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A Fixed Point Theorem for Asymptotic $\phi-$Contraction in Ultrametric Space

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Abstract

Recently, Kirk introduced an asymptotic form of Boyd-Wong fixed point theorem in metric space. In this paper, we have proved a fixed point theorem for asymptotic $\phi-$contraction in ultrametric space setting. We assume a weaker limit condition instead of boundedness of an orbit of the self map T, which was an assumption in the hypothesis of Kirk's theorem. Our result modifies the theorem given by Kirk in metric space.

Key words: Fixed point, Asymptotic contraction, Ultra metric space.
AMS classification: 47H10, 54H25.

1. Introduction

Banach Contraction Principle is astounding in its simplicity, and it may be the most extensively used fixed point theorem in nonlinear analysis. It has been modified and extended to various directions, because of its importance in nonlinear analysis. Several authors have obtained many interesting generalizations and some manuscripts introduced weaker contractive conditions (see [1], [16],[13],[4],[11] and references therein).

Among these, the generalization given by Rakotch is worth noting. He proved the following.

Theorem 1.1 [15] Let T be a self map on a complete metric space (X, d) such that

$$d(T(x), T(y)) \leq \alpha(d(x, y))d(x, y), \quad \text{for all } x, y \in X,$$

where $\alpha : \mathbb{R}^+ \to [0, 1)$ is monotonically non increasing. Then T has a unique fixed point and $\{T^n(x)\}$ converges to the fixed point for each $x \in X$.

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Later, in 1969, Boyd-Wong[12] proved more generalized version of Theorem (1.1), assuming the upper semicontinuity from right for the function involved in the contractive condition.

Theorem 1.2 [12] Suppose the self map T on a complete metric space (X, d) satisfies

$$d(Tx, Ty) \leq \xi(d(x, y)), \quad \text{for each } x, y \in X,$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semicontinuous from the right $\big($that is, $r \downarrow t \implies \limsup_{r \to t} \xi(r) \leq \xi(t)\big)$ and $0 \leq \xi(t) < t \; for \; t > 0$. Then T has a unique fixed point and the iterate sequence $\{T^n(x)\}$ converges to the fixed point for each $x \in X$.

Recently, Kirk[3] introduced an 'asymptotic' form of Boyd-Wong theorem with a nonconstructive proof(using ultrapower techniques) and in [2], authors have modified the result and gave a constructive proof for the same. In Kirk's result, the self map T is assumed to be continuous, whereas in [2], they presumed the uniform continuity of T. In this paper we have proved existence and uniqueness of fixed point for asymptotic $\phi-$contraction in ultrametric space setting in which we assumed only the continuity of the map involved. We have given a constructive proof for the main result. In recent years, ultrametric space have been studied rigorously and many fixed point theorems is proved in this setting. For more details we refer to [14], [5],[8] [6],[10],[9] and references therein.

2. Preliminaries

The definitions and results in this section will be used in following sequel. Throughout this paper $\mathbb{R}$, $\mathbb{R}^+$ denote the set of all real numbers and nonnegative real numbers respectively.

Definition 2.1 [7] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+$ is an ultrametric on X if, for all $x_1, x_2, x_3 \in X$,

(i) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$,

(ii) $d(x_1, x_2) = d(x_2, x_1)$,

(iii) $d(x_1, x_2) \leq \max\{d(x_1, x_3), d(x_3, x_2)\}$.

Then the pair (X, d) is known as an ultra metric space.

Example 2.2

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(1) Any nonempty set X with discrete metric, is an ultra metric.

(2) $(\mathbb{R}, d)$, where $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $d(x_1, x_2) = \max\{|x_1|, |x_2|\}$ is an ultrametric space.

Observe that each ultrametric is a metric. The converse need not hold. The metric $d(u, v) = |u - v|$ on $\mathbb{R}$ is not an ultrametric.
The following theorem plays a major role in the main result of this paper. It is an analogues of Cantor's intersection theorem in ultrametric space.

Theorem 2.3 Let (X, d) be a complete ultrametric space. Whenever $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$, is a nested sequence of nonempty closed sets with diam $(F_n) \to 0$, then $\bigcap\limits_{n=1}^{\infty} F_n = \{z\}$, for some $z \in X$, where $diam(F_n)$ denotes the diameter of the set F_n.

Definition 2.4 Let $T : X \to X$ be a self map and let $z \in X$, we define the orbit of T at z as

$$O_T(z) = \{z, Tz, T^2z, \cdots\}.$$

The next definition for asymptotic contraction was stated by Kirk in [3]. Let Φ denotes a class of functions, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, ϕ is continuous and $\phi(s) < s$ for all $s > 0$.

Definition 2.5 [3] Let (X, d) be a metric space. A mapping $T : X \to X$ is an asymptotic $\phi-$contraction if,

$$d(T^k(x), T^k(y)) \leq \phi_k(d(x, y)) \text{ for all } x, y \in X, \tag{1}$$

where $\phi_k : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi_k \to \phi \in \Phi$, uniformly on the range of the metric d.

3. Main Results

Let us prove some lemmas which will be used in the proof of main result.

Lemma 3.1 Let (X,d) be an ultrametric space and the self map T be an asymptotic $\phi-$contraction such that

$$\limsup_{s \to \infty}(s - \phi(s)) = \infty. \tag{2}$$

Then all orbits of T are bounded.

