



Fibonacci Sequence Generated from Three-Dimensional q -Difference Equation

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Abstract

In this research work, the authors define three dimensional q -difference operator and present some significant results using the inverse of the three dimensional q -difference operator. Also the summation solution and the closed form solutions of polynomials and logarithmic functions using three dimensional q -difference equation have been obtained. This work also includes generalized product formula for polynomial, reciprocal of polynomial and logarithmic functions using inverse of three dimensional q -difference operator. Consequently, relevant examples are being given to investigate the results..

Key words: Fibonacci numbers, higher order q -difference operator and Summation solution.

AMS classification: 39A10, 47B39, 39A70, 49M.

1. Introduction

The study of q -difference equations, initiated at the beginning of the twentieth century in intensive works especially by Jackson [6], Carmichael [5] and other authors such as Poincare, Picard, Ramanujan, is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics. In 1984, Jerzy Popenda [3] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. Recently, G.Britto Antony Xavier et al. [2] have got the solution of the generalized q -difference equation $\Delta_q^t v(k) = u(k)$, $k \in (-\infty, \infty)$ and $q \neq 1$. In [4], the authors introduced q -alpha difference operator defined as

$$\Delta_{(q)\alpha} u(k) = u(qk) - \alpha u(k), \quad (1)$$

and then extended to generalized higher order q -alpha difference equation

$$\Delta_{(q_1)\alpha_1} \left(\Delta_{(q_2)\alpha_2} \left(\cdots \Delta_{(q_t)\alpha_t} (v(k)) \cdots \right) \right) = u(k), \quad k \in (-\infty, \infty), \quad (2)$$

and obtained finite q -alpha multi-series formula and finite higher order q -alpha series formula.

With this background in this article, we introduce three dimensional q -difference operator and derive formula for finite series to polynomials and logarithmic functions by finding the numerical and closed form solutions of the three dimensional q -difference equation.

2. Three dimensional q -difference operator

Before stating and proving our results, we present basic definitions and preliminary results which will be used for the subsequent discussions.

Definition 2.1 Let p_1 and p_2 be the fixed reals and $(k_1, k_2, k_3) \in \mathfrak{R}^3$. Then the three-dimensional q -difference operator $\Delta_{(p_1, p_2)}^q$ is defined as

$$\Delta_{(p_1, p_2)}^q u(k_1, k_2, k_3) = u(q^2 k_1, q^2 k_2, q^2 k_3) - p_1 u(q k_1, q k_2, q k_3) - p_2 u(k_1, k_2, k_3) \quad (3)$$

and its inverse, denoted by $\Delta_{(p_1, p_2)}^{-1}$, is defined as below:

$$\text{if } \Delta_{(p_1, p_2)}^q v(k_1, k_2, k_3) = u(k_1, k_2, k_3), \text{ then } v(k_1, k_2, k_3) = \Delta_{(p_1, p_2)}^{-1} u(k_1, k_2, k_3). \quad (4)$$

Lemma 2.2 If $q^{2n} - p_1 q^n - p_2 \neq 0$ for $n = 0, 1, 2, \dots$, then

$$\Delta_{(p_1, p_2)}^{-1} (k_1 k_2 k_3)^n = \frac{(k_1 k_2 k_3)^n}{q^{2n} - p_1 q^n - p_2} \text{ and } \Delta_{(p_1, p_2)}^{-1} (1) = \frac{1}{1 - p_1 - p_2}. \quad (5)$$

Proof: Replacing $u(k_1, k_2, k_3)$ by $(k_1 k_2 k_3)^n$ in (3), we find

$$\begin{aligned} \Delta_{(p_1, p_2)}^q (k_1 k_2 k_3)^n &= (q^2 (k_1 k_2 k_3))^n - p_1 (q^n (k_1 k_2 k_3))^n - p_2 (k_1 k_2 k_3)^n \\ (k_1 k_2 k_3)^n &= (q^{2n} - p_1 q^n - p_2) \Delta_{(p_1, p_2)}^{-1} (k_1 k_2 k_3)^n \end{aligned} \quad (6)$$

Again replacing $u(k_1, k_2, k_3)$ by $(k_1 k_2 k_3)^0$ in (3),

$$\Delta_{q, (p_1, p_2)} (k_1 k_2 k_3)^0 = (q^2 (k_1 k_2 k_3))^0 - p_1 (q^0 (k_1 k_2 k_3))^0 - p_2 (k_1 k_2 k_3)^0$$

$$\Delta_{q, (p_1, p_2)} (1) = (1 - p_1 - p_2)(1) \tag{7}$$

Hence (6) and (7) yield (5).

