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OBC Fractional order stochastic differential equation driven by L'evy noise

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Abstract

In this article, we consider a class of stochastic fractional differential equations (SFDEs) driven by L'evy noise in the sense of a newly defined OBC-fractional derivative. This is a generalized Caputo type fractional derivative introduced recently by Zaid Odibat and Dumitru Baleanu. Under some suitable sufficient conditions, we have employed fixed point theorem to obtain existence and uniqueness results for the considered equation. We have also presented an example which illustrates the applicability of our obtained results.

Key words:Fractional derivatives and integrals, Initial value problems, Stochastic differential equations, Stochastic integral equations, Fixed-point theorems

AMS classification:26A33, 34A12, 60H10, 60H20, 47H10

1 Introduction

The topic fractional calculus has a very interesting historical background which starts from 1695 and speeds up only recently when computers came and we got to know how good the fractional operators can be in modeling the real world physical phenomena. The traces of that history can be seen in [22], a detailed and more advanced historical background is given in [18]. The literature for the topic can be found in the books [6, 13, 18, 20, 21]. The application part of the fractional calculus is the part which motivates many researchers to work on it. It has been applied in various fields of science and engineering including some practical problems like, public economics [3], study of damage [4], vibrations of heart valves [7], Schrödinger and wave equations [14, 28].

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There are a lot of physical phenomena which, while modeling, have some unavoidable environmental noises that impact greatly on the results obtained using ODEs, that's why stochastic differential equations [17] are needed. The most common noise is Gaussian noise but some phenomena, for example motion of a particle in a strongly heterogeneous medium involving large jumps, require a more generalized form, termed as L'evy noise [2]. For works concerning L'evy noise in ordinary stochastic can be found in [5, 10, 24]. Articles for fractional stochastic differential equations with Gaussian noise are [11, 23, 25, 26, 27]. For more advanced works on L'evy fractional stochastic one can look into [1, 16].

Huong et al. [9] have discussed well-posedness for solution of Caputo SFDEs in $\mathcal{L}^p$ spaces, we have worked on SFDEs driven by L'evy noise in the sense of newly define OBC derivative in $\mathcal{L}^p$ spaces. The motivation behind using the OBC fractional derivative [19] is that it is the extension of Caputo derivative, for $\rho = 1$ our result will give the Caputo SFDEs driven by L'evy noise. Motivated from above mentioned papers [9, 19] we have considered following SFDE driven by L'evy noise with OBC derivative,

$$\begin{aligned} {}_{0}^{OBC}\mathcal{D}_t^{\alpha,\rho}\mathcal{X}(t) =& \mathsf{f}(\mathcal{X}(t-),t) + \mathsf{g}(\mathcal{X}(t-),t)\frac{d\mathcal{B}}{dt} + \int_{|y|<c}\mathcal{H}(\mathcal{X}(t-),t,y)\frac{\tilde{\mathcal{N}}(dt,dy)}{dt},\ t\in[0,T], \\ \mathcal{X}(0) =& \xi, \end{aligned} \tag{1}$$

where, ${}_{0}^{OBC}\mathcal{D}_t^{\alpha,\rho}$ is the OBC fractional derivative $1/2 < \alpha < 1, \rho > 1/2$, 0 and t are terminals of the derivative, the drift coefficient $\mathsf{f} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, the diffusion coefficient $\mathsf{g} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times m}$ and the jump coefficient $\mathcal{H} : \mathbb{R}^d \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable. $\mathcal{B}(t) = (\mathcal{B}^1(t), \mathcal{B}^2(t), ..., \mathcal{B}^m(t))$ is Brownian motion with dimension m and the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathcal{P})$ is equipped with normal filtration satisfying usual conditions. $\mathcal{N} : \mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ is an independent $\mathcal{F}_t-$adapted Poisson random measure with a compensator $\tilde{\mathcal{N}}$ and intensity measure ν, which is called L'evy measure such that

$$\tilde{\mathcal{N}}(dt,dy) := \mathcal{N}(dt,dy) - \nu(dy)dt, \int_{\mathbb{R}^d\setminus\{0\}}\frac{y^2}{1+y^2}\nu(dy) < \infty.$$

ξ is the initial value satisfying $\mathbb{E}|\xi|^p < \infty$, and the constant $c \in [0,\infty)$ is the maximum allowable jump size.

In recent research, it is an established fact that the noise or stochastic

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perturbation is obvious and universal in nature as well as in man-made systems. Therefore, it is an importance aspect in various modeling problems to bring in the stochastic effects into the study of fractional differential systems. This is the reason, we have considered SFDE driven by L'evy noise with OBC fractional derivative in this article.

The rest of the paper is organized as follows, Section 2 has some useful definitions which we require to obtain our results, Section 3 discusses the motivation of the present work along with a detailed needed background, Section 4 consists our main results which is a theorem done using fixed point technique, Section 5 is dedicated to an example which is presented to support our established results and the paper is concluded in Section 6.

2 Preliminary Results

In an attempt to make this paper self-sufficient, we provide the following definitions and some well known results. For the general setting of fractional calculus and notations of stochastic calculus like Caputo's derivative ${}^{C}_{t_0}\mathcal{D}^{\alpha}_{t}$, Riemann-Liouville integral ${}_{t_0}\mathcal{J}_t^{-\alpha}$, Mittag Lefler function $E_{\alpha,\beta}$, see the the references cited in end of the article. Some basic definitions and function spaces that are the backbone of the article are given as follows.

