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Topological structure of solution sets for Hybrid impulsive fractional differential inclusion in Banach algebra

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Abstract

The aim of this paper is to study the topological structure of solution sets of hybrid impulsive fractional differential inclusions on the weighted space of piecewise continuous functions, we apply dhage fixed point theorem for multivalued operator with analysis tools to prove existence result. The paper concludes with an example to illustrate the feasibility of our main result.
Key words: Dhage fixed point theorem, differential inclusions, fractional integral, Impulsive, Riemann-Liouville fractional derivative, weight space, compactness.
AMS classification: 26A33, 34A08, 34A60, 47H10, 24A38, 34K34.

1 Introduction

During the last ten years, impulsive ordinary and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail see [19, 21], Covitz and Nadler [17], Lasota and Opial [23]. Fractional differential equations and inclusions have a large application in a variety of fields such as physics, mathematics, electrical networks, signal and image processing, aerodynamics, economics and do so on. Hence has increased more attention from both theoretical and the applied points of view in recent years, for for further details see [4, 5, 6, 7, 8, 9, 14]. Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Hybrid differential equations can be found in a series of papers [11, 13, 16]. In [20] Gabor deals with the R_δ structure of solution sets to some classes of impulsive differential inclusions with variable impulse times on the half-line.

$$\begin{cases} {}^{RL}D^{\alpha}y(t) \in F(t,y(t)), \text{a.e.} \quad t \in J = (0,+\infty],\ t \neq t_j(y(t)),\ j \in \mathbb{N}, \\ y(0) = y_0, \\ y(t^+) = y(t) + L_j(y(t)), \qquad t = t_j(y(t)),\ j \in \mathbb{N}, \end{cases}$$

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Sufficient conditions on barriers are given and discussed in detail. The paper is based on two different techniques : the inverse limit method and the definition of R_δ sets used in a new suitable Fréchet function space. In [26] Zhou deals with the question of the attractivity of solutions for a Cauchy problem of Riemann-Liouville type fractional evolution equations given by

$$D^{\alpha}_{0^+}y(t) = f(t,y(t)), \quad t \in J' = (0,+\infty), \tag{1}$$

$$I^{1-\alpha}_{0^+}y(0^+) = y_0, \tag{2}$$

where $0 < \alpha < 1$, $I^{1-\alpha}_{0^+}$ is Riemann-Liouville fractional integral of order $1-\alpha$, $f : [0,+\infty)\times E \to E$ is a continuous function satisfying some assumptions and y_0 is an element of the Banach space E. the author study the question of the attractivity of solutions for Cauchy problem (1)-(2). He establish sufficient conditions for the global attractivity for solutions of (1)-(2). In [27], Zhou and Peng considered fractional evolution inclusion with infinite delay

$$\begin{cases} \frac{d}{dt}[x(t) - h(t,x_t)] \in F(t,x_t), & t \in [0,b], \\ x(t) = \phi(t), & t \in [-\tau,0], \end{cases}$$

where the state $x(\cdot)$ takes value in Banach space E with norm $|.|$, F is a multimap defined on a subset of $[0,b]\times E$, A is the infinitesimal generator of an analytic semigroup $T(t)_{t\geq 0}$. The topological properties of the solution set is investigated. It is shown that the solution set is nonempty, compact and an R_δ-set which means that the solution set may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point. Additionally, fixed point theory can be used to develop the existence theory for the coupled systems of fractional hybrid differential equations [2, 3]. B. Ahmad et al. [2] discussed the existence of solution of the Dirichlet boundary value problem of coupled hybrid fractional differential equations

$$\begin{cases} {}^cD^{\delta}\left(\dfrac{x(t)}{f_1(t,x(t),y(t))}\right) = h_1(t,x(t),y(t)), & t \in I = [0,T],\ 1 < \delta \leq 2, \\ {}^cD^{\omega}\left(\dfrac{y(t)}{f_2(t,x(t),y(t))}\right) = h_2(t,x(t),y(t)), & t \in I = [0,T],\ 1 < \omega \leq 2, \\ x(0) = x(1) = 0, \qquad y(0) = y(1) = 0, \end{cases} \tag{3}$$

where ${}^cD^{\delta}, {}^cD^{\omega}$ denote the Caputo fractional derivative of orders δ,ω, respectively, $f_i \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $h_i \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R}), i = 1,2$. They deal with a standard Dhage fixed point theorem in Banach Algebra for hybrid differential equation to prove their main result. The purpose of the work is to investigate the topological structure of the solution set of an initial value problems for nonlinear fractional differential equations in Banach space. They proved that the solution set of (3) is nonempty, compact and, an R_δ-set. Motivated by the results cited above we deal in this paper with the study of topological structure of solution sets of a class of hybrid impulsive fractional differential inclusions,

$$ {}^{RL}D^{\alpha}\left(\frac{y(t)}{f(t,y(t))}\right) \in F(t,y(t)), \text{a.e. } t \in J=[0,T],\ t \neq t_k, \tag{4} $$

$$ \Delta^{*} y_{|t=t_k} = I_k(y(t_k^-)), \tag{5} $$

$$ \lim_{t\longrightarrow 0^+} t^{1-\alpha} y(t) = c. \tag{6} $$

$k=1,\ldots,m$, $0<\alpha\leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $F : J\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a given multivalued function with compact convex values, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $c\in\mathbb{R}$. $I_k : \mathbb{R}\to\mathbb{R}$ are the jump functions, $f : J\times\mathbb{R}\to\mathbb{R}0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T,\ T\geq 1.$ and $\Delta^{*}y|_{t_k} = y^{*}(t_k^+) - y(t_k^-)$, where $y^{*}(t_k^+) = \lim_{t\longrightarrow t_k^+}(t-t_k)^{1-\alpha}y(t)$ and $y(t_k^-) = \lim_{t\longrightarrow t_k^-} y(t)$. as a consequence for compliance we choose a space of piecewise continuous functions $PC_{*}([0,T],\mathbb{R})$ that will be defined later.

The paper is organized as follows, in Section 2 we give some general results and preliminaries multi-valued mapping and fractional calculous and in Section 3 we give our main result by using Dhage fixed point theorem for multivalued case in Banach algebra combining with multivalued analysis tools, we prove that the solution sets is nonempty and compact and in the last section we give an example to illustrate our main result.

