

Solution of Nonlinear Boundary Layer Problems with one Boundary Conditions at Infinity using Shooting Type Differential Transform Algorithm

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Received: 01 February 2021/ Accepted: 01 March 2022/ Published online: 22 March 2022

Sacred Heart Research Publications 2017

Abstract

In this paper, a novel variant of Differential Transform Method (DTM), Shooting Type Differential Transform Algorithm (STDTA) is used to solve some problems in fluid mechanics with one of the boundary conditions at infinity. The analytical solution obtained by using this method is effective in the sense that accuracy is found more. To apply the condition at infinity two procedures are used. STDTA is an effective method for solving these types of problems.

Key words: Nonlinear boundary layer problems, analytical solutions, STDTA, Pade Approximants, Boundary conditions at infinity, Final value theorem.

AMS classification: 35G30, 74G10.

1. Introduction

Most of the problems occurring in boundary layer theory turn out to be nonlinear ordinary differential equations with one of the conditions at infinity. A novel method, Shooting Type Differential Transform Algorithm (STDTA), a variation of Differential Transform Method (DTM), is applied to solve these type of problems. Shooting Type Adomian Method (STAM) [12] and Shooting Type Laplace Adomian Decomposition Algorithm (STLADA) [13] are used to solve these type of problems. The analytical solution obtained through DTM is a power series. Hence we use (n/n) Pade approximants in two different procedures to get the value of the parameter; first one being a direct application and the second uses the Final value theorem for Laplace transform. The convergence is very fast for a lower number of approximations. The convergence rate is more with a few approximations. Two boundary layer problems

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of different orders are considered with one of the boundary conditions at infinity.

2. Shooting Type Differential Transformation Algorithm

Differential Transform Method is an analytical method based on Taylor expansion. The concept of differential transform method is first proposed by Zhou. It is applied to electric circuit analysis problems for solving initial value problems. After words, it is applied to several systems and differential equations. Several authors have used this method and their variations to solve initial value problems [8], Integro-differential equations [11, 14], Difference equations [1], Partial Differential equations [2, 5] and system of differential equations [3].

Definition 2.1 The one-dimensional differential transform of a function $y(x)$ at the point $x = x_0$ is defined as follows [7]:

$$Y(k) = \frac{1}{k!} \left[\frac{d^k}{dx^k} y(x) \right] \Big|_{x=x_0} \quad (1)$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.

Definition 2.2 The differential inverse transform of $Y(k)$ is defined as follows:

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \quad (2)$$

From (1) and (2) we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dx^k} y(x) \right] (x - x_0)^k \quad (3)$$

The following theorems can be deduced from the above definitions:

Theorem 2.3 If $f(x) = g(x) \pm h(x)$, then

$$F(k) = G(k) \pm H(k). \quad (4)$$



Theorem 2.4 If $f(x) = \lambda g(x)$, then

$$F(k) = \lambda G(k) \quad (5)$$

where λ is a constant.

Theorem 2.5 If $f(x) = \frac{d^n}{dx^n}g(x)$, then

$$F(k) = \frac{(k+n)!}{k!}G(k+n). \quad (6)$$

Theorem 2.6 If $f(x) = g(x)h(x)$, then

$$F(k) = \sum_{k_1=0}^k G(k_1)H(k-k_1). \quad (7)$$

Theorem 2.7 If $f(x) = g(x)\frac{d^2}{dx^2}h(x)$, then

$$F(k) = \sum_{r=0}^k (k-r+1)(k-r+2)G(r)H(k-r+2). \quad (8)$$

Let X be a Banach space and consider the functional equation defined on the Banach space X , $Tu = b$ where T is an operator from X to X , b is a given function in X , and for each b satisfying the functional equation [4, 15] is the solution. Assume that the functional equation has a unique solution for each $b \in X$.

The operator T consists of non-linear and linear terms, and the linear term is split into $L_1 + L_2$, where L_1 is invertible and it contains the highest order derivative of the given problem and L_2 is the rest of the linear operator.

