

On Nehari solutions

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Received: 08 February 2021/ Accepted: 05 March 2022/ Published online: 30 March 2022

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Abstract

The Nehari solutions are solutions of a superlinear second order ordinary differential equation that possess remarkable properties. This property is the minimization of a certain functional associated with the equation. In this paper we review these properties, recall some problems about the Nehari solutions and solve some unsolved ones.

Key words: Superlinear ordinary differential equation, Nehari solutions, Nehari functional, multiple solutions

AMS classification: 34B15

1. Introduction

The second order ordinary differential equations are in the centre of mathematical physics, mathematical modelling and the theory of ordinary differential equations, both linear and nonlinear. When passing from linear case to the nonlinear one two kinds of model equations are to be considered, sublinear and superlinear. The typical feature of superlinear equations is dependence of the distance between two zeros on the initial conditions. The typical boundary value problems such as

$$x'' = -x^3, \quad x(a) = 0, \quad x(b) = 0 \quad (1)$$

are known to have infinitely many solutions. One of the first results of this kind belongs to Z. Nehari [16], [17], who considered solutions with special characteristics.

Sketch of the Nehari theory

In the works [16], [17] Z. Nehari has created the theory for superlinear differential equations of the second order. Let us provide some results from those papers for the reader convenience.

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Consider the equation

$$x'' + xF(t, x^2) = 0, \quad (2)$$

where function $F : R_+ \times R_+ \rightarrow R_+$ satisfies the conditions:

- (A1) $F(t, s)$ is continuous with respect to t and s for $t, s \geq 0$;
- (A2) $F(t, s) > 0$ for $t \geq 0, s > 0$;
- (A3) there exists $\varepsilon > 0$ such that the function $s^{-\varepsilon}F(t, s)$ is not decreasing on s for fixed t .

Equations of the form

$$x'' + p(t)|x|^{2\varepsilon}x = 0 \quad (3)$$

are included in this class of equations.

Equations (2) are considered together with the boundary conditions

$$x(a) = 0, \quad x(b) = 0. \quad (4)$$

The following existence theorem was formulated ([17, p.152]):

Theorem 1.1 Let Γ_n be the class of functions $x(t)$, possessing the following properties:

- $x(t)$ are continuous and piece-wise continuously differentiable in $[a, b]$;
- there exist numbers $a = a_0 < a_1 < \dots < a_n = b$ such that $x(a_\nu) = Q, \nu = 0, 1, \dots, n$, but $x(t)$ is not identically zero in any $[a_{\nu-1}, a_\nu]$;
-

$$\int_{a_{\nu-1}}^{a_\nu} x'^2 dt = \int_{a_{\nu-1}}^{a_\nu} x^2 F(t, x^2) dt. \quad (5)$$

Then the variational problem

$$H(x) = \int_a^b [x'^2 - G(t, x^2)] dt = \min = \lambda_n(a, b), \quad x \in \Gamma_n, \quad (6)$$

where $G(t, s) = \int_a^b F(t, \xi) d\xi$, has a solution $x_n(t)$ with continuous derivative.

The extremal function $x_n(t)$ vanishes exactly in $n - 1$ points of the interval (a, b) and solves the boundary value problem (BVP for brevity) (2), (4).

The numbers λ_n are strictly increasing together with n . They are called the characteristic numbers of the system (2), (4). We will call them **Nehari numbers**. It is an easy matter to check, that solutions of the BVP (2), (4) satisfy the equality (5) and are, therefore, the comparison functions for the variational problem (6). The Nehari theorem 1.1 defines a special subset of solutions of the BVP (2), (4). Thus, the Nehari's solutions are those solutions, that for given $(a, b), n$ minimize the functional $H(x)$.

For equation (3) one has, using (5), that

$$\begin{aligned} H(x) &= \int_a^b [x'^2 - (1 + \varepsilon)^{-1} p(t) x^{2+2\varepsilon}] dt \\ &= \frac{\varepsilon}{1+\varepsilon} \int_a^b p(t) x^{2+2\varepsilon} dt = \frac{\varepsilon}{1+\varepsilon} \int_a^b x'^2 dt. \end{aligned} \tag{7}$$

Our intent in this paper is to describe Nehari solutions and discuss their properties. Certainly, not all solutions of BVP (2), (4) possess the minimality property. Z. Nehari at the time of writing of his papers asked: 1) is any solution of the BVP (2), (4), which has exactly $n - 1$ zeros in (a, b) , the Nehari solution; 2) for given $(a, b), n$ is the Nehari solution $x_n(t)$ unique.