Definition 2.1 [17](Stochastic Process) A family $\{\mathcal{X}(t), t \in I\}$ of $\mathbb{R}^d$-valued random variable is called a stochastic process with parameter set (or index set) I and state space $\mathbb{R}^d$.

Definition 2.2 [2] (L'evy Process) Let $\mathcal{X} = (\mathcal{X}(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We say that $\mathcal{X}$ is a L'evy process if:

1. $\mathcal{X}(0) = 0$ $(a.s.)$;
2. $\mathcal{X}$ has independent and stationary increments;
3. $\mathcal{X}$ is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P(|\mathcal{X}(t) - \mathcal{X}(s)| > a) = 0.$$

Definition 2.3 [21] The Riemann-Liouville fractional integration of

$f \in L_{1,loc}([t_0, t], \mathbb{R})$ of order $\alpha > 0$ is defined as

$$ {}_{t_0}\mathcal{J}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau. $$

Definition 2.4 [21] The Caputo fractional derivative of a differentiable function f, such that $f' \in L_{1,loc}([t_0, t], \mathbb{R})$, of order $0 < \alpha < 1$ is defined as

$$ {}_{t_0}^{C}\mathcal{D}_t^{\alpha} f(t) = {}_{t_0}\mathcal{J}_t^{(1-\alpha)} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f'(\tau) d\tau. $$

Definition 2.5 [21] A two parameter Mittag-Leffler function for $z \in \mathbb{C}$ is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta > 0). $$

Theorem 2.6 [15] Each contraction map $g : X \to X$ on a complete metric space (X, d) has a unique fixed point.

Now, we turn our attention to the stochastic aspect of the problem, for $p \geq 2, t \in [0, \infty)$, let $\mathcal{L}^p(\Omega, \mathcal{F}_t, \mathcal{P})$ denote the space of all $\mathcal{F}_t$-measurable, p^{th} integrable function $h = (h_1, h_2, \ldots, h_d) : \Omega \to \mathbb{R}^d$ with

$$ \|h\|_p := \left(\sum_{i=1}^{d} \mathbb{E}(|h_i|^p) \right)^{1/p}. $$

Let $\mathbb{U}^p([0, T])$ be the space of all the process $\mathcal{X}$ which are measurable, right continuous with left limit, $\mathcal{F}_T := (\mathcal{F}_t)_{t\in[0,T]}$-adapted and satisfy that

$$ \|\mathcal{X}\|_{\mathbb{U}^p} := esssup_{t\in[0,T]} \|\mathcal{X}(t)\|_p < \infty. $$

Obviously, $(\mathbb{U}^p([0, T]), \|\cdot\|_{\mathbb{U}^p})$ is a Banach space.

Note: We will use following elementary inequalities [8] for the upcoming analysis in order to prove our main result.

1. If $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^d$, then

$$ |\mathcal{X} + \mathcal{Y}|_p^p \leq 2^{p-1}(|\mathcal{X}|_p^p + |\mathcal{Y}|_p^p). \tag{2} $$

2. Burkholder-Davis-Gundy inequality for white noise.
3. Burkholder-Davis-Gundy inequality for L'evy noise.

3 New OBC Fractional Derivative

In 2011, Katugampola [12] defined a new fractional order integral as a generalization of usual fractional integral by updating the kernel of the transformation. He added a new parameter $\rho > 0$. For this, the generalized fractional integral ${}^{K}_{t_0}\mathcal{J}^{\alpha}_{t}$ of $f \in L_{1,loc}([t_0, t], \mathbb{R})$ of order $\alpha > 0$ is defined as:

$$ {}^{K}_{t_0}\mathcal{J}^{\alpha,\rho}_{t} f(t) = [{}_{t_0^\rho/\rho}\mathcal{J}^{\alpha}_{t}(fog)](t^\rho/\rho) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} f(\tau) d\tau, \tag{3} $$

where, $g(t) = (\rho t)^{1/\rho}$. Using this, the Riemann-Katugampola (RK) derivative is defined as,

$$ {}^{RK}_{t_0}\mathcal{D}^{\alpha,\rho}_{t} f(t) = [D^m \ {}_{t_0^\rho/\rho}\mathcal{J}^{m-\alpha}_{t}(fog)](t^\rho/\rho) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^{m} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho - \tau^\rho)^{m-\alpha-1} f(\tau) d\tau. \tag{4} $$

Then, motivated by the above definition and the following identity for usual Caputo fractional derivative,

$$ {}_{t_0}\mathcal{J}^{\alpha}_{t} \ {}^{C}_{t_0}\mathcal{D}^{\alpha}_{t} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k \tag{5} $$

the Caputo-Katugampola (CK) derivative is defined as,

$$ {}^{CK}_{t_0}\mathcal{D}^{\alpha,\rho}_{t} f(t) = {}^{RK}_{t_0}\mathcal{D}^{\alpha,\rho}_{t} \left(f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k \right). \tag{6} $$