Lemma 2.3 Let $(k_1, k_2, k_3) \in \mathfrak{R}^3$ and $1 - p_1 - p_2 \neq 0$. Then we have

$$\Delta_{q, (p_1, p_2)}^{-1} \log(k_1 k_2 k_3) = \frac{\log(k_1 k_2 k_3)}{1 - p_1 - p_2} - \frac{(2 - p_1) \log q}{(1 - p_1 - p_2)^2}. \tag{8}$$

Proof: From (3), replacing $u(k_1, k_2, k_3)$ by $\log(k_1 k_2 k_3)$, we get

$$\Delta_{q, (p_1, p_2)} \log(k_1 k_2 k_3) = 2 \log q + \log(k_1 k_2 k_3) - p_1 \log q - p_1 \log(k_1 k_2 k_3) - p_2 \log(k_1 k_2 k_3)$$

$$\Delta_{q, (p_1, p_2)} \log k_1 k_2 k_3 = (2 - p_1) \log q + (1 - p_1 - p_2) \log(k_1 k_2 k_3).$$

Applying $\Delta_{q, (p_1, p_2)}^{-1}$ on both sides

$$\Delta_{q, (p_1, p_2)}^{-1} \left[\Delta_{q, (p_1, p_2)} \log(k_1 k_2 k_3) \right] = \Delta_{q, (p_1, p_2)}^{-1} ((2 - p_1) \log q) + \Delta_{q, (p_1, p_2)}^{-1} (1 - p_1 - p_2) \log(k_1 k_2 k_3)$$

$$\log(k_1 k_2 k_3) = \Delta_{q, (p_1, p_2)}^{-1} \log(k_1 k_2 k_3) (2 - p_1) \log q + \Delta_{q, (p_1, p_2)}^{-1} (1 - p_1 - p_2),$$

which gives (8).

Lemma 2.4 Let $(k_1, k_2, k_3) \in \mathfrak{R}^3$ and $q \neq 0$. Then we have

$$\Delta_{q, (2\alpha, -\alpha^2)} u(k_1, k_2, k_3) = u(q^2 k_1, q^2 k_2, q^2 k_3) - 2\alpha u(qk_1, qk_2, vk_3) + \alpha^2 u(k_1, k_2, k_3).$$

Proof: The proof completes by putting $p_1 = 2\alpha$ and $p_2 = -\alpha^2$ in (3).

3. Fibonacci sequence from three-dimensional q -difference equation

In this section, we introduce three dimensional Fibonacci sequence and its sum.

Definition 3.1 For each pair $(p_1, p_2) \in \mathbb{R}^2$, the three-dimensional Fibonacci sequence is defined as

$$F_{(p_1, p_2)} = \left\{ F_n \right\}_{n=0}^{\infty}, \quad (9)$$

where $F_0 = 1, F_1 = p_1$ and $F_n = p_1 F_{n-1} + p_2 F_{n-2}$ for $n \geq 2$.
 When $p_1 = p_2 = 1$, (9) becomes the usual Fibonacci sequence.