Thus $T = L_1 + L_2 + N$ where N is a non-linear operator. Hence the functional equation becomes

$$L_1u = b - L_2u - Nu.$$

Applying the Differential Transform to the above equation, the transformed equation is obtained as

$$U(k+n) = \frac{F(k)}{(k+n)!} \quad (9)$$

where $F(k)$ is the differential transform of $f(x, u, u', u'', \dots, u^{(n-1)}) = b - L_2 u - Nu$. Then transformed conditions given with the problem can be written as

$$U(k) = J, U(m) = \sum_{j=0}^N \prod_{i=1}^{m-1} (j-i) U(k) = I_m, (m < n), \quad (10)$$

where m is the order of the derivative in the boundary conditions and J, I_m are real constants. Using equations (9) and (10) the values of $U_i, i = 1, 2, 3, \dots$ can be determined and then using inverse differential transformation, the following approximate solution can be determined as

$$U_N = \sum_{k=0}^N U(k) x^k. \quad (11)$$

To begin with, initial value problems are solved by DTM. The authors [14] have introduced Shooting Type Differential Transform Algorithm (STDTA), to solve boundary value problems efficiently.

Shooting Type Differential Transform Algorithm (STDTA) consists of the following steps: (i) Converting the given boundary value problem into an initial value problem by assuming the missing initial conditions; (ii) applying the DTM to the converted initial value problem and (iii) finding the value(s) of the assumed constant(s) by applying the boundary condition(s) at the second point. If the second point is ∞ , then we have to resort diagonal Pade approximants. This can be done in two different procedures.

3. The Pade Approximants

Let $\sum_{i=0}^{\infty} a_i x^i$ be the series representation a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} a_i x^i. \quad (12)$$

The Pade approximant is a rational fraction and the $[L/M]$ Pade approximant [6] is taken as

$$[L/M] = \frac{N_L(x)}{D_M(x)} \quad (13)$$

where $N_L(x)$ is a polynomial of degree at most L and $D_M(x)$ is a polynomial of degree at most M .

Here

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots, \quad (14)$$

$$N_L(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_Lx^L, \quad (15)$$

$$D_M(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_Mx^M. \quad (16)$$

Notice that, in (13), the number of coefficients in the numerator and denominator are $L + 1$ and M , respectively, since the constant term in the denominator is taken as 1. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged.

If the notation of formal power series, [10]

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_Lx^L}{q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_Mx^M} + O(x^{L+M+1}) \quad (17)$$

Cross-multiplying (17), we find that

$$\begin{aligned} (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_Mx^M) \\ = (p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_Lx^L) + O(x^{L+M+1}). \end{aligned} \quad (18)$$

From (18) we obtain a set of equations, by equating like terms, and solving them gives the $[L/M]$ Padé approximant. We are using the Final value theorem for Laplace transform [12, 13].

4. Illustrative Examples

Example 4.1 Consider the second-order nonlinear ordinary differential equation [9, 10]

$$2u'' + u - u^2 = 0, u(0) = 0, u(\infty) = 1 \quad (19)$$

that governs the steady mixed convection flow past a plane of arbitrary shape under the boundary layer and Darcy-Boussinesq approximations. Equation (19) admits an exact solution given by

$$u(x) = \frac{1}{2} \left(-1 + 3 \tanh^2 \left[\frac{1}{4} (\sqrt{2}x + 4 \operatorname{arctanh} \left[\frac{1}{\sqrt{3}} \right]) \right] \right) \quad (20)$$

Applying STDTA to the equation (19), we get

$$U(k+2) = \frac{-U(k) + \sum_{k_1=0}^k U(k_1)U(k-k_1)}{2(k+1)(k+2)}, k \geq 0 \quad (21)$$

with $U(0) = 0$. We assume $u'(0) = \alpha$. Hence $U(1) = \alpha$ and the value of α to be found out by applying the condition $u(\infty) = 1$.

