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On Dimensions of Invariant Tensor Fields on MWH Spaces with Subgroup of Type $\mathrm{SL}(2) \otimes \mathrm{SO}(2n)$, $n > 3$

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Abstract

We consider invariant tensors fields on the Manturov–Wolf homogeneous spaces and calculate the dimensions of spaces of tensor fields valency 2, 3 and 4.

Key words: Homogeneous Riemannian space, invariant tensor field, Isotropy group, dimension

AMS classification: 53C30

1 Introduction

An isotropically irreducible homogeneous space is a homogeneous Riemannian space whose isotropy group is irreducible [2, 3, 9]. The application of our construction, described in part 3, allows us to construct tensors invariant with respect to the isotropy group [7, 8]. For these spaces in the tangent space of any his point, only trivial subspaces are invariants under the action of the isotropy group. Therefore, invariants with respect to the group of motion of isotropically irreducible homogeneous spaces, can be described in terms of invariant tensor fields on these spaces [6].

The paper is organized as follows. In Section 2 we give the necessary definitions and facts. In Section 3 we pose the problem of finding the dimensions of spaces invariant tensor fields on the MWH space. In Section 4 we give the formula of expansion of the tensor square of an isotropic representation of the given MWH space onto direct sum of irreducible representations and prove Theorem 4.1, we also comput the dimensions of representation spaces. In Section 5 prove our main result Theorem 5.1 on the dimensions of spaces of invariant tensor fields valency 2, 3 and 4.

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2 Preliminaries

Let $M = \mathfrak{G}/\hbar$ is homogeneous Riemann spaces of the left adjoint classes $g\hbar$, $g \in \mathfrak{G}$, where $\mathfrak{G}$ is Lee group and $\hbar \subset \mathfrak{G}$ – compact subgroup. The action of the group $\mathfrak{G}$ on homogeneous spaces M is transitively, i.e. for each pair of points $g_1\hbar, g_2\hbar \in M$ there exists an element $g \in \mathfrak{G}$, such that $g_2 = gg_1$.

The element $h \in \mathfrak{G}$ will mapped the point of homogeneous space $e\hbar \in M$ in yourself if and only if $h \in \hbar$:

$$h(e\hbar) = (he)\hbar = h\hbar = \hbar = e\hbar.$$

Here $e \in \mathfrak{G}$ is unit group element. Each action of the subgroup $\hbar$ on the homogeneous space $M = \mathfrak{G}/\hbar$ generates a linear transformation of the tangent space of this homogeneous space at the $e\hbar$ point. The set of such transformations, relative to the operation of composing linear transformations, is a linear group called by the isotropy group of the homogeneous space M.

Denote by G and H the compact Lie algebras of the Lie groups $\mathfrak{G}$ and $\hbar$, respectively. For each $h \in H$ we consider the linear map $\mathrm{Ad}_h : G \to G$ defined by the formula:

$$\mathrm{Ad}_h(g) = [h, g] = hg - gh, \quad h \in H, \quad g \in G.$$

The conversion $h \to \mathrm{Ad}_h$, $h \in H$, is a representation of the Lie algebra H in the linear space G and the linear subspace $H \subset G$ is invariant under this representation.

The subgroup $\hbar$ is compact, therefore each of its representations is completely reducible, i.e. expanses as a direct sum of irreducible representations. Consequently, there exist subspace $B \subset G$ that is invariant with respect to the $h \to \mathrm{Ad}_h$, $h \in H$ representation, which is orthogonal to the linear space H with respect to the Cartan metric in G and there exist direct sum expansion:

$$H = H \oplus B.$$

The linear space B can be identified with the tangent space of the homogeneous space $M = \mathfrak{G}/\hbar$ at the point $e\hbar$. The Lie algebra of the transformations $h \to \mathrm{Ad}_h$, $h \in H$, of the linear space B is called by the isotropy algebra of the homogeneous space. The corresponding linear group is an isotropy group of homogeneous spaces.