In the sequel we will consider these and related questions for the equation (2).

2. Problems and questions concerning the Nehari solutions

The Nehari solutions, given the interval (a, b) and a positive integer n , supply a minimal value for the functional $J_n(x)$. This minimal value, up to multiplication with the certain constant, coincides with the Nehari number $\lambda_n(a, b)$.

A number of questions about the Nehari numbers, arises.

Let us mention some of them. We put the symbol ♣ before any formulation of the question concerning the Nehari solutions.

♣1 Is the Nehari solution unique? More generally, how many Nehari solutions can exist for a, b, n given?

♣2 Is the Nehari solution extendable saving the property of minimality?

♣3 Are there other solutions of BVP (2), (4) with the same number of zeros?

♣3 What is the distribution of zeros of Nehari's solutions?

Some of this questions were answered before and some are waiting solving.

Properties of Nehari solutions

The following is true for the Nehari numbers and solutions ([17], Lemma 3.1 and Remark on page 153).

Proposition 2.1 If $a \leq a' < b' \leq b$ then $\lambda_n(a, b) \leq \lambda_n(a', b')$.

Proposition 2.2 $\lambda_n(a, b) \rightarrow +\infty$ as $b - a \rightarrow 0$.

Proposition 2.3 $\lambda_n(a, b) > \lambda_{n-1} > \dots > \lambda_2(a, b) > \lambda_1(a, b) =: \lambda(a, b)$.

Proposition 2.4 $\lambda_n(a, b)$ is continuously dependent on both a and b .

The following comparison result is valid. It was proved in [16, page 115] for $n = 1$ and for any n in [10, Proposition 1].

Proposition 2.5 If $F(t, x) \leq F^1(t, x)$, $t \geq 0$, $x > 0$, then $\lambda_n(a, b) \geq \lambda_n^1(a, b)$, where λ_n and λ_n^1 are the Nehari numbers, corresponding to F and F^1 .

Direct calculations for the unique Nehari solution with $n - 1$ zeros in (a, b) yield the following formulas, p is constant.

Proposition 2.6 For the equation $x'' = -p|x|^{2\varepsilon}x$

$$\lambda_n(a, b) = p^{-1/\varepsilon} A[(b - a)^{-1-1/(1+\varepsilon)} n^2]^{1+1/\varepsilon}, \quad (8)$$

where $A = \frac{\varepsilon}{(1+\varepsilon)^{1/\varepsilon}} (2 + \varepsilon)^{-1} (4\beta^2)^{1+1/\varepsilon}$, $\beta = \int_0^1 \frac{ds}{\sqrt{1-s^{2\varepsilon+2}}}$.

For $\varepsilon = 1$ one has

Proposition 2.7

$$\lambda_n(a, b) = \frac{1}{p} \frac{32}{3} \beta^4 (b - a)^{-3} n^4, \quad (9)$$

where $\beta = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$.

Extendability preserving the property of extremality

Definition 2.8 [10] The Nehari solution on an interval (a, b) is *non-extendable Nehari solution* (not to be mixed with non-extendable solution in a general sense) if it has a zero c next to b , but it is not the Nehari solution in (a, c) .

Definition 2.9 [10] Let $x_\infty(t)$ be a solution of (2) which has infinitely many zeros $t_i, t_0 = a$. If it is the Nehari solution on any interval (a, t_i) , then it is an *oscillatory Nehari solution*.

Remark 2.10 If $x_n(a, b)$ is the Nehari solution, then $-x_n(a, b)$ is also the Nehari solution. When we speak about two different Nehari solutions, we do not distinguish between x and $-x$.

Proposition 2.11 Suppose on the interval $[a, b]$ there exist two Nehari solutions. Then both are non-extendable Nehari solutions.