Substituting $k = 0, 1, 2, 3, \dots$, in (21), we get

$$U(2) = 0, U(3) = \frac{-\alpha}{12}, U(4) = \frac{\alpha^2}{24}, U(5) = \frac{\alpha}{480}, U(6) = \frac{-\alpha^2}{288}, U(7) = \frac{1}{84} \left[\frac{-\alpha}{480} + \frac{\alpha^3}{12} \right],$$

$$U(8) = \frac{\alpha^2}{7680}, U(9) = \frac{1}{144} \left[\frac{\alpha}{40320} - \frac{30\alpha^3}{2016} \right], U(10) = \frac{1}{180} \left[\frac{-114\alpha^2}{322560} + \frac{15\alpha^4}{4032} \right].$$

$$\text{Therefore } u_{10}(x) = ax - \frac{\alpha}{12}x^3 + \frac{\alpha^2}{24}x^4 + \frac{\alpha}{480}x^5 - \frac{\alpha^2}{288}x^6 + \frac{1}{84} \left[\frac{-\alpha}{480} + \frac{\alpha^3}{12} \right]x^7$$

$$+ \frac{\alpha^2}{7680}x^8 + \frac{1}{144} \left[\frac{\alpha}{4032} - \frac{30\alpha^3}{2016} \right]x^9 + \frac{1}{180} \left[\frac{-114\alpha^2}{322560} + \frac{15\alpha^4}{4032} \right]x^{10}. \quad (22)$$

Diagonal Pade approximants are used to estimate the parameter α , through two procedures. Note that the condition at infinity applies to diagonal approximants produces a finite value.

In the first procedure, Pade approximant is directly applied to $u(x)$. Let

$$u(x) = \frac{c_0 + c_1x}{1 + c_2x} \quad (23)$$

be the $[1/1]$ Pade. Cross multiplying, expanding and equating (23) up to the term x^2 , we get $c_0 = 0, c_1 = \alpha, c_2 = 0$. Applying the condition $u(\infty) = 1$ does not yield any result about α . Then

$$u(x) = \frac{c_0 + c_1x + c_2x^2}{1 + c_3x + c_4x^2} \quad (24)$$

be the $[2/2]$ Pade. Cross multiplying, expanding and equating (24) up to the term x^4 , we get $c_2 = 0.5\alpha^2, c_4 = 0.083333333$. Using these values of c_2 and c_4 we get $\alpha = 0.40824829$.

Similarly applying the condition $u(\infty) = 1$ to $[3/3]$ and $[4/4]$ Pade give the values of α as $\alpha = 0.43969686$ and $\alpha = 0.419133314$. We see that the value of α is converging. The exact value of α is 0.40824829, which is the value obtained in $[2/2]$

Pade. Comparison of the exact solution with u_n is shown in Table 1.

x	$u_1 = u_2$	u_3	u_4	u_5	Exact
0.2	0.08164966	0.08137749	0.08138860	0.08138888	0.08138884
0.4	0.16329932	0.16112199	0.16129976	0.16130848	0.16130621
0.6	0.24494897	0.23760050	0.23850050	0.23856664	0.23854154
0.8	0.32659863	0.30918004	0.31202448	0.31230318	0.31216625
1.0	0.40824829	0.37422750	0.38117204	0.38202256	0.38151634

Table 1: Convergence of u_n with Exact solution

In the second procedure, Laplace transform of the dependent variable is taken, multiplied by s and Final value Theorem is used to get the value of the parameter. Here, we get

$$su(s) = \frac{\alpha}{s} - \frac{\alpha}{2s^3} + \frac{\alpha^2}{s^4} + \frac{\alpha}{4s^5} - \frac{5\alpha^2}{2s^6} + \frac{60}{s^7} \left[\frac{-\alpha}{480} + \frac{\alpha^3}{12} \right] + \frac{21\alpha^2}{4s^8} + \dots \quad (25)$$

The first diagonal Pade is assumed as

$$sL[u(x)] = \frac{c_0 + \frac{c_1}{s}}{1 + \frac{c_2}{s}} \quad (26)$$

Cross multiplying, expanding and equating (26) up to the term $\frac{1}{s^2}$, we get $c_0 = 0$, $c_1 = \alpha$, $c_2 = 0$. The condition $u(\infty) = 1$ turns as $\lim_{s \rightarrow 0} su(s) = 1$. The application to first diagonal Pade approximant gives the value of α as 0. The successive diagonal Pade approximants of $su(s)$ give the value of α respectively as 0.5, 0.44557584, 0.42970601. Table 2 presents a comparison with the exact solution.

x	u_4	u_6	u_8	Exact
0.2	0.09968330	0.08883160	0.08566728	0.08138888
0.4	0.19760000	0.17607237	0.17208865	0.16130621
0.6	0.29235000	0.26043724	0.25112800	0.23850049
0.8	0.38293333	0.34096125	0.32872558	0.31216624
1.0	0.46875000	0.41695581	0.40193699	0.38151634

Table 2: Convergence of u_n with Exact solution

Example 4.2 Consider the nonlinear boundary-layer problem $\phi''' + \phi\phi'' - \phi'^2 + 1 = 0$ with boundary conditions $\phi(0) = 0, \phi'(0) = 0, \phi'(\infty) = 1$. This equation arises in plane stagnation point flow [12, 13].