But this derivative has a drawback, unlike the usual Caputo derivative, it doesn't satisfy the generalized form of the identity (5) which makes the calculation of the fractional derivatives easy. In an attempt to remove this drawback, Odibat-Baleanu-Caputo (OBC) derivative was defined as follows,

$$ \begin{aligned} {}^{OBC}_{t_0}\mathcal{D}^{\alpha,\rho}_{t} f(t) &= [{}_{t_0^\rho/\rho}\mathcal{J}^{m-\alpha}_{t} D^m(fog)](t^\rho/\rho) \\ &= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho - \tau^\rho)^{m-\alpha-1} \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^{m} f(\tau) d\tau \end{aligned} \tag{7} $$

this derivative satisfies the following generalization of the identity (5),

$$ {}_{t_0}^{K}\mathcal{J}_t^{\alpha,\rho}\ {}_{t_0}^{OBC}\mathcal{D}_t^{\alpha,\rho} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{(t^\rho - \tau^\rho)^k}{\rho^k k!} \left[\left(t^{1-\rho} \frac{d}{dt} \right)^k f(t) \right]_{t=t_0}. \tag{8} $$

We have the following remarks for the indorsee of above definition of fractional derivative and integral in stochastic sense:

1. A function f defined on any interval I is absolutely continuous on I if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of point-wise disjoint subintervals (x_k, y_k) of I satisfies $\sum_k |y_k - x_k| < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.
2. A function f defined on a closed interval $[a, b]$ is absolutely continuous if and only if f has a derivative almost everywhere, the derivative is Lebesgue integrable and

$$ f(x) = f(a) + \int_a^x f'(t)dt, \forall x \in [a, b]. $$

   A class of all absolutely continuous functions is denoted by $AC([a, b])$.
3. A function f defined on a open interval Ω such that $\int_K |f|dx < +\infty$, that is, its Lebesgue integral is finite on all compact subsets K of Ω, then f is called locally integrable. A class of all locally integrable functions on open interval Ω is denoted by $L_{1,loc}(\Omega)$.
4. A class of all absolutely continuous function $f : [a, b] \to \mathbb{R}$ such that its Lebesgue integral is finite on all compact subsets K of $[a, b]$ is denoted by $f' \in L_{1,loc}([a, b], \mathbb{R})$.

Definition 3.1 A measurable process $\mathcal{X} : [0, T] \to \mathcal{L}^p(\Omega, \mathcal{F}, \mathcal{P})$ is said to be $\mathcal{F}$-adapted if $\mathcal{X}(t) \in \mathcal{L}^p(\Omega, \mathcal{F}_t, \mathcal{P})$ for all $t \geq 0$. For each $\xi \in \mathcal{L}^p(\Omega, \mathcal{F}_0, \mathcal{P})$ a $\mathcal{F}$-adapted process $\mathcal{X}$ is called a solution of (1) if $\mathcal{X}(0) = \xi$ and the following equation holds

$$
\begin{aligned}
\mathcal{X}(t) = \xi &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\mathcal{X}(\tau-), \tau) d\tau \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{g}(\mathcal{X}(\tau-), \tau) d\mathcal{B}(\tau) \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \int_{|y|<c} \mathcal{H}(\mathcal{X}(\tau-), \tau, y) \tilde{\mathcal{N}}(d\tau, dy), \; t \in (0, T].
\end{aligned} \tag{9}
$$

4 Existence and Uniqueness Results for the Solution

We give the following theorem with some sufficient condition on the IVP (1) stated below.

Theorem 4.1 Let K and $\mathcal{M}$ be two positive constants such that

1. For all $\mathcal{X}_1, \mathcal{X}_2, y \in \mathbb{R}^d$ and $t \in [0, T]$

$$
\begin{aligned}
&|\mathsf{f}(\mathcal{X}_1(t), t) - \mathsf{f}(\mathcal{X}_2(t), t)|_p \vee |\mathsf{g}(\mathcal{X}_1(t), t) - \mathsf{g}(\mathcal{X}_2(t), t)|_p \\
&\vee \int_{|y|<c} |\mathcal{H}(\mathcal{X}_1(t), t, y) - \mathcal{H}(\mathcal{X}_2(t), t, y)|_p \nu(dy) \leq K|\mathcal{X}_1 - \mathcal{X}_2|_p.
\end{aligned} \tag{10}
$$

2. For all $y \in \mathbb{R}^d$ and $t \in [0, T]$

$$
\begin{aligned}
&esssup_{t\in[0,T]} |\mathsf{f}(0, t)|_p < \mathcal{M}, esssup_{t\in[0,T]} |\mathsf{g}(0, t)|_p < \mathcal{M}, \\
&esssup_{t\in[0,T]} \int_{|y|<c} |\mathcal{H}(0, t, y)|_p \nu(dy) < \mathcal{M},
\end{aligned} \tag{11}
$$

then there exists a unique solution to the IVP (1).