2 Preliminaries

This section presents the notations and definitions used throughout this paper, and give some preliminaries facts from multivalued analysis. Let J be a compact interval in R, $C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm

$$ \|y\|_\infty = \sup\{\|y(t)\|, t\in J\}, $$

and $L^1(J,\mathbb{R})$ the Banach space of all functions $y : J\to\mathbb{R}$ which are lesbegue integrable with the norm

$$ \|y\|_{L^1} = \int_0^T |y(t)|dt. $$

Let X be a metric space. Define $\mathscr{P}(X) = \{Y\subset X, Y\neq\emptyset\}$, $\mathscr{P}_{cl}(X) = \{Y\in\mathscr{P}(X), Y \text{ closed}\}$, $\mathscr{P}_b(X) = \{Y\in\mathscr{P}(X), Y \text{ bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y\in\mathscr{P}(X), Y \text{ compact}\}$ and $\mathscr{P}_{cp,cv}(X) = \{Y\in\mathscr{P}(X), Y \text{ compact and convex}\}$. Let (X,d) and (Y,ρ) be two metric spaces. A multivalued map $G: X \to \mathscr{P}(Y)$ is convex (resp. closed) valued if $G(y)$ is convex (resp. closed) for all $y \in X$, G is bounded on bounded sets if $G(B) = \bigcup_{y\in B} G(y)$ is bounded in Y for all $B \subset \mathscr{P}_b(X)$ (i.e. $\sup_{y\in B}\{\sup\{\|v\| : v \in G(y)\}\} < \infty$). The fixed point set of the multivalued operator G will be denoted by FixG. A single-valued map $v : X \to Y$ is said to be a selection of G and we write $v \subset G$ whenever $v(y) \in G(y)$ for all $y \in X$. Define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,X) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}.$$

The multifunction G is called closed if its graph $\Gamma_G = \{(x,y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. The multifunction G is called quasicompact if its restriction to any compact subset $M \subset X$ is compact.

Lemma 2.1 [22] If $G: X \to P_{cp}(Y)$ is compact and has a closed graph, then G is u.s.c. For further reading and details on multivalued analysis, we refer the reader to the books of [22]. In the following, we give some definitions of the theory of fractional computation. Let $\alpha > 0$, $n = \lceil \alpha \rceil$ (the least integer greater than or equal to α) and $u \in L^1(J,\mathbb{R})$. The Riemann-Liouville fractional integral is defined by

$$I^{\alpha}_{0+}u(t) = g_\alpha(t) * u(t) = \int_0^t g_\alpha(t-s)u(s)ds,\ t > 0,$$

where $*$ denotes convolution and $g_\alpha(t) = t^{\alpha-1}/\Gamma(p)$. In case $\alpha = 0$, we set $g_0(t) = \delta(t)$, the Dirac measure concentrated at the origin. For $u \in C(J,\mathbb{R})$, the Riemann-Liouville fractional derivative is defined by

$$^{RL}D^{\alpha}_{0+}u(t) = \frac{d^n}{dt^n}(g_{n-\alpha}(t) * u(t))$$

and the Caputo fractional derivative can be defined by

$$^{C}D^{\alpha}_{0+}u(t) = g_{n-\alpha}(t) * \frac{d^n u(t)}{dt^n}$$

for all $t > 0$. For more details, see [1]. The following fixed point theorem due to Dhage [12] is fundamental in the proof of our main result.

Theorem 2.2 [12] Let S be a closed convex and bounded subset of a B Banach algebra X and let $\mathfrak{A} : PC_*(J,\mathbb{R}) \to PC_*(J,\mathbb{R})$, and $B : PC_*(J,\mathbb{R}) :\to \mathscr{P}_{cl,cv}(PC_*(J,\mathbb{R}))$ be two multi-valued operators satisfying

(a) $\mathfrak{A}$ is single-valued Lipschitz with a Lipschitz constant K,
(b) $\mathfrak{B}$ is compact and upper semi-continuous,
(c) $\mathfrak{A}x\mathfrak{B}x$ is a convex subset of X for each $x \in X$,

(d) $2MK < 1$, where $M = \|B(S)\|_{\mathscr{P}} = Sup_{x\in S}\|B(x)\|$.

Then either

(i) the the operator inclusion $x \in \mathfrak{A}x\ \mathfrak{B}x$ has a solution,
or
(ii) the set $\mathscr{E} = \{u \in PC_*(J,\mathbb{R})\mu u \in \mathfrak{A}u\ \mathfrak{B}u, \mu > 1\}$is unbounded.

3 Main Results

In order to define a solution of problem (4)-(6) , we shall consider the space

$$PC_*(J,R) = \Big\{ y : J \to R : y_k \in C(t_k,t_{k+1}], k = 0,\dots,m \quad \text{and there exist} \quad y(t_k^-), y_\alpha(t_k^+), k = 1,\dots,m \text{ with } y(t_k) = y(t_k^-) \Big\},$$

which is a Banach space with the norm

$$\|y\|_* = \max_{k=1,\dots m.} \|y_k\|_\alpha,$$

where y_k is the restriction of y to $J_k = (t_k,t_{k+1}]$, $k = 0,\dots,m+1$.

$$\|y_k\|_\alpha = \sup_{t\in[t_k,t_{k+1}]} \|(y_k)_\alpha(t)\|,$$

for every $k = 0\dots m+1$, where,

$$y_\alpha(t) = \begin{cases} (t-t_k)^{1-\alpha}y(t), & \text{if } t \in (t_k,t_{k+1}] \\ \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}y(t), & \text{if } t = t_k^+. \end{cases}$$

For A a subset of the space $PC_*(J,R)$, define $\mathscr{A}_\alpha$ by

$$\mathscr{A}_\alpha = \{y_\alpha,\ y \in A\}.$$

It is clear that $y_\alpha \in PC(J,\mathbb{R})$. We note the following Ascoli-Arzela type criteria. The following definitions are used in the subsequent sections.