Theorem 3.2 [Three-Dimensional Finite q -Series] Let $F_n \in F_{(p_1, p_2)}$ and $(k_1, k_2, k_3) \in \mathbb{R}^3$. Then we have

$$\begin{aligned} \sum_{d=0}^m F_d u\left(\frac{k_1}{q^{d+2}}, \frac{k_2}{q^{d+2}}, \frac{k_3}{q^{d+2}}\right) &= \Delta_q^{-1} u(k_1, k_2, k_3) - F_{m+1} \Delta_q^{-1} u\left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\right) \\ &\quad - p_2 F_m \Delta_q^{-1} u\left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right). \end{aligned} \quad (10)$$

Proof: Taking $\Delta_q^{-1} u(k_1, k_2, k_3) = v(k_1, k_2, k_3)$
 (p_1, p_2)

$$\Delta_q v(k_1, k_2, k_3) = u(k_1, k_2, k_3) \quad (p_1, p_2)$$

By (3), we write

$$v(q^2 k_1, q^2 k_2, q^2 k_3) = u(k_1, k_2, k_3) + p_1 v(qk_1, qk_2, qk_3) + p_2 v(k_1, k_2, k_3) \quad (11)$$

Replacing (k_1, k_2, k_3) by $\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right)$ in (11), we obtain

$$v(qk_1, qk_2, qk_3) = u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + p_1 v(k_1, k_2, k_3) + p_2 v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \quad (12)$$

Substituting the value of $v(qk_1, qk_2, qk_3)$ in (11), we get

$$\begin{aligned} v(q^2 k_1, q^2 k_2, q^2 k_3) &= u(k_1, k_2, k_3) + p_1 \left[v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \right. \\ &\quad \left. + p_1 v(k_1, k_2, k_3) + p_2 \left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \right] + p_2 v(k_1, k_2, k_3) \\ v(q^2 k_1, q^2 k_2, q^2 k_3) &= u(k_1, k_2, k_3) + p_1 u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \\ &\quad + (p_1^2 + p_2) v(k_1, k_2, k_3) + p_1 p_2 v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right). \end{aligned} \quad (13)$$

Replace k_1, k_2, k_3 by $\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right)$ in (12) and putting the value of $v(k_1, k_2, k_3)$ in (13), we obtain

$$\begin{aligned}
 v(q^2k_1, q^2k_2, q^2k_3) &= u(k_1, k_2, k_3) + p_1u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + (p_1^2 + p_2)\left[u\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right)\right. \\
 &\quad \left.+ p_1v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + p_2v\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right)\right] + p_1p_2v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \\
 &\quad + p_1(p_1^2 + p_2)v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + p_2(p_1^2 + p_2)v\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right) + p_1p_2v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) \\
 v(q^2k_1, q^2k_2, q^2k_3) &= u(k_1, k_2, k_3) + p_1u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + (p_1^2 + p_2)u\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right) \\
 &\quad + \left\{p_1(p_1^2 + p_2) + p_1p_2\right\}v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + p_2(p_1^2 + p_2)v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right). \quad (14)
 \end{aligned}$$

Since $F_n \in F_{(p_1, p_2)}$ we get

$$\begin{aligned}
 v(q^2k_1, q^2k_2, q^2k_3) &= F_0u(k_1, k_2, k_3) + F_1u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + F_2u\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right) \\
 &\quad + F_3v\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + p_2F_2v\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right)
 \end{aligned}$$

Proceeding like this, we arrive

$$\begin{aligned}
 v(q^2k_1, q^2k_2, q^2k_3) &= F_0u(k_1, k_2, k_3) + F_1u\left(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\right) + \dots + F_mu\left(\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}\right) \\
 &\quad + F_{m+1}v\left(\frac{k_1}{q^{m-1}}, \frac{k_2}{q^{m-1}}, \frac{k_3}{q^{m-1}}\right) + p_2F_mv\left(\frac{k_1}{q^m}, \frac{k_2}{q^m}, \frac{k_3}{q^m}\right). \quad (15)
 \end{aligned}$$