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Obviously, the isotropy algebra of a homogeneous space is determined by some representation χ of the algebra H:

$$\chi : h \to \mathrm{Ad}_h \,|_B, \quad h \in H, \quad \text{where} \quad \mathrm{Ad}_h(b) = [h, b], \quad h \in H, b \in B.$$

The representation χ is called by the isotropic representation of the homogeneous Riemannian space $M = \mathfrak{G}/\hbar$.

Each tensor in a linear space B invariant under the isotropy algebra of a homogeneous Riemannian space induces a tensor field, invariant under the action of the group $\mathfrak{G}$ on a homogeneous space.

In further, homogeneous Riemannian spaces $M = \mathfrak{G}/\hbar$ whose isotropy group is irreducible, i.e. isotropically irreducible homogeneous Riemannian spaces, we will called by the Manturov–Wolf homogeneous (MWH) spaces. Obviously, for the MWH spaces the isotropic representation is irreducible. The isotropy group is a linear group defined by the corresponding isotropic representation of the Lie algebra [9], with a certain high weight (Coxeter–Dynkin diagram).

3 Formulation of the problem

We consider the MWH space $M = \mathfrak{G}/\hbar$, where the subgroup $\hbar$ has of type $\mathrm{SL}(2) \otimes \mathrm{SO}(2n)$, $n > 3$ and the inclosure of the subgroup $\hbar$ in the linear group $\mathfrak{G} = \mathrm{S}p(N)$ is given by the transformation which has following high weight:

$$\begin{pmatrix} \overset{1}{0} & & & & & & \\ & & & & & & \\ \otimes & & & & & & \\ \overset{1}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0} \end{pmatrix}.$$

This transformation has dimension $N = 4n$ [6].

The isotropic representation χ of this space is given by the following highest

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weight [9]:

$$\begin{pmatrix} \overset{2}{0} \\ \\ \otimes \\ \\ \overset{2}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0} \end{pmatrix}.$$

We calculate the dimension of the tensor field of valency 2 on the MWH space $M = \mathfrak{G}/\hbar$, invariant with respect to the group $\mathfrak{G}$, generated by the tensors in the tangent space, invariants with respect to the isotropy group. For this, we consider the tensor square $\chi \otimes \chi$ of the isotropic representation χ.

On the whole, $\chi \otimes \chi$ is completely reducible representation, i.e. it can always be represented as a direct sum of irreducible representations:

$$\chi \otimes \chi = \bigoplus_{i=1}^{m} k_i \cdot \varphi_i,$$

where φ_i, $i = 1, 2, \ldots, m$, – irreducible representations of Lie algebra H and k_i, $i = 1, 2, \ldots, m$, – non-negative integers showing the multiplicity of representations φ_i into expansion of the tensor square. The multiplicity of the one-dimensional, trivial representation in the direct sum expansion of the tensor square is equal to the dimension of the space of invariant tensor fields of valencies 2. Similarly, the expansions of $\chi \otimes \chi \otimes \chi$ and $\chi \otimes \chi \otimes \chi \otimes \chi$ into direct sums of irreducible representations will determine the dimensions of the spaces of invariant tensor fields of valencies 3 and 4, respectively.

To the solve problem, it is sufficient to solve the problem of expansion into irreducible representations of the tensor square $\chi \otimes \chi$ for isotropic representations and to calculate the coefficients of the components in this expansion. Really, by Shur's lema, tensor product of two irreducible representations in expansion into a direct sum of irreducible representations contain one-dimensional, trivial component if and only if when this two representations are contra-gradient to each other. Since the isotropic representation is self-contracting, it follows that the dimension of the space of invariant tensors field of valence 3 is equal to the coefficient with which the isotropic representation χ is included in the expansion $\chi \otimes \chi$. The dimension of the space of invariant tensors valence 4 is equal to the sum of the product of multiplicities of pairs of mutually contra-gradient representations, that is, the sum of the squared coefficients k_i, $i = 1, 2, \dots, m$, into the expansion of the tensor square of the isotropic representation [7, 8].

Thus, to solve the problem it is sufficient calculating the multiplicities of the irreducible representations in the expansion of the tensor square $\chi \otimes \chi$, for the isotropic representation χ of the corresponding homogeneous space.