Proof: We convert our problem (1) into fixed point problem. So, we define a operator $\mathcal{T}_\xi : \mathbb{U}^p([0, T]) \to \mathbb{U}^p([0, T])$ such that

$$
\begin{aligned}
\mathcal{T}_\xi \mathcal{X}(t) = \xi &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\mathcal{X}(\tau-), \tau) d\tau \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{g}(\mathcal{X}(\tau-), \tau) d\mathcal{B}(\tau) \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \int_{|y|<c} \mathcal{H}(\mathcal{X}(\tau-), \tau, y) \tilde{\mathcal{N}}(d\tau, dy), \ t \in (0, T].
\end{aligned} \tag{12}
$$

Now, first we'll show that mapping is well-defined. By using the inequality (2), we get,

$$
\begin{aligned}
||\mathcal{T}_\xi \mathcal{X}(t)||_p^p \leq & 2^{2p-2}||\xi||_p^p + 2^{2p-2} \frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p} \left\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\mathcal{X}(\tau-), \tau) d\tau \right\|_p^p \\
&+ 2^{2p-2} \frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p} \left\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{g}(\mathcal{X}(\tau-), \tau) d\mathcal{B}(\tau) \right\|_p^p \\
&+ 2^{2p-2} \frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p} \left\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \int_{|y|<c} \mathcal{H}(\mathcal{X}(\tau-), \tau, y) \tilde{\mathcal{N}}(d\tau, dy) \right\|_p^p.
\end{aligned}
$$

First considering the drift term and using Hölder's inequality, we obtain,

$$
\begin{aligned}
&\left\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\mathcal{X}(\tau-), \tau) d\tau \right\|_p^p \\
&\quad \leq \sum_{i=1}^d \mathbb{E} \left| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}_i(\mathcal{X}(\tau-), \tau) d\tau \right|^p \\
&\quad \leq \sum_{i=1}^d \mathbb{E} \left\{ \Big(\int_0^t \tau^{\frac{p(\rho-1)}{p-1}} (t^\rho - \tau^\rho)^{\frac{p(\alpha-1)}{p-1}} d\tau \Big)^{p-1} \Big(\int_0^t |\mathsf{f}_i(\mathcal{X}(\tau-), \tau)|^p d\tau \Big) \right\} \\
&\quad = \left(\frac{t^{\frac{p\alpha\rho-1}{p-1}}}{\rho} B\Big(\frac{\alpha p - 1}{p-1}, \frac{\rho p - 1}{\rho(p-1)} \Big) \right)^{p-1} \sum_{i=1}^d \mathbb{E} \Big(\int_0^t |\mathsf{f}_i(\mathcal{X}(\tau-), \tau)|^p d\tau \Big) \\
&\quad \leq C_1 \Big(\int_0^t \|\mathsf{f}(\mathcal{X}(\tau-), \tau)\|_p^p d\tau \Big),
\end{aligned}
$$

where $B(m,n) := \int_0^1 t^{m-1}(1-t)^{n-1} dt$ is the Beta function and $C_1 = \left(\frac{T^{\frac{p\alpha\rho-1}{p-1}}}{\rho} \frac{\Gamma\big(\frac{\alpha p-1}{p-1}\big)\Gamma\big(\frac{\rho p-1}{\rho(p-1)}\big)}{\Gamma\Big(\frac{\alpha p-1}{p-1} + \frac{\rho p-1}{\rho(p-1)}\Big)} \right)^{p-1}$.

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Using (10) and (11), we get,

$$\begin{aligned}|\mathsf{f}(\tau,\mathcal{X}(\tau))|_p^p &\le 2^{p-1}|\mathsf{f}(\tau,\mathcal{X}(\tau))-\mathsf{f}(\tau,0)|_p^p+2^{p-1}|\mathsf{f}(\tau,0)|_p^p\\ &\le 2^{p-1}K^p|\mathcal{X}(\tau)|_p^p+2^{p-1}|\mathsf{f}(\tau,0)|_p^p\end{aligned}$$

$$\begin{aligned}\int_0^t\|\mathsf{f}(\mathcal{X}(\tau-),\tau)\|_p^p d\tau &\le 2^{p-1}K^p(esssup_{\tau\in[0,T]}\|\mathcal{X}(\tau-)\|_p)^p\int_0^t d\tau+2^{p-1}\int_0^t|\mathsf{f}(\tau,0)|_p^p d\tau\\ &\le 2^{p-1}K^pT\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}T\mathcal{M}^p\end{aligned}$$

$$\left\|\int_0^t\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\mathcal{X}(\tau-),\tau)d\tau\right\|_p^p\le C_1(2^{p-1}K^pT\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}T\mathcal{M}^p). \tag{13}$$

Now, taking the diffusion term into account and using Burkholder-Davis Gundy and Hölder's inequalities to simplify the following,

$$\begin{aligned}&\left\|\int_0^t\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{g}(\mathcal{X}(\tau-),\tau)d\mathcal{B}(\tau)\right\|_p^p\\ &\le\sum_{i=1}^d\mathbb{E}\left|\int_0^t\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{g}_i(\mathcal{X}(\tau-),\tau)d\mathcal{B}(\tau)\right|^p\\ &\le\sum_{i=1}^d\mathbb{C}_p\mathbb{E}\left|\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)|^2d(\tau)\right|^{p/2}\\ &\le\sum_{i=1}^d\mathbb{C}_p\mathbb{E}\bigg\{\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)|^pd\tau\\ &\quad\Big(\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}d\tau\Big)^{\frac{p-2}{2}}\bigg\}\\ &=\mathbb{C}_p\left(\frac{t^{2\alpha\rho-1}}{\rho}B(\Gamma(2\alpha-1),\Gamma\big(\frac{2\rho-1}{\rho}\big))\right)^{\frac{p-2}{2}}\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)\|_p^pd\tau\\ &\le\mathbb{C}_pC_2^{\frac{p-2}{2}}\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)\|_p^pd\tau,\end{aligned}$$

where $C_2=\dfrac{T^{2\alpha\rho-1}}{\rho}\dfrac{\Gamma(2\alpha-1)\Gamma\big(\frac{2\rho-1}{\rho}\big)}{\Gamma(2\alpha-1)+\Gamma\big(\frac{2\rho-1}{\rho}\big)}$ and $\mathbb{C}_p$ is a constant arose due to the use of Burkholder-Davis Gundy inequality.