Theorem 3.1 Let A be a bounded set in $PC_*(J,\mathbb{R})$. Assume that $\mathscr{A}_\alpha$ is equi-continuous on $PC(J,\mathbb{R})$ and the set $\{y_\alpha(t),\ y_\alpha \in \mathscr{A}_\alpha\}$, is relatively compact in $\mathbb{R}$ then A is relatively compact in $PC_*([0,T],\mathbb{R})$.

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Let $\{y_n\}_{n=1}^{\infty} \subset A$, then $\{(y_\alpha)_n\}_{n=1}^{\infty} \subset PC(J,\mathbb{R})$, from Arzela-Ascoli theorem, the set

$$K_0 = \{(y_\alpha)_n : n \in N^*\}$$

is relatively compact in $PC(J,\mathbb{R})$, thus there exists a subsequence of $(y_\alpha)_{n\in N}$, still denoted by $(y_\alpha)_{n\in N}$, which converges to $y \in (PC(J,\mathbb{R}), \|.\|_{PC})$.
Hence

$$\|(y_\alpha)_n - y\|_\alpha = \sup_{t\in[t_k,t_{k+1}]} (t-t_k)^{1-\alpha}\|y_n(t) - y(t)\| \to 0 \text{ as } n \to +\infty.$$

Therefore

$$\{y_n\}_{n=1}^{\infty} \longrightarrow y \; on \; PC_*(J,\mathbb{R}).$$

We need the following auxiliary lemmas that will be used throughout the paper.

Lemma 3.2 [25] Let $\alpha > 0$, then the differential equation

$${}^{RL}D^{\alpha}_{a^+}h(t) = 0,$$

has solutions $h(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$
for some, $c_i \in R$, $i = 1\ldots n$, where $n = [\alpha]+1$.

Lemma 3.3 [25] Let $\alpha > 0$, then

$$I^{\alpha RL}D^{\alpha}_{a^+}h(t) = h(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$$

for some $c_i \in R$, $i = 0,\ldots,n$, where $n = [\alpha]+1$.

Lemma 3.4 Let $0 < \alpha < 1$ and let h be a continuous function. A function y is a solution of the fractional integral equation

$$y(t)=\begin{cases} f(t,y(t))\left(t^{\alpha-1}\frac{c}{f(t_0,y(t_0))}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds\right) & \text{if } t\in[0,t_1],\\ f(t,y(t))\Big((t-t_1)^{\alpha-1}t_1^{\alpha-1}c+\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds & \\ +\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\frac{I_1(y(t_1^-))}{f(t_1,y(t_1))}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)ds\Big) & \text{if } t\in(t_1,t_2],\\ f(t,y(t))\Big((t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c_0 & \\ +\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}h(s)ds+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)ds\right] & \\ +\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))}+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right] & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}h(s)ds\Big), & \text{if } t\in(t_k,t_{k+1}],\ k=2,\dots,m. \end{cases} \tag{7}$$

if and only if y is a solution of the fractional initial-value problem

$$D^{\alpha}\left(\frac{y(t)}{f(t,y(t))}\right)=h(t),\quad t\in J' \tag{8}$$

$$\Delta^* y|_{t_k}=I_k(y(t_k^-)),\ k=1\dots m, \tag{9}$$

$$\lim_{t\to 0}t^{1-\alpha}y(t)=c. \tag{10}$$

Proof Assume y satisfies (8)-(10). If $t\in[0,t_1]$ then ${}^{RL}D^{\alpha}\left(\frac{y(t)}{f(t,y(t))}\right)=h(t)$. Lemmas 3.2 and 3.3 imply

$$\frac{y(t)}{f(t,y(t))}=t^{\alpha-1}c_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds.$$

Hence $c_0=c$. Thus

$$y(t)=[f(t,y(t))]\ \left(\frac{ct^{\alpha-1}}{f(0,y(0))}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds\right).$$

If $t \in (t_1, t_2]$, then Lemmas 3.2 and 3.3 imply

$$\begin{aligned}\frac{y(t)}{f(t,y(t))} &= (t-t_1)^{\alpha-1}\frac{y^*(t_1^+)}{f(t_1,y(t_1))} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}h(s)ds \\ &= (t-t_1)^{\alpha-1}\left(\frac{I_1(y(t_1^-)+y(t_1^-)}{f(t_1,y(t_1))}\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}h(s)ds \\ &= (t-t_1)^{\alpha-1}\frac{t_1^{\alpha-1}c}{f(0,y(0))} + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}h(s)ds + \frac{(t-t_1)^{\alpha-1}}{f(t_1,y(t_1))}I_1(y(t_1^-)).\end{aligned}$$

Thus

$$\begin{aligned}y(t) = [f(t,y(t))]\Bigg((t-t_1)^{\alpha-1}\frac{t_1^{\alpha-1}c}{f(0,y(0))} + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds \\ + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}h(s)ds + \frac{(t-t_1)^{\alpha-1}}{f(t_1,y(t_1))}I_1(y(t_1^-))\Bigg).\end{aligned}$$

If $t \in (t_2, t_3]$, then Lemmas 3.2 and 3.3 imply

$$\begin{aligned}\frac{y(t)}{f(t,y(t))} &= (t-t_2)^{\alpha-1}\frac{y^*(t_2^+)}{f(t_2,y(t_2))} + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}h(s)ds \\ y(t) &= (t-t_2)^{\alpha-1}[\frac{y(t_2^-)+I_2(y(t_2^-))}{f(t_2,y(t_2))}] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}h(s)ds \\ &= (t-t_2)^{\alpha-1}\frac{(t_2-t_1)^{\alpha-1}t_1^{\alpha-1}c}{f(0,y(0))} + \frac{(t-t_2)^{\alpha-1}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds \\ &+ \frac{(t-t_2)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(t)ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}h(s)ds \\ &+ (t-t_2)^{\alpha-1}\left[(t_2-t_1)^{\alpha-1}\frac{I_1(y(t_1^-))}{f(t_1,y(t_1))} + \frac{I_2(y_2(t_2^-))}{f(t_2,y(t_2))}\right].\end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.2 and 3.3, we get (7). Conversely, assume that y satisfies the impulsive fractional integral equation (7). If $t \in [0, t_1]$ then $\lim_{t\to 0} t^{1-\alpha}y(t) = c_0$ and using the fact that ${}^{RL}D^{\alpha}$ is the left inverse of I^{α} we get

$$ {}^{RL}D^{\alpha}y(t) = h(t), \quad \text{for each } t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$ and using the fact that ${}^{RL}D^{\alpha}C = 0$, where C is a constant,

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we get

$$^{RL}D^{\alpha}y(t) = h(t), \text{for each } t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta^* y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

Set

$$J' := J \backslash \{t_1, \dots, t_m\}.$$

Let us define what we mean by a solution of problem $(4)-(6)$.

