_{28} : p \geq 0,256794$
$E_2 : p \geq 0,344187$	$E_{11} : p \geq 0,272578$	$E_{20} : p \geq 0,260926$	$E_{29} : p \geq 0,256449$
$E_3 : p \geq 0,313208$	$E_{12} : p \geq 0,270518$	$E_{21} : p \geq 0,260228$	$E_{30} : p \geq 0,256130$
$E_4 : p \geq 0,303476$	$E_{13} : p \geq 0,268732$	$E_{22} : p \geq 0,259594$	$E_{31} : p \geq 0,255834$
$E_5 : p \geq 0,295935$	$E_{14} : p \geq 0,267173$	$E_{23} : p \geq 0,259018$	$E_{32} : p \geq 0,255560$
$E_6 : p \geq 0,289948$	$E_{15} : p \geq 0,265804$	$E_{24} : p \geq 0,258493$	$E_{33} : p \geq 0,255304$
$E_7 : p \geq 0,285100$	$E_{16} : p \geq 0,264594$	$E_{25} : p \geq 0,258013$	$E_{34} : p \geq 0,255065$
$E_8 : p \geq 0,281110$	$E_{17} : p \geq 0,263519$	$E_{26} : p \geq 0,257572$	$E_{35} : p \geq 0,254843$

In this tabular we see that the sequence of the minimum values of p approaches the value $0,25$, which is the exact value of p generating oscillatory solutions. Indeed, as it is known from the basic theory of linear difference equations, if the discriminant of the characteristic equation $x^2 - x + px = 0$ is negative, the solutions are oscillatory

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and this happens when $p > 0,25$.

4. Discussion

In this paper we provide two kinds of conditions: An exact condition in Theorem 2.1, and a one-parameter condition in Theorem 2. Obviously, Theorem 2.1 can be applied or not to a difference equation, because condition (9) can be immediately checked. And if this condition is satisfied, then the answer is positive. But, for Theorem 2.2, the case is a little bit more complicated. Indeed, given a difference equation we have to check whether or not there exists a value of the parameter λ, namely, a term of the sequence B_λ which satisfies condition (E_λ). Of course, this fact is not so easy to be checked for all values λ, even for simple types of difference equations. These facts are demonstrated in the applications given in Section 3.

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