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From (10) and (11) we get,

$$\begin{aligned}|\mathsf{g}(\tau,\mathcal{X}(\tau))|_p^p &\leq 2^{p-1}|\mathsf{g}(\tau,\mathcal{X}(\tau))-\mathsf{g}(\tau,0)|_p^p+2^{p-1}|\mathsf{g}(\tau,0)|_p^p\\ &\leq 2^{p-1}K^p|\mathcal{X}(\tau)|_p^p+2^{p-1}|\mathsf{g}(\tau,0)|_p^p\end{aligned}$$

$$\begin{aligned}&\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)\|_p^p d\tau\\ \leq& 2^{p-1}K^p(esssup_{\tau\in[0,T]}\|\mathcal{X}(\tau-)\|_p)^p\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}d\tau\\ &+2^{p-1}\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{g}(\tau,0)|_p^p d\tau\\ \leq& 2^{p-1}C_2K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}C_2\mathcal{M}^p.\end{aligned}$$

Hence,

$$\left\|\int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{g}(\mathcal{X}(\tau-),\tau)d\mathcal{B}(\tau)\right\|_p^p \leq \mathbb{C}_p C_2^{p/2}2^{p-1}(K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+\mathcal{M}^p). \tag{14}$$

Lastly the jump term is considered and Burkholder-Davis Gundy inequality for L'evy and Hölder's inequality are used as follows,

$$\begin{aligned}&\left\|\int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\int_{|y|<c}\mathcal{H}(\mathcal{X}(\tau-),\tau,y)\tilde{\mathcal{N}}(d\tau,dy)\right\|_p^p\\ &\leq \sum_{i=1}^d \mathbb{E}\left|\int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\int_{|y|<c}\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)\tilde{\mathcal{N}}(d\tau,dy)\right|^p\\ &\leq \bar{\mathbb{C}}\Bigg[\mathbb{E}\bigg(\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^p\nu(dy)d\tau\bigg)\\ &\quad+\mathbb{E}\left|\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^2\nu(dy)d\tau\right|^{p/2}\Bigg]\\ &\leq \bar{\mathbb{C}}\Bigg[\mathbb{E}\bigg(\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^p\nu(dy)d\tau\bigg)\\ &\quad+\mathbb{E}\bigg\{\bigg(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}d\tau\bigg)^{\frac{p-2}{2}}\\ &\quad\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^p\nu(dy)d\tau\bigg\}\Bigg]\end{aligned}$$

$$\leq \bar{\mathbb{C}}\Bigg[\mathbb{E}\bigg(\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^p\nu(dy)d\tau\bigg)$$
$$+C_2^{\frac{p-2}{2}}\mathbb{E}\bigg(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)|^p\nu(dy)d\tau\bigg)\Bigg].$$

Where $\bar{\mathbb{C}}$ is again a constant arose due to the use of Burkholder-Davis Gundy inequality for L'evy [8]. Now, from (10) and (11), we obtain,

$$\begin{aligned}\int_{|y|<c}|\mathcal{H}(\mathcal{X}(\tau),\tau,y)|_p^p\nu(dy) \leq &\int_{|y|<c}[2^{P-1}|\mathcal{H}(\mathcal{X}(\tau),\tau,y)-\mathcal{H}(0,\tau,y)|_p^p\\ &+2^{P-1}|\mathcal{H}(0,\tau,y)|_p^p]\nu(dy)\\ \leq &2^{p-1}K^p|X(\tau)|_p^p+2^{p-1}\int_{|y|<c}|\mathcal{H}(0,\tau,y)|_p^p\nu(dy),\end{aligned}$$

$$\begin{aligned}\int_{|y|<c}\|\mathcal{H}(\mathcal{X}(\tau),\tau,y)\|_p^p\nu(dy) \leq &2^{p-1}K^p esssup_{\tau\in[0,T]}\|X(\tau)\|_p^p\\ &+2^{p-1}esssup_{\tau\in[0,T]}\int_{|y|<c}\|\mathcal{H}(0,\tau,y)\|_p^p\nu(dy)\\ \leq &2^{p-1}K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}\mathcal{M}^p.\end{aligned}$$