Definition 3.5 A function $y \in PC_* \bigcap \bigcup_{k=0}^{m} AC(J_k, \mathbb{R})$ is said to be a solution of problem $(4)-(6)$ if there exists $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, y(t))$ a.e. $t \in J$, $\lim_{t \longrightarrow 0} t^{1-\alpha} y(t) = c$, $\Delta y_{|t=t_k} = I_k(y(t_k^-))$ and $^{RL}D^{\alpha}y(t) = v(t)$, a.e. $t \in J'$.

Let us introduce the following assumptions.

(H_0) There exist a nonnegative constant K, $\overline{M}$ such that one has,

$$|f(t,u_2) - f(t,u_1)| \leq K|u_2 - u_1| \text{ for each } u_2, \ u_1 \in \mathbb{R}, \text{ and } t \in [0,T].$$

$$|f(t,u)| \leq \overline{M} \text{ for each } u \in \mathbb{R}, \text{ and } t \in [0,T].$$

(H1) $F : J \times \mathbb{R} \longrightarrow \mathscr{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multi-valued map;

(H2) There exist $p \in C(J, R_+)$ and $q \in C(J, \mathbb{R}^+)$ continuous such that

$$\|F(t,u)\|_{\mathscr{P}} \leq p(t)|u| + q(t), \text{ for } t \in J \text{and } u \in \mathbb{R}.$$

(H3) There exists $l \in L^1(J, \mathbb{R})$, with $I^{\alpha} l < \infty$ such that

$$H_d(F(t,u), F(t,\overline{u})) \leq l(t)|u - \overline{u}| \text{ for every } u, \overline{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq l(t), \text{ a.e. } t \in J.$$

(H_4) There exists a constant r

$$\begin{aligned} & T^{m(\alpha-1)} \frac{c}{\overline{\overline{M}}} + \frac{(T^{\alpha}\|q\|_{\infty} + \alpha b^*)(md^{m(\alpha-1)} + 1) + T\|q\|_{\infty}}{\Gamma(\alpha+1)} \\ & + r \left(\frac{(md^{m(\alpha-1)} + 1)(d^{\alpha-1} \frac{a^*}{\overline{\overline{M}}})}{\Gamma(\alpha)} + \frac{\Gamma(\alpha)\|p\|_{\infty}\mathfrak{D}}{\Gamma(2\alpha)} \right) \\ & \leq r. \end{aligned} \tag{11}$$

and

$$2T^{m(\alpha-1)}\frac{cK}{\overline{M}}+2K\frac{(T^{\alpha}\|q\|_{\infty}+\alpha b^{*})(md^{m(\alpha-1)}+1)+T\|q\|_{\infty}}{\Gamma(\alpha+1)}$$
$$+2Kr\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^{*}}{\overline{M}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_{\infty}\mathfrak{D}}{\Gamma(2\alpha)}.\right)$$
$$<1. \tag{12}$$

where $a^{*}=\max_{i=\overline{1,m}}a_{i},\ b^{*}=\max_{i=\overline{1,m}}b_{i},\ d=\max_{i=1,\dots,m}|t_{i}-t_{i+1}|$

$$\mathfrak{D}=md^{m(\alpha-1)}T^{2\alpha-1}+T^{\alpha}+T^{2\alpha-1}.$$

(H5) There exist constants $a_k,b_k\in R^{+}$ such that

$$|I_k(u)|\le a_k|u|+b_k \quad \text{for } u\in\mathbb{R}.$$

Theorem 3.6 Assume that the conditions (H_1)-(H_5) hold if

$$\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^{*}}{\overline{M}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_{\infty}\mathfrak{D}}{\Gamma(2\alpha)}\right)<1,$$

the problem $(4)-(6)$ has at least one solution.

Remark 3.7 [22] Under conditions (H_1) for every continuous function

$$y:[\alpha,\beta]\subset J\to\mathbb{R},$$

the multifunction $t\to F(t,y(t))$ admits a mesurable selection $v\in L^1([\alpha,\beta],\mathbb{R})$.

It follows then,

$$S_{F,y}=\{v\in L^1(J,\mathbb{R}):v(t)\in F(t,y(t)),\quad \text{a.e.}\ \ t\in J\},$$

is nonempty set (see [22]), and then the superposition multioperator

$$S_F:PC_{*}(J,\mathbb{R})\to\mathscr{P}(L^1(J,\mathbb{R})),$$

defined by

$$S_{F,y}=\{v\in L^1(J,\mathbb{R}):v(t)\in F(t,y(t)),\quad \text{a.e.}\ \ t\in J\}$$

is correctly defined from remark(3.7). Notice that the space $PC_*(J,\mathbb{R})$ has also structure of Banach algebra, where the multiplication is defined as the usual product of real functions. Set $X = PC_*(J,\mathbb{R})$ and define a subset S of X as

$$S = \{x \in X : \|x\|_\alpha \leq r\},$$

where r satisfies inequality (11). Clearly S is closed, convex and bounded subset of the Banach space X. Transform the problem (4)-(6)into a fixed point problem. Consider the operator

$$\mathfrak{A},\ PC_*(J,\mathbb{R}) \to PC_*(J,\mathbb{R}),\ \mathfrak{B} : PC_*(J,\mathbb{R}) \to \mathscr{P}(PC_*(J,\mathbb{R})),$$