Now replace (k_1, k_2, k_3) by $\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right)$ in (15) we get

$$\begin{aligned}
 v(k_1, k_2, k_3) &= F_0u\left(\frac{k_1}{q^2}, \frac{k_2}{q^2}, \frac{k_3}{q^2}\right) + F_1u\left(\frac{k_1}{q^3}, \frac{k_2}{q^3}, \frac{k_3}{q^3}\right) + \dots + F_mu\left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right) \\
 &\quad + F_{m+1}v\left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\right) + p_2F_mv\left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right)
 \end{aligned}$$

$$\begin{aligned} \Delta_{(p_1, p_2)}^{-1} u(k_1, k_2, k_3) - F_{m+1} \Delta_{(p_1, p_2)}^{-1} u\left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\right) \\ - p_2 F_m \Delta_{(p_1, p_2)}^{-1} u\left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right) = \sum_{d=0}^m F_d u\left(\frac{k_1}{q^{d+2}}, \frac{k_2}{q^{d+2}}, \frac{k_3}{q^{d+2}}\right), \end{aligned}$$

which completes the proof of the theorem.

Example 3.3 Considering $p_1 = 2, p_2 = 4, m = 3, n = 2, q = 5, k_1 = 1, k_2 = 3, k_3 = 2, u(k_1, k_2, k_3) = (k_1 k_2 k_3)^n$, then we are able to find $F_0 = 1, F_1 = p_1 = 2, F_2 = 8, F_3 = 24$, and $F_4 = 80$.

$$\begin{aligned} \text{From (10), L.H.S} = \sum_{d=0}^3 F_d u\left(\frac{k_1}{q^{d+2}}, \frac{k_2}{q^{d+2}}, \frac{k_3}{q^{d+2}}\right) = \sum_{d=0}^3 F_d \left(\frac{k_1}{q^{d+2}} \cdot \frac{k_2}{q^{d+2}} \cdot \frac{k_3}{q^{d+2}}\right)^n \\ = 0.063033753 \end{aligned}$$

$$\begin{aligned} \text{Now,} \quad \Delta_{(p_1, p_2)}^{-1} u(k_1, k_2, k_3) &= \Delta_{(p_1, p_2)}^{-1} (k_1 k_2 k_3)^n \\ &= \frac{(k_1 k_2 k_3)^n}{q^{2n} - p_1 q^n - p_2} = \frac{(1 * 3 * 2)^2}{5^4 - (2 * 5)^2 - 4} \\ &= 0.063047285 \end{aligned}$$

$$\begin{aligned} \Delta_{(p_1, p_2)}^{-1} u\left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\right)^n &= \frac{\left(\frac{k_1}{q^{m+1}} \cdot \frac{k_2}{q^{m+1}} \cdot \frac{k_3}{q^{m+1}}\right)^n}{q^{2n} - p_1 q^n - p_2} \\ &= \frac{\left(\frac{6}{5^4}\right)^2}{571} = 1.614010508 \times 10^{-7} \end{aligned}$$

$$\Delta_{(p_1, p_2)}^{-1} u\left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right)^n = \frac{\left(\frac{6}{5^5}\right)^2}{571} = 6.456042032 \times 10^{-9}$$

$$\begin{aligned} R.H.S &= 0.063047285 - F_4(1.614010508 \times 10^{-7}) - (4)F_3(6.456042032 \times 10^{-9}) \\ &= 0.063033753 \end{aligned}$$

Hence the theorem is verified.

The following corollary is a special case of Theorem 3.2.

Corollary 3.4 Assume that $p_1 + p_2 \neq 1$ and $F_n \in F_{(p_1, p_2)}$. Then we have

$$\sum_{d=0}^m F_d = \frac{1 - F_{m+1} - p_2 F_m}{1 - p_1 - p_2}. \quad (16)$$

Proof: From (10) replacing $u(k)$ by k^0 , we find

$$\begin{aligned} \sum_{d=0}^m F_d \left(\frac{k_1}{q^{d+2}} \cdot \frac{k_2}{q^{d+2}} \cdot \frac{k_3}{q^{d+2}} \right)^0 &= \Delta_q^{-1} (k_1 k_2 k_3)^0 \\ &\quad - F_{m+1} \Delta_q^{-1} \left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}} \right)^0 - p_2 F_m \Delta_q^{-1} \left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}} \right)^0 \\ \sum_{d=0}^m F_d &= \left(\begin{array}{ccc} \Delta_q^{-1} & -F_{m+1} \Delta_q^{-1} & -p_2 F_m \Delta_q^{-1} \\ (p_1, p_2) & (p_1, p_2) & (p_1, p_2) \end{array} \right) (1), \end{aligned}$$

which yields (16). An example is given below to illustrate Corollary 3.4 :

Example 3.5 Considering $m = 5$, $p_1 = 10$, $p_2 = 15$, then from (16) we get $F_0 = 1$, $F_1 = p_1 = 10$, $F_2 = 115$, $F_3 = 1300$, $F_4 = 14725$, $F_5 = 166750$ and $F_6 = 1888375$.