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Proof: Since T is an asymptotic $\phi-$contraction, there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ such that, $\phi_n \to \phi \in \Phi$ uniformly. Then, for $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\phi_n(s) < \phi(s) + \epsilon \quad \forall\ n \geq N_0.$$

In particular for $\epsilon = 1$, we have $k \in \mathbb{N}$ such that

$$\phi_k(s) < \phi(s) + 1.$$

Define $\xi(s) = \phi(s) + 1$, also observe that

$$d(T^k x_1, T^k x_2) \leq \xi(d(x_1, x_2)) \tag{3}$$

and from (2)

$$\limsup_{s\to\infty}(s - \xi(s)) = \infty. \tag{4}$$

Since, for $z \in X$, the orbit, $O_T(z) = \bigcup_{i=0}^{k-1} O_{T^k}(T^i z)$, it is enough to prove that O_{T^k} is bounded for fixed $k \in \mathbb{N}$. Now, fix $x_0 \in X$. Then from (4), there is $M > 0$ such that

$$M - \xi(M) \geq d(x_0, T^k x_0).$$

We use induction to prove

$$d(x_0, T^{kn} x_0) \leq M \tag{5}$$

for each $n \in \mathbb{N}$. The case $n = 1$ follows from the choice of M. Assume now (5) is true for some $n \in \mathbb{N}$. Then

$$d(x_0, T^{k(n+1)} x_0) \leq \max\left\{d(x_0, T^k x_0), d(T^k x_0, T^k(T^{kn})x_0)\right\}.$$

If $\max\left\{d(x_0, T^k x_0), d(T^k x_0, T^k(T^{kn})x_0)\right\} = d(x, T^k x_0)$, then by induction argument the result follows.

Else, if $\max\left\{d(x_0, T^k x_0), d(T^k x_0, T^k(T^{kn})x_0)\right\} = d(T^k x_0, T^k(T^{kn})x_0)$, then

$$d(T^k x_0, T^k(T^{kn})x_0) \leq \xi(d(x_0, T^{kn} x_0)) \leq \xi(M) \leq M.$$

Hence, $O_{T^k}(x_0) \subseteq B_M(x_0)$, ball of radius M centered at x_0.

Lemma 3.2 Let (X,d) be an ultrametric space and T : X $\to$ X be an asymptotic ϕ-

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contraction and ϕ satisfies the condition (2) . Then

$$\lim_{k\to\infty} d(T^k x, T^k y) = 0 \quad for \quad all\ x, y \in X. \tag{6}$$

Proof: The proof proceeds using the same arguments as in Lemma 4 of [2].
Next, we shall prove the existence and uniqueness of fixed point for an asymptotic $\phi-$contraction. In [3], Kirk assumed the continuity of T as well as the continuity of ϕ_n, along with that, the boundedness of an orbit of T was essential. In our result the continuity of ϕ_n is not required and the boundedness of orbit is replaced by the limit condition (2).

Theorem 3.3 Let (X,d) be a complete ultrametric space and T is a continuous asymptotic ϕ-contraction, where ϕ satisfies the condition

$$\lim_{s\to\infty} (s - \phi(s)) = \infty.$$

Then T has a unique fixed point.

Proof: The idea of the proof is to construct a decreasing sequence of closed sets, as in Theorem (1.1). Without loss of generality assume that ϕ is nondecreasing. Then by Lemma (3.2), we have, $d(T^n z, T^{n+1} z) \to 0, \quad for \quad all\ z \in X$. Thus we get

$$\inf\{d(z, Tz) : z \in X\} = 0. \tag{7}$$

Now, define $A_n := \{z \in X \ : \ d(z, Tz) \le \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then (7) gives that, A_n's are nonempty. Since T is continuous, A_n's are closed.
Claim 1: diam $(A_n) = 0$.
Suppose not. Then diam $(A_n) = \delta > 0$. Since A_n's are nonempty, we can choose points z_n, y_n in A_n satisfying $d(z_n, y_n) \ge \dfrac{\delta}{3}$. But

$$d(z_n, y_n) \le \max\{d(z_n, Tz_n), d(Tz_n, Ty_n), d(Ty_n, y_n)\}.$$

In either of the three cases, it is evident that, $d(z_n, y_n) \to 0 \ as \ n \to \infty$. Thus we get a contradiction. Hence $\lim\limits_{n\to\infty} diam(A_n) = 0$. We are now in a situation to use Theorem (2.3). Since X is a complete ultrametric space, we have $\bigcap\limits_{n=1}^{\infty} A_n = \{z_0\}$. By the construction of A_n, it follows that z_0, inevitably, is the unique fixed point for T.

Example 3.4 Consider $X = l^2$ and the metric defined by $d(\hat{x}, \hat{y}) = \max\{\|\hat{x}\|, \|\hat{y}\|\}$, where

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$\|\hat{x}\| = \left(\sum |x_i|^2\right)^{1/2}$ and $\hat{x} = (x^1, x^2, \cdots, x^n, \cdots)$, $\hat{y} = (y^1, y^2, \cdots, y^n, \cdots)$. Then (X, d) is an ultrametric space.
Define $T : l^2 \to l^2$ by $T(\hat{x}) = \dfrac{\hat{x}}{3}$. Note that T is continuous. Let $\phi_n(s) = \dfrac{1}{2^n}$. Here $\phi_n \to \phi$ uniformly, where $\phi(s) = 0$, for all $s \in [0, \infty]$, ϕ is continuous and nondecreasing and $\phi(s) < s$ for all $s > 0$. So T satisfies all the assumptions of Theorem (3.3), and hence it has a unique fixed point, which is $\hat{x} = \hat{0} = (0, 0, \cdots)$.

Remark 3.5 Theorem (3.3) is proved in metric space presuming the uniform continuity of the self map T in [2]. The question of relaxing the continuity of T with upper semicontinuity is still open. We can also check whether the limit condition is suffices to assert the existence of fixed point. Modification of the hypothesis of Theorem (3.3) can be considered, as it may give fixed point theorems for an extensive class of self maps.

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