Hence, we get,

$$\begin{aligned}&\bigg\|\int_0^t\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\int_{|y|<c}\mathcal{H}(\mathcal{X}(\tau-),\tau,y)\tilde{\mathcal{N}}(d\tau,dy)\bigg\|_p^p\\ &\leq\bar{\mathbb{C}}\bigg[\int_0^t\tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}(2^{p-1}K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}\mathcal{M}^p)d\tau\\ &\quad+C_2^{\frac{p-2}{2}}\int_0^t\tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}(2^{p-1}K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}\mathcal{M}^p)d\tau\bigg]\\ &\leq\bar{\mathbb{C}}(2^{p-1}K^p\|\mathcal{X}\|_{\mathbb{U}^p}^p+2^{p-1}\mathcal{M}^p)(C_3+C_2^{p/2}),\end{aligned} \tag{15}$$

where $C_3=\dfrac{T^{\alpha p\rho-p+1}}{\rho}\dfrac{\Gamma\big(p(\alpha-1)+1\big)\Gamma\big(\frac{p(\rho-1)+1}{\rho}\big)}{\Gamma\big(p(\alpha-1)+1+\frac{p(\rho-1)+1}{\rho}\big)}$.
From (13), (14) and (15), we get,

$$||\mathcal{T}_\xi\mathcal{X}(t)||_p^p<\infty$$

Hence, operator $\mathcal{T}_\xi$ is well-defined.

Now, we'll show that the operator $\mathcal{T}_\xi$ is contraction. For arbitrary $\mathcal{X}, \bar{\mathcal{X}} \in \mathbb{U}^p([0,T])$ and $t \in [0,T]$, consider,

$$\begin{aligned}||\mathcal{T}_\xi\mathcal{X}(t) - \mathcal{T}_\xi\bar{\mathcal{X}}(t)||_p^p \leq{}& 2^{p-1}\frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p}\bigg\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1}[\mathsf{f}(\mathcal{X}(\tau-),\tau) - \mathsf{f}(\bar{\mathcal{X}}(\tau-),\tau)]d\tau\bigg\|_p^p \\ &+ 2^{2p-2}\frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p}\bigg\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1}[\mathsf{g}(\mathcal{X}(\tau-),\tau) - \mathsf{g}(\bar{\mathcal{X}}(\tau-),\tau)]d\mathcal{B}(\tau)\bigg\|_p^p \\ &+ 2^{2p-2}\frac{\rho^{p-p\alpha}}{(\Gamma(\alpha))^p}\bigg\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1}\int_{|y|<c}[\mathcal{H}(\mathcal{X}(\tau-),\tau,y) - \mathcal{H}(\bar{\mathcal{X}}(\tau-),\tau,y)]\tilde{\mathcal{N}}(d\tau,dy)\bigg\|_p^p,\end{aligned}$$

where we have used elementary inequality (2). Now, as we have done above, considering the drift term and using Hölder's inequality and (10), we get,

$$\begin{aligned}&\bigg\| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1}[\mathsf{f}(\mathcal{X}(\tau-),\tau) - \mathsf{f}(\bar{\mathcal{X}}(\tau-),\tau)]d\tau\bigg\|_p^p \\ &\leq \sum_{i=1}^d \mathbb{E}\bigg| \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1}[\mathsf{f}_i(\mathcal{X}(\tau-),\tau) - \mathsf{f}_i(\bar{\mathcal{X}}(\tau-),\tau)]d\tau\bigg|^p \\ &\leq \sum_{i=1}^d \mathbb{E}\Bigg[\bigg(\int_0^t \tau^{\frac{(p-2)(\rho-1)}{p-1}}(t^\rho-\tau^\rho)^{\frac{(p-2)(\alpha-1)}{p-1}}d\tau\bigg)^{p-1} \\ &\quad\times\bigg(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{f}_i(\mathcal{X}(\tau-),\tau) - \mathsf{f}_i(\bar{\mathcal{X}}(\tau-),\tau)|^p d\tau\bigg)\Bigg] \\ &\leq \Bigg(\frac{t^{\frac{\alpha\rho(p-2)-2p+1}{p-1}}}{\rho}B\bigg(\frac{\alpha(p-2)+1}{p-1}, \frac{\rho(p-2)+1}{\rho(p-1)}\bigg)\Bigg) \\ &\quad\times\sum_{i=1}^d \mathbb{E}\bigg(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{f}_i(\mathcal{X}(\tau-),\tau) - \mathsf{f}_i(\bar{\mathcal{X}}(\tau-),\tau)|^p d\tau\bigg) \\ &\leq C_4K^p\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathcal{X}(\tau-) - \bar{\mathcal{X}}(\tau-)\|_p^p d\tau, \end{aligned} \tag{16}$$

where $C_4 = \left(\dfrac{T^{\frac{\alpha\rho(p-2)-2p+1}{p-1}}}{\rho}\dfrac{\Gamma\left(\frac{\alpha(p-2)+1}{p-1}\right)\Gamma\left(\frac{\rho(p-2)+1}{\rho(p-1)}\right)}{\Gamma\left(\frac{\alpha(p-2)+1}{p-1}+\frac{\rho(p-2)+1}{\rho(p-1)}\right)}\right)^{p-1}$.