Now $L.H.S = \sum_{d=0}^m F_d = 182901$ and

$$R.H.S = \frac{1 - F_{m+1} - p_2 F_m}{1 - p_1 - p_2} = \frac{1 - F_6 - p_2 F_5}{1 - p_1 - p_2} = 182901.$$

4. Product formula of three dimensional q -difference equation

Theorem 4.1 For the real valued functions $u(k_1, k_2, k_3)$ and $v(k_1, k_2, k_3)$, we have

$$\begin{aligned} \Delta_q^{-1} (u(k_1, k_2, k_3)v(k_1, k_2, k_3)) &= \frac{1}{p_2} \left\{ u(k_1, k_2, k_3) \Delta_q^{-1} v(k_1, k_2, k_3) \right. \\ &\quad \left. - \Delta_q^{-1} \left(\Delta_q u(k_1, k_2, k_3) \Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) \right) \right. \\ &\quad \left. - p_1 \Delta_q^{-1} \left(u(qk_1, qk_2, qk_3) \Delta_q \left(\Delta_q^{-1} v(k_1, k_2, k_3) \right) \right) \right\}. \quad (17) \end{aligned}$$

Proof: From (1), we find that

$$\Delta_q u(k_1, k_2, k_3) = u(q^2 k_1, q^2 k_2, q^2 k_3) - p_1 u(qk_1, qk_2, qk_3) - p_2 u(k_1, k_2, k_3)$$

$$\Delta_q u(k_1, k_2, k_3)w(k_1, k_2, k_3) = u(q^2 k_1, q^2 k_2, q^2 k_3)w(q^2 k_1, q^2 k_2, q^2 k_3)$$

$$\begin{aligned}
 & -p_1 u(qk_1, qk_2, qk_3) w(qk_1, qk_2, qk_3) - p_2 u(k_1, k_2, k_3) w(k_1, k_2, k_3) \\
 u(k_1, k_2, k_3) w(k_1, k_2, k_3) &= \Delta_q^{-1} \left[w(q^2 k_1, q^2 k_2, q^2 k_3) [\Delta_q u(k_1, k_2, k_3)] \right. \\
 & \quad \left. + p_1 u(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} w(k_1, k_2, k_3)) + p_2 u(k_1, k_2, k_3) v(k_1, k_2, k_3) \right] \\
 u(k_1, k_2, k_3) \Delta_q^{-1} v(k_1, k_2, k_3) &= \Delta_q^{-1} \left[\Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) [\Delta_q u(k_1, k_2, k_3)] \right. \\
 & \quad \left. + p_1 u(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} w(k_1, k_2, k_3)) + p_2 u(k_1, k_2, k_3) v(k_1, k_2, k_3) \right] \\
 u(k_1, k_2, k_3) \Delta_q^{-1} v(k_1, k_2, k_3) &= \Delta_q^{-1} \left[\Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) [\Delta_q u(k_1, k_2, k_3)] \right. \\
 & \quad \left. + p_1 u(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} w(k_1, k_2, k_3)) + p_2 u(k_1, k_2, k_3) v(k_1, k_2, k_3) \right] \\
 u(k_1, k_2, k_3) \Delta_q^{-1} v(k_1, k_2, k_3) &= \Delta_q^{-1} \left[\Delta_q u(k_1, k_2, k_3) \Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) \right. \\
 & \quad \left. + p_1 \Delta_q^{-1} \left(u(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} v(k_1, k_2, k_3)) \right) + p_2 \Delta_q^{-1} u(k_1, k_2, k_3) v(k_1, k_2, k_3) \right] \\
 -p_2 \Delta_q^{-1} \left(u(k_1, k_2, k_3) v(k_1, k_2, k_3) \right) &= -u(k_1, k_2, k_3) \Delta_q^{-1} v(k_1, k_2, k_3) \\
 & \quad + \Delta_q^{-1} \left(\Delta_q u(k_1, k_2, k_3) \Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) \right) \\
 & \quad + p_1 \Delta_q^{-1} \left(u(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} v(k_1, k_2, k_3)) \right),
 \end{aligned}$$