4 The expansion of the tensor square into a direct sum of irreducible representation

Theorem 4.1 If the irreducible representation ψ given by the following highest weight:

$$\overset{2}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0},$$

where $n > 3$, then the tensor square $\psi \otimes \psi$ expansion into direct sum of irreducible representations as follows:

$$\psi \otimes \psi = k_1\psi_1 \oplus k_2\psi_2 \oplus k_3\psi_3 \oplus k_4\psi_4 \oplus k_5\psi_5 \oplus k_6\psi_6,$$

where representations are given by the following highest weight:

$$\begin{aligned}
\psi_1 &: \ \overset{4}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0},\\
\psi_2 &: \ \overset{2}{0} - \overset{1}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},\\
\psi_3 &: \ 0 - \overset{2}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},\\
\psi_4 &: \ \overset{2}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0},\\
\psi_5 &: \ 0 - \overset{1}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},\\
\psi_6 &: \ 0 - 0 - 0 - 0 - \cdots - 0^{-0}_{-0}
\end{aligned}$$

and $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1$.

Proof: To expansion the tensor square into direct sum, we use the algorithm for finding all the weights of the representation by its highest weight. Highest weight of the representation $\psi \otimes \psi$ are given by the scheme:

$$\overset{4}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0}.$$

We apply inductance with respect to the number n - the dimension of the Cartan subalgebra.

For $n = 4$, the representation ψ are given by the scheme: $\overset{2}{0} - 0^{-0}_{-0}$ and the representation $\psi \otimes \psi$ will have the scheme $\overset{4}{0} - 0^{-0}_{-0}$. The weights Λ_{ψ_i} of the irreducible representations ψ_i into expansion of tensor square will have following form:

$$\Lambda_{\psi_i} = 2 \cdot \Lambda_\psi - m_1\alpha_1 - m_2\alpha_2 - m_3\alpha_3 - m_4\alpha_4,$$

where m_i, $i = 1, 2, 3, 4$, – non-negative integers and α_i, $i = 1, 2, 3, 4$, – simple roots having the following coordinates:

$$\alpha_1 = (2; -1; 0; 0), \quad \alpha_2 = (-1; 2; -1; 0),$$
$$\alpha_3 = (0; -1; 2; -1), \quad \alpha_4 = (0; -1; 0; 1).$$

We obtain the following system of linear inequalities:

$$(4; 0; 0; 0) - m_1(2; -1; 0; 0) - m_2(-1; 2; -1; 0) \\ - m_3(0; -1; 2; -1) - m_4(0; -1; 0; 1) \geq 0.$$

The results of the system solution are given in the following table:

i	$(m_i; m_2; m_3; m_4)$	Λ_{ψ_i}	$\dim \psi_i$	k_i
1	(0; 0; 0; 0)	(4; 0; 0; 0)	294	1
2	(1; 0; 0; 0)	(2; 1; 0; 0)	567	1
3	(2; 0; 0; 0)	(0; 2; 0; 0)	300	1
4	(2; 2; 1; 1)	(2; 0; 0; 0)	35	1
5	(3; 2; 1; 1)	(0; 1; 0; 0)	28	1
6	(4; 4; 2; 2)	(0; 0; 0; 0)	1	1

Comparing the values of the dimensions of the representations spaces, we obtain that for $n = 4$ in expansion of tensor square there exist the components given in table and only them.

In case $n = 5$ we have the following system of linear inequalities:

$$(4;0;0;0;0) - m_1(2;-1;0;0;0) - m_2(-1;2;-1;0;0) \\ - m_3(0;-1;2;-1;0) - m_4(0;0;-1;2;-1) - m_5(0;0;-1;0;1) \geq 0.$$

We have the following table of results:

i	$(m_i;m_2;m_3;m_4;m_5)$	Λ_{ψ_i}	$\dim\psi_i$	k_i
1	$(0;0;0;0;0)$	$(4;0;0;0;0)$	660	1
2	$(1;0;0;0;0)$	$(2;1;0;0;0)$	1386	1
3	$(2;0;0;0;0)$	$(0;2;0;0;0)$	770	1
4	$(2;2;2;1;1)$	$(2;0;0;0;0)$	54	1
5	$(3;2;2;1;1)$	$(0;1;0;0;0)$	45	1
6	$(4;4;4;2;2)$	$(0;0;0;0;0)$	1	1

A comparison of the dimensions of the representations spaces shows that for $n = 5$ the set of irreducible representations ψ_i in the expansion of the tensor square is completely exhausted by the components given in the table.