Now, considering term with diffusion coefficient, using Hölder's inequality, Burkholder-Davis Gundy inequality and (10),

$$\begin{aligned}
&\left\| \int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}[\mathsf{g}(\mathcal{X}(\tau-),\tau)-\mathsf{g}(\bar{\mathcal{X}}(\tau-),\tau)]d\mathcal{B}(\tau)\right\|_p^p \\
&\quad\le \sum_{i=0}^d \mathbb{E}\left| \int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}[\mathsf{g}_i(\mathcal{X}(\tau-),\tau)-\mathsf{g}_i(\bar{\mathcal{X}}(\tau-),\tau)]d\mathcal{B}(\tau)\right|^p \\
&\quad\le \sum_{i=0}^d \mathbb{C}_p\mathbb{E}\left| \int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)-\mathsf{g}_i(\bar{\mathcal{X}}(\tau-),\tau)|^2 d\tau\right|^{p/2} \\
&\quad\le \sum_{i=0}^d \mathbb{C}_p\mathbb{E}\Bigg[\left(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}d\tau\right)^{\frac{p-2}{2}} \\
&\qquad\times\left(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}|\mathsf{g}_i(\mathcal{X}(\tau-),\tau)-\mathsf{g}_i(\bar{\mathcal{X}}(\tau-),\tau)|^p d\tau\right)\Bigg] \\
&\qquad\le \mathbb{C}_p C_2^{\frac{p-2}{2}} K^p \int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p d\tau. \qquad (17)
\end{aligned}$$

Now, we will simplify the term with jump using Hölder's inequality, Burkholder-Davis Gundy inequality for L'evy and (10),

$$\begin{aligned}
&\left\| \int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\int_{|y|<c}[\mathcal{H}(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}(\bar{\mathcal{X}}(\tau-),\tau,y)]\tilde{\mathcal{N}}(d\tau,dy)\right\|_p^p \\
&\quad\le \sum_{i=0}^d \mathbb{E}\left| \int_0^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\int_{|y|<c}[\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}_i(\bar{\mathcal{X}}(\tau-),\tau,y)]\tilde{\mathcal{N}}(d\tau,dy)\right|^p \\
&\quad\le \bar{C}\sum_{i=0}^d \Bigg[\mathbb{E}\left(\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}_i(\bar{\mathcal{X}}(\tau-),\tau,y)|\nu(dy)d\tau\right) \\
&\qquad+\mathbb{E}\left|\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}_i(\bar{\mathcal{X}}(\tau-),\tau,y)|^2\nu(dy)d\tau\right|^{p/2}\Bigg] \\
&\quad\le \bar{\mathbb{C}}\sum_{i=0}^d \Bigg[\mathbb{E}\left(\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}_i(\bar{\mathcal{X}}(\tau-),\tau,y)|\nu(dy)d\tau\right) \\
&\qquad+\mathbb{E}\Bigg[\left(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}d\tau\right)^{\frac{p-2}{2}} \\
&\qquad\times\left(\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\int_{|y|<c}|\mathcal{H}_i(\mathcal{X}(\tau-),\tau,y)-\mathcal{H}_i(\bar{\mathcal{X}}(\tau-),\tau,y)|^p\nu(dy)d\tau\right)\Bigg]\Bigg]
\end{aligned}$$

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$$\begin{aligned}&\leq \bar{\mathbb{C}} K^p \int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p d\tau \\ &\quad + \bar{\mathbb{C}} C_2^{\frac{p-2}{2}} K^p \int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p d\tau.\end{aligned} \tag{18}$$

From (16), (17) and (18), we get,

$$\begin{aligned}||\mathcal{T}_\xi \mathcal{X}(t)-\mathcal{T}_\xi \bar{\mathcal{X}}(t)||_p^p \leq & \mathcal{A}\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p d\tau \\ & + \bar{\mathbb{C}} K^p \int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p d\tau,\end{aligned}$$

where $\mathcal{A} = K^p(C_4 + \mathbb{C}_p C_2^{\frac{p-2}{2}} + \bar{\mathbb{C}} C_2^{\frac{p-2}{2}})$.

We choose a positive constant λ such that

$$\frac{\mathcal{A} T^{\frac{\rho(2\rho+1)}{2}}\Gamma(2\alpha-1)}{1-C_3\bar{\mathbb{C}}K^p} < \lambda, \tag{19}$$

and define a weighted norm $\|.\|_\lambda$ on the space $\mathbb{U}^p([0,T])$ such that

$$\|X\|_\lambda := esssup_{\tau\in[0,T]}\left(\frac{\|X(\tau)\|_p^p}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})}\right)^{1/p}. \tag{20}$$

Clearly, the two norms are equivalent, since $(\mathbb{U}^p([0,T]), \|.\|_{\mathbb{U}^p})$ is a Banach space, this implies $(\mathbb{U}^p([0,T]), \|.\|_\lambda)$ is also a Banach space. Consider the following,

$$\begin{aligned}\frac{||\mathcal{T}_\xi \mathcal{X}(t)-\mathcal{T}_\xi \bar{\mathcal{X}}(t)||_p^p}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)})} \leq & \frac{\mathcal{A}\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)}\frac{\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})}E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)})} \\ & + \frac{\bar{\mathbb{C}}K^p\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)}\frac{\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})}E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)})}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(esssup_{\tau\in[0,T]} \left(\frac{\|\mathcal{X}(\tau-)-\bar{\mathcal{X}}(\tau-)\|_p^p}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})} \right)^{1/p} \right)^p \\
&\quad \times \left[\frac{\mathcal{A}\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)} E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)})} \right. \\
&\quad \left. + \frac{\bar{\mathbb{C}}K^p\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)} E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau}{E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)})} \right].
\end{aligned} \tag{21}
$$