defined by

$$\mathfrak{A}(t) = f(t,y(t))$$

and multivalued operator,

$$\mathfrak{B} : PC_*(J,\mathbb{R}) \to \mathscr{P}(PC_*(J,\mathbb{R})),$$

$$\mathfrak{B}(y) = \left\{ h \in PC_*, h(t) = \left\{ \begin{array}{l} (t-t_k)^{\alpha-1} \prod\limits_{i=1}^{i=k} (t_i - t_{i-1})^{\alpha-1} \frac{c}{f(0,y(0)} \\ + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} h(s) ds \right. \\ \left. + \sum\limits_{i=1}^{k-1} \prod\limits_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds \right] \\ + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))} \right. \\ \left. + \sum\limits_{i=1}^{k-1} \prod\limits_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \frac{I_i(y(t_i^-))}{f(t_i,y(t_i))} \right] \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s) ds, \end{array} \right. \right., \ .$$

where $v \in S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}$. It is clear that the operators $\mathfrak{A}$ and multi-valued operator $\mathfrak{B}$ are well defined and the set of all solution for the problem $(4)-(6)$ on J is the set $\mathscr{F}ix\mathfrak{A}\mathfrak{B} = \{y : y \in \mathfrak{A}(y)\mathfrak{B}(y)\}$.
Clearly, from Lemma (3.4), fixed points of $\mathfrak{A}\mathfrak{B}$ are solutions to (4)-(6). We shall show that $\mathfrak{A}$ and $\mathfrak{B}$ satisfies the assumptions of Dhage fixed point(2.2). For the sake of clarity, we split the proof into a sequence of steps

Step 1 $\mathfrak{A}$ is Lipschitz .

We first show that $\mathfrak{A}$ is Lipschitz operator on $PC_*(J,R)$. Let $x,y \in PC_*(J,R)$. Then

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by (H_0) for each $t \in (t_k, t_{k+1}]$ we have,

$$t^{1-\alpha}|\mathfrak{A}x(t) - \mathfrak{A}x(t)| \leq Kt^{1-\alpha}|x(t) - y(t)|$$
$$K\|x - y\|_*$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|\mathfrak{A}x - \mathfrak{A}x\|_* \leq K\|x - y\|_*$$

for all $x, y \in PC_*(J, R)$. So $\mathfrak{A}$ is a Lipschitz on $PC_*(J, R)$ with a Lipschitz constant K.

Step 2 $\mathfrak{B}(y)$ is convex for each $y \in PC_*(J, \mathbb{R})$.

Indeed, if h_1, h_2 belong to $\mathfrak{B}(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in (t_k, t_{k+1}]$ we have

$$h_i(t) = (t - t_k)^{\alpha-1} \prod_{i=1}^{i=k} (t_i - t_{i-1})^{\alpha-1} \frac{c}{f(0, y(0))} \tag{13}$$

$$+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} v_i(s) ds \right.$$
$$\left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v_i(s) ds \right]$$
$$+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\frac{I_k(y(t_k^-))}{f(t_k, y(t_k))} + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \frac{I_i(y(t_i^-))}{f(t_i, y(t_i))} \right]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} v_i(s) ds.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1 - d)h_2)(t) = \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} (dv_1(s) + (1 - d)v_2(s)) ds \right.$$
$$\left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} (dv_1(s) + (1 - d)v_2(s)) ds \right]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} (dv_1(s) + (1 - d)v_2(s)) ds$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1 - d)h_2 \in \mathfrak{B}(y).$$

Step 3: $\mathfrak{B}$ maps bounded sets into bounded sets in $PC_*([0,T],R)$.
Indeed, it is enough to show that there exists a positive constant l such that for each $y \in B_\eta = \{y \in PC_*([0,T],R) : \|y\|_{PC_*} \leq \eta\}$ one has $\|\mathfrak{B}(y)\|_{PC_*} \leq l$.
Then using condition (H_3) we deduce that for each $h \in \mathfrak{B}(y)$, there exists $v \in S_{F,y}$ such that

$$|(t-t_k)^{1-\alpha}h(t)| \leq \prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{f(0,y(0))}\right| + \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\left(\left|\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))}\right| + \sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\left|\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right|\right)$$
$$+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_n(s)|ds$$

$$\leq d^{m(\alpha-1)}\frac{c}{f(0,y(0))} + \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left|\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right| + \frac{1}{\Gamma(\alpha)}\left|\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))}\right| + \frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_n(s)|ds$$
$$\leq d^{m(\alpha-1)}\left|\frac{c}{f(0,y(0))}\right| + \frac{\|q\|_\infty T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}(\|p\|_\infty\|y(.)\|_{PC_*})\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})^{\alpha-1}ds$$
$$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\left[\frac{mT^\alpha\|q\|_\infty}{\alpha} + (\|p\|_\infty\|y(.)\|_{PC_*})\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_{i-1})^{\alpha-1}ds\right]$$
$$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left((s-t_i)^{\alpha-1}a_i\frac{\|y(.)\|_{PC_*}}{f(t_i,y(t_i))} + b_i\right) + \frac{1}{\Gamma(\alpha)}\left(a_k(s-t_i)^{\alpha-1}\frac{\|y(.)\|_{PC_*}}{f(t_k,y(t_k)} + b_k\right)$$
$$+\frac{T\|q\|_\infty}{\Gamma(\alpha+1)} + \frac{T^{1-\alpha}}{\Gamma(\alpha)}(\|p\|_\infty\|y(.)\|_{PC_*})\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds,$$

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Thus

$$\|\mathfrak{B}(y)\|_{PC_*} \leq T^{m(\alpha-1)}\frac{c}{\overline{\overline{M}}} + \frac{(T^{\alpha}\|q\|_{\infty}+\alpha b^*)(md^{m(\alpha-1)}+1)+T\|q\|_{\infty}}{\Gamma(\alpha+1)}$$
$$+\eta\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^*}{\overline{\overline{M}}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_{\infty}\mathfrak{D}}{\Gamma(2\alpha)}.\right) := \ell.$$

Step 4: $\mathfrak{B}$ maps bounded sets into equicontinuous sets of $PC_*([0,T],\mathbb{R})$.