As we see, when passing to the value $n = 5$, new terms in the expansion $\psi \otimes \psi$ do not appear. Thus, for $n > 3$, in the expansion of the tensor square there will be irreducible representations ψ_i, $i = 1,2,3,4,5,6$, which are given by the highest weights:

$$\psi_1 : \overset{4}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0},$$
$$\psi_2 : \overset{2}{0} - \overset{1}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},$$
$$\psi_3 : 0 - \overset{2}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},$$

$$\psi_4: \overset{2}{0} - 0 - 0 - 0 - \cdots - 0^{-0}_{-0},$$
$$\psi_5: 0 - \overset{1}{0} - 0 - 0 - \cdots - 0^{-0}_{-0},$$
$$\psi_6: 0 - 0 - 0 - 0 - \cdots - 0^{-0}_{-0}$$

and only they. Obviously, for the dimensions spaces of the representations ψ_i, $i = 1, 2, 3, 4, 5, 6$, the following conditions hold:

$$\dim(\psi \otimes \psi) = \sum_{i=1}^{6} \dim \psi_i.$$

The dimensions of the spaces of representations of irreducible components and their multiplicities in the expansion of the tensor square $\psi \otimes \psi$ are given in the following table:

i	$\dim \psi_i$	k_i
1	$\frac{n(n+1)(2n+1)(2n+7)}{6}$	1
2	$\frac{n(n+1)(2n-1)(2n+5)}{2}$	1
3	$\frac{(n-1)(n+1)(2n+1)(2n+3)}{3}$	1
4	$n(2n+3)$	1
5	$n(2n+1)$	1
6	1	1

The theorem is proved.

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5 Calculating the dimensions of spaces invariant tensor fields valency 2, 3 and 4

Theorem 5.1 The dimensions of the spaces of invariant tensor fields valency 2, 3 and 4 on the MWH space $M = \mathfrak{G}/\hbar$, which we consider, are equal to the numbers 1, 1 and 18, respectively.

Proof: The isotropic representation χ of the given homogeneous space is gives with: $\chi = \begin{pmatrix} \overset{2}{0} \\ \otimes \\ \psi \end{pmatrix}$, where ψ is the representation with following highest weight:

$$\overset{2}{0} - 0 - 0 - 0 - \cdot - 0^{-0}_{-0}.$$

Obviously, in order to expansion the representation $\chi \otimes \chi$ into a direct sum, it is sufficient to know the decompositions into direct sum of the representations $\psi \otimes \psi$ and $\overset{2}{0} \otimes \overset{2}{0}$.

Using the Clebsch–Gordan formula:

$$\overset{k}{0} \otimes \overset{l}{0} = \overset{k+l}{0} \oplus \overset{k+l-2}{0} \oplus \cdots \oplus \overset{k-l}{0}, \quad k \geq l,$$

we have the following expansion:

$$\overset{2}{0} \otimes \overset{2}{0} = \overset{4}{0} \oplus \overset{2}{0} \oplus 0.$$

By the theorem proved above, we obtain that the components of the direct sum of the expansion $\chi \otimes \chi$ will has following forms: $\varphi_i^j = \begin{pmatrix} \overset{j}{0} \\ \otimes \\ \psi_i \end{pmatrix}$, where indices independently of one another take values: $j = 0, 2, 4$ and $i = 1, 2, 3, 4, 5, 6$.

In the expansion $\chi \otimes \chi$ the multiplicities of the one-dimensional trivial representation and the isotropic representation χ are equal to one, therefore, the dimensions of the spaces of invariant tensor fields of valency 2 and 3 are equal to one.

In case of valency 4, the corresponding dimension is equal to the sum of the squares of the multiplicities of the expansion:

$$\sum_{i=1}^{18} k_i^2 = \sum_{i=1}^{18} 1 = 18.$$

The theorem is proved.

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