Applying STDTA to the above equation, we get

$$\phi(k+3) = \frac{1}{(k+1)(k+2)(k+3)} \left\{ -\sum_{r=0}^k (k-r+2)(k-r+1)\phi(r)\phi(k-r+2) \right. \\ \left. + \sum_{r=0}^k (r+1)(k-r+1)\phi(r+1)\phi(k-r+1) - \delta(k-0) \right\}, k \geq 0 \quad (27)$$

$\phi(0) = 0, \phi'(0) = 0$ and let $\phi''(0) = \alpha$. Then $\Phi(0) = 0, \Phi(1) = 0$ and $\Phi(2) = \frac{\alpha}{2}$.

Proceeding as in the previous problem, we get

$$\phi(x) = \frac{\alpha x^2}{2} - \frac{x^3}{6} + \frac{\alpha^2 x^5}{120} - \frac{\alpha x^6}{360} + \frac{x^7}{2520} - \frac{\alpha^3 x^8}{40320} + \frac{\alpha^2 x^9}{90720} + \dots \quad (28)$$

Therefore

$$\phi'(x) = \alpha x - \frac{x^2}{2} + \frac{\alpha^2 x^4}{24} - \frac{\alpha x^5}{60} + \frac{x^6}{360} - \frac{\alpha^3 x^7}{5040} + \frac{\alpha^2 x^8}{10080} - \dots \quad (29)$$

In the first procedure, Pade approximant is directly applied to ϕ' . Let $\phi'(x) = \frac{c_0 + c_1 x}{1 + c_2 x}$, where $c_i, 0 \leq i \leq 2$ to be determined. Cross multiplying, equating and simplifying we get $c_0 = 0, c_1 = \alpha, c_2 = \frac{1}{2\alpha}$.

Applying the condition $\phi'(\infty) = 1$, we get $\alpha = \pm 0.7071068$. Similarly applying the condition $\phi'(\infty) = 1$ to second, third and fourth diagonal Pade give the values of α as $\alpha = \pm 1.41421356, \alpha = 1.24466595$ and $\alpha = 1.22403746$ respectively.

In the second procedure, Laplace transform of the dependent variable is taken, multiplied by s and Final Value Theorem for Laplace transform is used to get the value of the parameter. Here, we get

$$s\phi(s) = \frac{\alpha}{s} - \frac{1}{s^2} + \frac{\alpha^2}{s^4} - \frac{2\alpha}{s^5} + \frac{2}{s^6} - \frac{\alpha^3}{s^7} + \frac{4\alpha^2}{s^8} + \dots$$

The first diagonal Pade approximant for this function is taken as $sL[\phi'(x)] = \frac{c_0 + \frac{c_1}{s}}{1 + \frac{c_2}{s}}$, $c_i, 0 \leq i \leq 2$ are constants to be determined. The condition $\phi'(\infty) = 1$ turns as $\lim_{s \rightarrow 0} s\phi(s) = 1$. The application to first diagonal Pade approximant gives the value of α as ± 1 . The successive diagonal Pade approximants of $s\phi(s)$ give the values of α respectively as 1.27201965, ± 1 and 1.23285265. The exact value of α is 1.2326 [12, 13]. Thus, we observe that in the second procedure faster convergence is occurring.

5. Conclusion

In the present paper, we applied STDTA to two different boundary layer problems of different orders and used two different procedures of handling the boundary condition at infinity. Among all these methods this has received remarkable attention due to its simplicity and easy iterative procedure. Advantage of this method is to minimize the tedious computational work when providing the series solution with a fast convergence rate. Computations are performed using Sage Math.

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