Let $\tau_1,\tau_2 \in (0,T],\ \tau_1<\tau_2$ and B_η be a bounded set of $PC_*([0,T],\mathbb{R})$ as in step 2, let $y\in B_\eta$ and $h\in\mathfrak{B}(\eta)$, then

$|(\tau_2-t_k)^{1-\alpha}h(\tau_2)-(\tau_1-t_k)^{1-\alpha}h(\tau_1)| \leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{f(0,y(0))}\right|$

$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds)$

$|(\tau_2-t_k)^{1-\alpha}h(\tau_2)-(\tau_1-t_k)^{1-\alpha}h(\tau_1)| \leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{f(0,y(0))}\right|$

$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds)$

$+\frac{|(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}v_n(s)ds$

$+\frac{(\tau_1-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]v_n(s)ds$

$+\frac{(\tau_2-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}v_n(s)ds$

$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\left|\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right|)$

$\leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{f(0,y(0))}\right|$

$+\frac{\|q\|_\infty}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds)$

$+\frac{\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_i)^{\alpha-1}ds)$

$+\frac{[(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}]\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}ds$

$+\frac{[(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}]\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds$

$+\frac{(\tau_1-t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds$

$+\frac{(\tau_1-t_k)^{1-\alpha}\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}](s-t_k)^{\alpha-1}ds$

$+\frac{(\tau_2-t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds$

$+\frac{(\tau_2-t_k)^{1-\alpha}\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds$

$+\frac{d^{\alpha-1}}{\Gamma(\alpha)}\left(a^*\frac{\eta}{M}+b^*\right)\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1})$

$\leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{M}\right|$

$+\frac{(t_i-t_{i-1})^{2\alpha-1}B(\alpha,\alpha)}{\Gamma(\alpha)}(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1})$

$+\frac{|(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}|(\tau_1-t_k)^{2\alpha-1}B(\alpha,\alpha)}{\Gamma(\alpha)}(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)$

$+\frac{|(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}|(\tau_2-t_k)^{2\alpha-1}B(\alpha,\alpha)}{\Gamma(\alpha)}(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)$

$+\frac{(\tau_2-\tau_1)^{\alpha}}{\Gamma(\alpha+1)}(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)$

$+\frac{d^{\alpha-1}}{\Gamma(\alpha)}\left(a^*\frac{\eta}{M}+b^*\right)\sum_{0<t_i<\tau_2-\tau_1}(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1})$. As $\tau_2\longrightarrow\tau_1$ the right-hand side of the above inequality tends to zero, then $\mathfrak{B}(B_\eta)$ is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $\mathfrak{B}: PC_*(J,R)\longrightarrow(\mathscr{P}(PC_*(J,R))$ is completely continuous.

Step 5 $\mathfrak{B}$ has a closed graph.

Let $y_n\to y_*$, $h_n\in\mathfrak{B}(y_n)$ which satisfies (13) and $h_n\to h_*$, then we need to show that $h_*\in\mathfrak{B}(y_*)$ associated with $h_n\in\mathfrak{B}(y_n)$, there exists $v_n\in S_{F,y_n}$ and $v_*\in S_{F,y_*}$ such that for each $t\in J$

$$\begin{aligned}
h_*(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}\frac{c}{f(0,y(0))}\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_*(s)ds+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_*(s)ds\right]\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y_*(t_k^-))}{f(t_k,y_*(t_k))}+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\frac{I_i(y_*(t_i^-))}{f(t_i,y_*(t_i))}\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_*(s)ds
\end{aligned}$$

Since $F(t,.)$ is upper semicontinuous by (H_1), then for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t,y_n(t)) \subset F(t,y_*(t)) + \varepsilon B(0,1) \quad a.e.\ t \in J.$$

Since $F(.,.)$ has compact values then there exists a subsequence $v_{n_k}(.)$ such that

$$v_{n_k}(.) \longrightarrow v_*(.) \quad as \quad k \to \infty,$$

$$v_*(t) \in F(t,y_*(t)) \quad a.e. \quad t \in J.$$

For every $w \in F(t,y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t,y_*(t)).$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_* and using condition (H_2), it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t,y_n(t)), F(t,y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$|h_n(t) - h_*(t)| = \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}|v_n(t) - v_*(s)|ds\right.$$

$$\left. + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1} - t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}|v_n(t) - v_*|(s)ds\right]$$

$$+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y_n(t_k^-))}{f(t_k,y_n(t_k))} + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1} - t_{k-j})^{\alpha-1}\frac{I_i(y_n(t_i^-))}{f(t_i,y_n(t_i))}\right]$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_n(t) - v_*(s)|ds$$

$$- \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y_*(t_k^-))}{f(t_k,y_*(t_k))} - \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1} - t_{k-j})^{\alpha-1}\frac{I_i(y_*(t_i^-))}{f(t_i,y_*(t_i))}\right]$$

$$\leq \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\|y_{n_m}(t) - y_*(s)\|ds\right.$$

$$\left. + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1} - t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\|y_{n_m}(s) - y_*(s)\|ds\right]$$

$$+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{I_k(y_n(t_k^-))}{f(t_k,y_n(t_k))} - \frac{I_k(y_*(t_k^-))}{f(t_k,y_*(t_k))}\right| + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|y_{n_m}(t) - y_*(s)|ds$$

$+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\left|\frac{I_i(y_n(t_i^-))}{f(t_i,y_n(t_i))}-\frac{I_i(y_*(t_i^-))}{f(t_i,y_*(t_i))}\right|$ Since $\|y_{n_m}-y_*\|\to 0$ as $m\to 0$, one has $\|h_n-h_*\|\to 0$ as $n\to 0$.