Proof: Let $x_n(a, b)(t)$ and $y_n(a, b)$ be two different Nehari solutions. Suppose that $x_n(a, b)(t)$ can be extended to $t = c > b$, preserving the extremality property, so $x_N(a, c)$ is the Nehari solution, $N > n$. Construct the comparison function for the Nehari problem as follows. Take $y_n(a, b)$ and continue it with $x_N(a, c)$. Denote this new function $z_N(a, c)$. The Nehari numbers $\lambda_N(a, c)$ for $x_N(a, c)$ and for $z_N(a, c)$ are the same. Therefore $z_N(a, c)$ is also the Nehari solution in the interval (a, c) and it is smooth. Then $x'_n(a, b)(b) = z'_N(a, c)(b) = y'_n(a, b)(b)$. But then, by uniqueness of solutions of Cauchy problems, $x_n(a, b)(t)$ and $y_n(a, b)$ must coincide. This contradicts the assumptions. \square

Examples of multiple Nehari solutions were constructed in the papers [1], [2], [13], [3], [4], [5].

In the papers [1], [2] an example of three solutions for the equation

$$x'' = -q(t)x^3 \tag{10}$$

on the interval $[-1, 1]$ was constructed. The coefficient $q(t)$ was a U-shaped continuous positive valued function. These solutions are depicted in Fig. 1. Both side solutions are the Nehari solutions, the central one is not. Therefore two solutions cannot be continued saving the extremality property by Nehari. Later, in

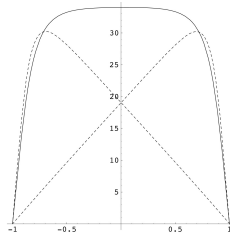


Figure 1: Three solutions, the central one is not the Nehari solution, two side ones are.

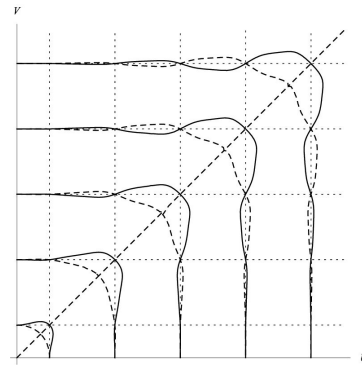


Figure 2: The cross points correspond to solutions of the BVP.

the work [13] R. Kajikiya has studied the same problem, making theoretical analysis, which confirmed the existence of two Nehari solutions. They were called in [14] “non-even least energy solutions.”

In the work [3] equation of the form (10) was studied. The coefficient $q(t)$ was constructed as $q(t) = \frac{k}{\xi(t)^6}$, where $\xi(t)$ was positive valued function. Calculations were made for solutions with exactly 4 zeros in the interval $(-1, 1)$. They were three, and the symmetric solution had greater value in the Nehari problem. So the example of two Nehari solutions $x_5(t)$ and $y_5(t)$ was constructed.

Further investigation of equation (10) with the coefficient $q(t) = \frac{k}{\xi(t)^6}$ revealed the fact that for “high” enough Λ -shaped function $\xi(t)$ there are infinitely many solutions of the boundary value problem (10), $x(-1) = x(1) = 0$. It was possible to express solutions through the lemniscatic functions ([6]). The technical problem of smoothly cluing solutions lead to solving the system of equations of the form

$\Phi(u, v) = 0$, $\Psi(u, v) = 0$. The solution curves of both equations behave like shown in Fig. 2. Any component of these set of curves is associated with the number of zeros of a solution. The cross points mean a solution of the BVP (10), $x(-1) = x(1) = 0$.

The example by R. Moore and Z. Nehari [12] for the equation (10) with piece-wise constant coefficient $q(t)$ was considered in depth in the work [5]. The equation (10), where

$$q(t) = \begin{cases} 2, & t \in [-1, -1 + \varepsilon], \\ 0, & t \in [-1 + \varepsilon, 1 - \varepsilon], \\ 2, & t \in [1 - \varepsilon, 1]. \end{cases} \quad (11)$$

was shown to have infinite number of solutions, vanishing at $t = -1$ and $t = 1$, for any $\varepsilon \in (0, 1)$.

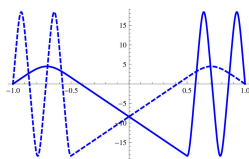


Figure 3:

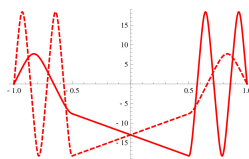


Figure 4:

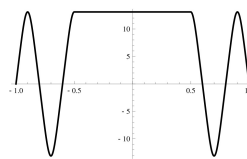


Figure 5:

Several solutions with four internal zeros are depicted in Fig. 3 to Fig. 5.