Consider the following integral,

$$
\begin{aligned}
&\frac{\lambda}{\Gamma(2\alpha-1)}\int_0^t \tau^{2(\rho-1)}(t^\rho-\tau^\rho)^{2(\alpha-1)} E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau \\
&\qquad \leq T^{\frac{\rho(2\rho+1)}{2}} E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)}).
\end{aligned} \tag{22}
$$

Again, consider the following integral and applying Mean Value theorem

$$
\int_0^t \tau^{p(\rho-1)}(t^\rho-\tau^\rho)^{p(\alpha-1)} E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda\tau^{\rho(2\alpha-1)})d\tau \leq C_3 E_{2\alpha-1,\frac{2\rho-1}{2}}(\lambda t^{\rho(2\alpha-1)}). \tag{23}
$$

Using (22) and (23) in (21), we get

$$
\|\mathcal{T}_\xi(\mathcal{X})-\mathcal{T}_\xi(\bar{\mathcal{X}})\|_\lambda^p \leq \left(\frac{\mathcal{A}T^{\frac{\rho(2\rho+1)}{2}}\Gamma(2\alpha-1)}{\lambda} + C_3\bar{\mathbb{C}}K^p \right) \|\mathcal{X}-\bar{\mathcal{X}}\|_\lambda^p
$$

$$
\|\mathcal{T}_\xi(\mathcal{X})-\mathcal{T}_\xi(\bar{\mathcal{X}})\|_\lambda \leq \left(\frac{\mathcal{A}T^{\frac{\rho(2\rho+1)}{2}}\Gamma(2\alpha-1)}{\lambda} + C_3\bar{\mathbb{C}}K^p \right)^{1/p} \|\mathcal{X}-\bar{\mathcal{X}}\|_\lambda.
$$

We want $C_3\bar{\mathbb{C}}K^p < 1$, that is

$$
\begin{aligned}
&\frac{T^{\alpha p\rho-p+1}}{\rho} \frac{\Gamma\big(p(\alpha-1)+1\big)\Gamma\big(\frac{p(\rho-1)+1}{\rho}\big)}{\Gamma\big(p(\alpha-1)+1+\frac{p(\rho-1)+1}{\rho}\big)} \bar{\mathbb{C}}K^p < 1 \\
&\Longrightarrow 0 < T < \left(\frac{\rho\Gamma\big(p(\alpha-1)+1+\frac{p(\rho-1)+1}{\rho}\big)}{\bar{\mathbb{C}}K^p\Gamma\big(p(\alpha-1)+1\big)\Gamma\big(\frac{p(\rho-1)+1}{\rho}\big)} \right)^{\alpha p\rho-p+1},
\end{aligned}
$$

which gives us the interval of existence for the solution. Now, staying in this interval and using (19), we get,

$$\frac{\mathcal{A}T^{\frac{\rho(2\rho+1)}{2}}\Gamma(2\alpha-1)}{\lambda}+C_3\bar{\mathbb{C}}K^p<1.$$

Hence, $\mathcal{T}_\xi$ is contraction with respect to the weighted norm defined in (19) and by Banach fixed point Theorem 2.6 $\mathcal{T}_\xi$ has a unique fixed point in $\mathbb{U}^p([0,T])$ which is the solution of IVP (1).

5 Example

Consider the following nonlinear OBC fractional order differential equation with given initial condition driven by L'evy noise to illustrate the established results:

$$\begin{aligned}
{}_0^{OBC}\mathcal{D}_t^{3/4,5/6}\begin{pmatrix}\mathcal{X}_1(t)\\ \mathcal{X}_2(t)\end{pmatrix}=&\begin{pmatrix}4\mathcal{X}_1(t)+t\\ 4\mathcal{X}_2(t)+2t\end{pmatrix}+\begin{pmatrix}\sqrt{5}\mathcal{X}_1(t)+5t\\ \sqrt{5}\mathcal{X}_2(t)+\sqrt{3}t\end{pmatrix}\frac{d\mathcal{B}}{dt}\\
&+\int_{|y|<1}y+\begin{pmatrix}\mathcal{X}_1(t)+t\\ \mathcal{X}_2(t)+4t\end{pmatrix}\frac{\tilde{\mathcal{N}}(d\tau,dy)}{dt},\ t\in[0,T]\\
\mathcal{X}(0)=&\xi,
\end{aligned}\tag{24}$$

where $\mathcal{B}(t)$ is a one-dimensional Brownian motion, intensity measure follows $\nu(dy)=\dfrac{dy}{1+|y|^2}$ and the initial condition satisfies $\mathbb{E}|\xi|^p<\infty$.

Comparing this example (24) with the equations (10) and (11) of the Theorem 4.1 and calculating $K,\mathcal{M}$ which turn out to be $K=4$ and $\mathcal{M}=\pi(\sqrt{17}T+1/2)$. Existence of K and $\mathcal{M}$ suggest, by our Theorem 4.1, that the solution of (24) exists and is unique.

6 Conclusion

We have studied the existence and uniqueness of the solution for a class of nonlinear fractional stochastic differential equation with a newly defined OBC-fractional derivative under the suitable restrictions as sufficient conditions on nonlinear functions which are stated in the respective problem. We have used mentioned stochastic inequalities and fixed point technique to obtain our results. At last an example is presented satisfying the stated conditions in support of our established results.

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