which completes the proof.

Corollary 4.2 For real valued function $v(k_1, k_2, k_3)$ and for $k_1, k_2, k_3 > 0$, we have

$$\begin{aligned}
 \Delta_q^{-1} \left(v(k_1, k_2, k_3) \log(k_1 k_2 k_3) \right) &= \frac{1}{p_2} \left[\log(k_1 k_2 k_3) \Delta_q^{-1} v(k_1, k_2, k_3) \right. \\
 & \quad \left. - \Delta_q^{-1} \left(\Delta_q \log(k_1 k_2 k_3) \Delta_q^{-1} v(q^2 k_1, q^2 k_2, q^2 k_3) \right) \right. \\
 & \quad \left. - p_1 \Delta_q^{-1} \left(\log(qk_1, qk_2, qk_3) \Delta_q (\Delta_q^{-1} v(k_1, k_2, k_3)) \right) \right] \quad (18)
 \end{aligned}$$

Proof: The proof is trivial by replacing $u(k_1, k_2, k_3)$ by $\log(k_1 k_2 k_3)$ in (17).

Corollary 4.3 For $q, k_1, k_2, k_3 > 0$, $1 - p_1 q - p_2 q^2 \neq 0$ and $F_n \in F_{(p_1, p_2)}$, we have

$$\begin{aligned}
 \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1 k_2 k_3} \log(k_1 k_2 k_3) \right) &= \frac{q^2}{(1 - p_1 q - p_2 q^2) k_1 k_2 k_3} \\
 & \quad \left\{ \log(k_1 k_2 k_3) - \frac{(2 - p_1 q) \log(q)}{(1 - p_1 q - p_2 q^2)} \right\} \quad (19)
 \end{aligned}$$

and hence

$$(1 - F_{m+1} q^{m+1} - a_2 F_m q^{m+2}) \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1 k_2 k_3} \log k_1 k_2 k_3 \right)$$

$$\begin{aligned}
 & + \left((m+1)F_{(m+1)}q^{m+1} + p_2(m+2)F_mq^{m+2} \right) \\
 & \times \frac{q^2 \log q}{(1-p_1q-p_2q^2)k_1k_2k_3} = \sum_{d=0}^m F_d \frac{q^{d+2}}{k_1k_2k_3} \log \left(\frac{k_1k_2k_3}{q^{d+2}} \right) \quad (20)
 \end{aligned}$$