As to extendability in the general sense it was observed that equations of the form (3) with positive continuous coefficients $p(t)$ can have non-extendable to infinity solutions. All solutions $x(t)$ satisfy $x(t)x''(t) < 0$ in non-zero points, that is, the graphs are concave for $x > 0$, convex for $x < 0$. For small $p(t) > 0$ solutions with finite number of zeros can exist. If a solution has infinite number of zeros, it is called *oscillatory*. Oscillatory solutions can be *singular*. Several examples were constructed in [9], [15], [8], [11].

The following theorem (the generalization of [10], Proposition 7) is valid.

Proposition 2.12 Suppose $p_0(t) \leq \frac{F(t, s)}{s^\varepsilon} \leq p_1(t)$ for $(t, s) \in \{[a, b] \times (0, +\infty)\}$, where p_0 and p_1 are positive valued continuous functions. A singular on $[a, b]$ solution $x(t)$ cannot be oscillatory Nehari solution.

Proof: Let $x(t)$ be a singular on $[a, b)$ solution of equation (2). Consider equation (2) and similar equations

$$x'' = -p_0(t)|x|^{2\varepsilon}x \tag{12}$$

and

$$x'' = -p_1(t)|x|^{2\varepsilon}x \tag{13}$$

Let $\lambda_n^0, \lambda_n, \lambda_n^1$ be the characteristic numbers (the Nehari numbers) for the equations (12), (2), (13) respectively. Then, by Proposition 2.5,

$$\lambda_n^1(a', b') \leq \lambda_n(a', b') \leq \lambda_n^0(a', b') \text{ for any } [a', b'] \subset [a, b]. \tag{14}$$

Let $b_k \rightarrow b$ be consecutive zeros of $x(t)$. One has then

$$\begin{aligned} 1 &= \lambda_{2n}(a, b_{2n})\lambda_{2n}^{-1}(a, b_{2n}) = [\lambda_n(a, b_n) + \lambda_n(b_n, b_{2n})]\lambda_{2n}^{-1} \\ &\geq [\lambda_n^1(a, b_n) + \lambda_n^1(b_n, b_{2n})\lambda_{2n}^0(a, b_{2n})^{-1}] \\ &= p_1^{-1/\varepsilon} A n^{2(1+1/\varepsilon)} [(b_n - a)^{-(1+2/\varepsilon)} + (b_{2n} - b_n)^{-(1+2/\varepsilon)}] \\ &\times [p_0^{-1/\varepsilon} A (2n)^{2(1+1/\varepsilon)} (b_{2n} - a)^{-(1+2/\varepsilon)}]^{-1} \\ &= 4^{-(1+1/\varepsilon)} (p_0 p_1^{-1})^{1/\varepsilon} \left[\left(\frac{b_{2n} - a}{b_n - a} \right)^{1+2/\varepsilon} + \left(\frac{b_{2n} - a}{b_{2n} - b_n} \right)^{1+2/\varepsilon} \right] \end{aligned} \tag{15}$$

Since the second addend in the last row is not bounded by n , the contradiction is obtained. \square

Corollary 2.13 The oscillatory Nehari solution for the equation $x'' = -p(t)|x|^{2\varepsilon}x$, where p is a continuous positive valued function, is extendable to $+\infty$.

3. Conclusions

Superlinear equations of the form (2) possess many interesting properties.

- They have solutions of oscillatory nature with any number of zeros;
- The Dirichlet boundary value problems $x(a) = x(b) = 0$ always have solutions with any number of internal zeros;
- For any n there is a solution that minimizes the Nehari functional in the class of functions possessing $n - 1$ internal zeros;
- It is possible to have a unique (up to multiplication by -1) Nehari solution for

- any n and (a, b) ;
- The Nehari solutions can coexist with other solutions with the same number of zeros;
 - The case that any solution is the Nehari one is trivial (the problem (1));
 - Several Nehari solutions for the same n can exist;
 - They possess the remarkable property of non-extendability saving the extremality;
 - Singular (non-extendable to infinity) solutions are of oscillatory type and they cannot be the oscillatory Nehari solutions;
 - For equations of the form (3) the oscillatory Nehari solutions must extend to infinity.

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