Step 6 $\mathfrak{A}y\mathfrak{B}y$ is a convex subset of PC_* for each $y\in PC_*$.
arguing as in step 1 one has easily,

$$\mathfrak{A}y\mathfrak{B}y \text{ is a convex subset of } PC_* \text{ for each, } y\in PC_*.$$

Step 7. Now we shall show that $2MK\leq 1$, where $M=\|\mathfrak{B}(S)\|=\sup_{y\in S}|\mathfrak{B}(y)|$ and K the constant Lipschitz from condition Step 3 one obtain,

$$2MK=2M\|\mathfrak{B}(y)\|_{PC_*}\leq 2M\left[T^{m(\alpha-1)}\frac{c}{\overline{\overline{M}}}+\frac{(T^{\alpha}\|q\|_{\infty}+\alpha b^*)(md^{m(\alpha-1)}+1)+T\|q\|_{\infty}}{\Gamma(\alpha+1)}\right.$$
$$\left.+r\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^*}{\overline{\overline{M}}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_{\infty}\mathfrak{D}}{\Gamma(2\alpha)}\right)\right]$$

it follows then from condition (H_4) inequality (13)

$$2MK=2M\|\mathfrak{B}(y)\|_{PC_*}<1,$$

Step 8 A priori bounds on solutions.
We shall prove now that the set $\mathscr{E}=\{u\in PC_*,\ \mu y\in\mathfrak{A}y\ \mathfrak{B}y,\mu>1\}$is bounded. Let y be $\mu y\in\mathfrak{A}y\ \mathfrak{B}y,\ \mu>1$. Then, there exists $v\in S_{F,y}$ such that, for each $t\in J$,

$$|(t-t_k)^{1-\alpha}y(t)|\leq\frac{1}{\mu}|f(t,y(t))|\left(\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}\left|\frac{c}{f(0,y(0))}\right|\right.$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_n(s)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\left(\left|\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))}\right|+\sum_{0<t_i<t}\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\left|\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right|\right)$$
$$\left.+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_n(s)|ds\right)$$
$$\leq\frac{\overline{M}}{\mu}\left(d^{m(\alpha-1)}\frac{c}{f(0,y(0))}+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_n(s)|ds\right.$$

$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_n(s)|ds$

$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left|\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right|+\frac{1}{\Gamma(\alpha)}\left|\frac{I_k(y(t_k^-))}{f(t_k,y(t_k))}\right|+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_n(s)|ds\Big)$

$\leq\frac{\overline{M}}{\mu}\Big(d^{m(\alpha-1)}\left|\frac{c}{f(0,y(0))}\right|+\frac{\|q\|_\infty T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}(\|p\|_\infty\|y(.)\|_{PC_*})\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})^{\alpha-1}ds$

$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\left[\frac{mT^\alpha\|q\|_\infty}{\alpha}+(\|p\|_\infty\|y(.)\|_{PC_*})\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_{i-1})^{\alpha-1}ds\right]$

$+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left((s-t_i)^{\alpha-1}a_i\frac{\|y(.)\|_{PC_*}}{f(t_i,y(t_i))}+b_i\right)+\frac{1}{\Gamma(\alpha)}\left(a_k(s-t_i)^{\alpha-1}\frac{\|y(.)\|_{PC_*}}{f(t_k,y(t_k))}+b_k\right)$

$+\frac{T\|q\|_\infty}{\Gamma(\alpha+1)}+\frac{T^{1-\alpha}}{\Gamma(\alpha)}(\|p\|_\infty\|y(.)\|_{PC_*})\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds\Big)$, Thus

$$\|y\|_{PC_*}\leq cT^{m(\alpha-1)}+M\frac{T^\alpha(md^{m(\alpha-1)}+1)+T}{\Gamma(\alpha+1)}\|q\|_\infty+M\frac{b^*(md^{m(\alpha-1)}+1)}{\Gamma(\alpha)}$$
$$+\|y\|_{PC_*}\overline{M}\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^*}{\overline{M}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_\infty\mathfrak{D}}{\Gamma(2\alpha)}\right)$$

Finally

$$\|y\|_{PC_*}\leq\frac{cT^{m(\alpha-1)}+\overline{M}\frac{T^\alpha(md^{m(\alpha-1)}+1)+T}{\Gamma(\alpha+1)}\|q\|_\infty+\overline{M}\frac{b^*(md^{m(\alpha-1)}+1)}{\Gamma(\alpha)}}{1-\overline{M}\left(\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}\frac{a^*}{\overline{M}})}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)\|p\|_\infty\mathfrak{D}}{\Gamma(2\alpha)}\right)}$$

This shows that the set $\mathscr{E}$ is bounded. The he operators $\mathfrak{A}:PC_*(J,\mathbb{R})\to PC_*(J,\mathbb{R})$ and $\mathfrak{B}PC_*(J,\mathbb{R}):\to\mathscr{P}_{cp,cv}(PC_*(J,\mathbb{R}))$ satisfy all conditions of theorem (2.2). Then we deduce that the solution set $S(F,c)$ is nonempty. We shall prove in the second approach that is, the compactness of solution.

3.1 Compactness of the solution set

For each $c\in\mathbb{R}$, let $S(F,c):=\{y\in PC_*(J,\mathbb{R}):y$ is a solution of problem (4)-(6) $\}$. From the previous consideration, there exists $\widetilde{M}$ such that for every $y\in S(F,c)$, $\|y\|_\alpha\leq\widetilde{M}$. Since $\mathfrak{AB}$ is completely continuous, $(S(F,c)$ is relatively compact in PC_*. Let $y\in S(F,c)$, then $y\in\mathfrak{A}(y)\mathfrak{B}(y)$ and hence $S(F,c)\subset\mathfrak{AB}(S(F,c))$. It remains to prove that $S(F,c)$ is a closed subset in PC_*. Let $\{y_n:n\in\mathbb{N}\}\subset S(F,c)$ be such that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n\in\mathbb{N}$, there exists v_n such that $v_n(t)\in$

$F(t,y_n(t))$, a.e. $t\in J$ and

$$y_n(t)=f(t,y_n(t))\Bigg((t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}\frac{c}{f(0,y(0))}$$
$$+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds\right]$$
$$+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y_n(t_k^-))}{f(t_k,y_n(t_k))}\right)+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\frac{I_i(y_n(t_i^-))}{f(t_i,y_n(t_i))}$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds\Bigg)$$

Arguing as in **Step 4**, we can prove that there exists v such that $v(t)\in F(t,y(t))$ and

$$y(t)=f(t,y(t))\Bigg((t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}\frac{c}{f(0,y(0))}$$
$$+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)ds+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\right]$$
$$+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{I_k(y_*(t_k^-))}{f(t_k,y(t_k))}+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\frac{I_i(y(t_i^-))}{f(t_i,y(t_i))}\right]$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds\Bigg)$$