Proof: Taking $v(k_1, k_2, k_3) = \frac{1}{k_1k_2k_3}$ in (18), we get

$$\begin{aligned}
 \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1k_2k_3} \log(k_1k_2k_3) \right) &= \frac{1}{p_2} \left\{ \log(k_1k_2k_3) \Delta_{(0,1)}^{-1} \frac{1}{k_1k_2k_3} \right. \\
 & \quad - \Delta_{(p_1, p_2)}^{-1} \left(\Delta_{(p_1, p_2)} \log(k_1k_2k_3) \Delta_{(0,1)}^{-1} \frac{1}{q^2(k_1k_2k_3)} \right) \\
 & \quad \left. - p_1 \Delta_{(p_1, p_2)}^{-1} \left(\log(q(k_1k_2k_3)) \Delta_{(1,0)} \Delta_{(0,1)}^{-1} \frac{1}{k_1k_2k_3} \right) \right\} \\
 \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1k_2k_3} \log(k_1k_2k_3) \right) &= \frac{1}{p_2} \left(\frac{q^2}{1-q^2} \right) \left\{ \log(k_1k_2k_3) \frac{1}{k_1k_2k_3} \right. \\
 & \quad - \frac{1}{q^2} \Delta_{(p_1, p_2)}^{-1} \left((2-p_1) \log q \frac{1}{k_1k_2k_3} + (1-p_1-p_2) \log(k_1k_2k_3) \frac{1}{k_1k_2k_3} \right) \\
 & \quad \left. - p_1 \left(\frac{1-q}{q^2} \right) \Delta_{(p_1, p_2)}^{-1} \left(\log(q(k_1k_2k_3)) \frac{1}{k_1k_2k_3} \right) \right\} \\
 \left[1 + \left(\frac{1}{p_2} \right) \left(\frac{q^2}{1-q^2} \right) \frac{1}{q^2} (1-p_1-p_2) + \left(\frac{1}{p_2} \right) \left(\frac{q^2}{1-q^2} \right) p_1 \frac{1-q}{q^2} \right] & \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1k_2k_3} \log(k_1k_2k_3) \right) \\
 &= \left(\frac{1}{p_2} \right) \left(\frac{q^2}{1-q^2} \right) \frac{1}{k_1k_2k_3} \left\{ \log(k_1k_2k_3) - (2-p_1) \log q \left(\frac{q^2}{1-p_1q-p_2q^2} \right) \right. \\
 & \quad \left. - p_1 \left(\frac{1-q}{q^2} \right) \log q \left(\frac{q^2}{1-p_1q-p_2q^2} \right) \right\} \\
 \Delta_{(p_1, p_2)}^{-1} \left(\frac{1}{k_1k_2k_3} \log(k_1k_2k_3) \right) &= \frac{q^2}{1-p_1q-p_2q^2(k_1k_2k_3)} \\
 \left\{ \log(k_1k_2k_3) - \left(\frac{1}{1-p_1q-p_2q^2} \right) \left\{ 2 \log q - p_1 \log q + p_1 \log q - qp_1 \log q \right\} \right\}, &
 \end{aligned}$$

which gives (19).

Again (20) is obvious by replacing $u(k_1, k_2, k_3)$ by $\frac{1}{k_1 k_2 k_3}(\log(k_1 k_2 k_3))$ in (10). The numerical verifications for Fibonacci series using real valued functions is given below:

Example 4.4 Considering $m = 2, k_1 = -10, k_2 = -15, k_3 = -19, q = -20, p_1 = 2, p_2 = 4$, then we find $F_0 = 1, F_1 = p_1 = 2, F_2 = 8, F_3 = 24$ and $F_4 = 80$.

$$L.H.S = F_0 \left(\frac{(-20)^2}{-2850} \right) \log \left(\frac{-2850}{(-20)^2} \right) + F_1 \left(\frac{(-20)^3}{-2850} \right) \log \left(\frac{-2850}{(-20)^3} \right) \\ + F_2 \left(\frac{(-20)^4}{-2850} \right) \log \left(\frac{-2850}{(-20)^4} \right)$$

$$= -788.0361283 \quad \text{and}$$

$$R.H.S = (1 - 24((-20)^3) - 4(8)((-20)^4)) \left\{ \frac{(-20)^2}{(-1559)(-2850)} \right\} \\ \left\{ (-3.45484486) - 0.03505098 \right\} + \left\{ 19904000(-1.171268128 \times 10^{-4}) \right\} \\ + \left((3)(72)(-20)^3 - (5)(4)(13)(-20)^4 \right) \left[\frac{(-20)^2(1.041392685)}{(-648)(-2850)} \right]$$

$$= -788.0361285.$$

Hence the theorem is verified.

5. Conclusion

In this research work, the summation solution and the closed form solutions of polynomials and logarithmic functions using three dimensional q -difference equation have been obtained. Also several results using three dimensional q -difference operator are derived. Moreover generalized product formula for polynomial, reciprocal of polynomial and logarithmic functions using inverse of three dimensional q -difference operator is obtained. Consequently, relevant examples are being given to investigate the results.

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