Therefore $y\in S(F,c)$ which yields that $S(F,c)$ is closed, and hence compact in PC_*.□

4 Example

As an application of our first main result, we consider the hybrid impulsive fractional differential inclusion

$$^{RL}D^{\frac{1}{2}}\left(\frac{y(t)}{f(t,y(t))}\right)\in F(t,y(t)),\ \text{a.e. } t\in J=(0,1],\ t\neq\frac{1}{2}, \tag{14}$$

$$\lim_{t\longrightarrow 0^+}t^{\frac{1}{2}}y(t)=\frac{1}{4}, \tag{15}$$

$$\Delta y|_{t=\frac{1}{2}}=1+y(\frac{1^-}{2}), \tag{16}$$

where $T=1,\ t_0=0,\ t_1=\frac{1}{2},\ t_2=1$. Set Consider the multivalued map $F_2:(0,1]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ given by

$$F(t,x)=\left[0,\ \frac{1}{3}\sin x+\frac{|x|}{t+9}+\frac{1}{9}\right], \tag{17}$$

$$f(t,u)=\exp^{-t}\frac{|u|}{10(1+|u|)}+\frac{9}{10},\ I_1=1+u.$$

Then IVP (14)-(16) takes the form (4)-(6). We shall prove now that conditions H_0-H_5 are satisfied, we begin with condition H_0, it easy to see that $|f(t,u)|\leq 1.,\ \forall\, t\in[0,T],\ u\in\mathbb{R}$.
in another hand

$$\begin{aligned}|\exp^{-t}\frac{|u|}{10(1+|u|)}-\exp^{-t}\frac{|v|}{10(1+|v|)}| &= |\frac{1}{10(1+|v|)}-\frac{1}{10(1+|u|)}|\\ &\leq \frac{1}{10}|u-v|\end{aligned}$$

Hence f is Lipshitz funtion with lipshitz constant $K=\frac{1}{10},\ \overline{M}=1.$. Clearly F is compact and convex valued, in addition $\mathbb{R}$ is separable Banach and $F(.,.)\in P_{cp,cv}(R)$, we deduce that $F(.,)$ has measurable selection [24], The multivalued function $F(.,u)$ is upper semi continuous multifunction (see [18]), which implies that (H_1) is satisfied.

$$\begin{aligned}\|F(t,x)\| &= \sup\{|v|: v\in F(t,x)\}\\ &\leq \frac{8}{9}|\sin x|+\frac{9|x|}{t+9}+\frac{1}{9}\\ &\leq \frac{9}{100(t+9)}|x|+1,\end{aligned}$$

Clearly condition (H_2) holds from the previous fact with $p(t)=\frac{9}{t+9},\ q(t)=1$ and

$$H_d(F(t,x),F(t,y))\leq\left(\frac{8}{9}+\frac{9}{100(t+9)}\right)|x-y|.$$

Let $\ell(t)=\frac{8}{9}+\frac{9}{100(t+9)}$. Then
$I^{\frac{1}{2}}\ell(s)ds=\frac{1}{\Gamma(\frac{1}{2})}\int_0^1(\frac{8}{9}+\frac{9}{100(t+9)})ds=\frac{2}{\sqrt{\pi}}\left[\frac{8}{9}s+\frac{9}{100}\ln(s+9)\right]_0^1$

$=\frac{2}{\sqrt{\pi}}\left[\frac{8}{9}+\frac{9}{100}(\ln 10-9\ln 9)\right]$

this shows that $I^{\frac{1}{2}}\ell(s)ds<\infty$.

For the jump function, we have

$$I_1 = 1+u, \text{ so } |I_1| \le 1+|u|, \text{it followqs then } a_1 = 1 \text{ and } b_1 = 1$$

which implies that condition H_4 holds. let us now verify condition (H_5).

$$T=1,\ t_0=0,\ T=1,\ t_0=0,\ t_1=\frac{1}{2},\ t_2=1, c=\frac{1}{4},\ T=1, \Gamma(\frac{1}{2})=\sqrt{\pi},$$

$$m=1,\ d=1,\ \alpha=\frac{1}{2},\ \|q\|_\infty=1,\ \|p\|_\infty=\frac{1}{100},\ \Gamma(1))=1, \mathfrak{D}=3.$$

Condition (11) becomes,

$$\frac{1}{4}+\frac{8}{\pi}+r(\frac{1}{\sqrt{\pi}}+3\frac{\sqrt{\pi}}{100}) \le r.$$

It easy to verify that condition (11) is satisfied by taking $r=5$. Let us now verify the condition (13), since $K=\frac{1}{10}$ and $r=5$ condition (13) becomes,

$$\frac{1}{10}\left(\frac{1}{4}+\frac{8}{\pi}+\frac{5}{\sqrt{\pi}}+3\frac{\sqrt{\pi}}{20}\right)=0.841$$
$$<1.$$

Hence all assumptions of theorem (2.2) are satisfied which infer us from the same theorem that the solution set of the problem (14)-(16) is nonempty set and compact.

5 Conclusion

In this work, we deal with the problem concerning topological structure of solution sets for Riemman-Liouville hybrid impulsive fractional differential inclusion modeled by equation (4)-(6). The obstruction of this topic lies in the fact that the discontinuity of the state y is not defined at t_k^+ howover $\lim_{t\to t_k^+}(t-t_k)^{1-\alpha}y(t)$ exists without being null , to overcome this situation we have defined a special weight space of piecewise continuous function $PC_*(J,\mathbb{R})$. The constructed space is in a natural way, in the sense that we have reacquired Banach structure and defined the jump function $y(t_k^+) = \lim_{h\to 0^+} h^{1-\alpha}y(t_k+h)$ unlike the usual case.
The assumed hypotheses have as goals :
In this work we have assumed conditions (H_1), to ensure that the superposition multi-operator is correctly defined. we deal also with a more general affine condition (H_2) contrary as in the literature. (H_3) to prove that the operator solution has a closed graph so it is upper semi-continuous, these conditions are optimal in the sense that no condition implies the other. we make use in our approach a recent fixed point theorem for hybrid multi-valued function due to Dhage combining with analysis

functional tools. The paper concludes with an example to illustrate the feasibility of our main result.

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