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Stochastic Fractional Differential Equations With Generalized Caputo's Derivative and Impulsive Effects

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Abstract

In this paper, impulsive stochastic fractional differential equations (ISFDEs) in $L^p(p \geq 2)$ space are introduced. We present a general framework for finding solution for ISFDEs. Then, by using the Burkholder - Davis - Gundy inequality and Holder's inequality, we prove the existence and uniqueness of solution to ISFDE by fixed point theorem. We also investigate Lipschitz continuity of solutions with respect to initial values by using Gronwall inequality. Finally, we provide an application to illustrate the results we obtained.

Key words: Stochastic fractional differential equations, Impulsive condition, Generalized Caputo's derivative, Existence and uniqueness of solutions, Continuity of solutions.

AMS classification: [2010] 26A33, 34A12, 60H10, 60H20, 47H10

1 Introduction

Fractional differential equations (FDEs) involve the derivatives of the form $\frac{d^\alpha}{dt^\alpha}$, where $\alpha > 0$ is not necessarily an integer, have received much attention from researchers. This rising interest is motivated both by important applications of the theory, and by the difficulties involved in the mathematical structure. Fractional evolution equations appear in many physical phenomena arising from various scientific fields including analysis of viscoelastic materials, electrical engineering, control theory of dynamical systems, electrodynamics with memory, quantum mechanics, heat conduction in materials with memory, signal processing, economics, and many other

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fields. For comprehensive study of fractional differential equations, we refer the books by Podlubny [20], Hilfer [16] and the papers [10, 11, 8, 9, 2] and references there in.

In order to describe and forecast a real phenomenon, it is necessary to introduce a component that captures the random behaviour caused by a major source of uncertainty, that usually propagates in time. When we add such a component, the model obtain is now governed by stochastic fractional differential equations (SFDEs). SFDEs are emerging in various fields of science and engineering such as economics, control physics, mechanics and many other areas (see [22, 23, 1, 12, 18, 17, 30]).

Recently, impulsive SFDEs (ISFDEs) naturally emerging in the models to describe the case where deterministic changes with impulses are interwoven with noisy fluctuations. In the nature there are lot of systems in which the time evolution of the state variable depend on trajectory subject to abrupt changes are modeled in ISFDEs. For more details on existence and uniqueness results for the ISFDEs can be found in [1, 12] and reference therein. However, the theory of mild solution of IFDEs are studying in two aspacts, one is based on classical Caputo's derivative and other is genralized caputo derivative. Under classical Caputo's derivative, authors (see [21, 26, 27, 5, 19, 13]) described mild solution as integrals over $(t_k, t_{k+1}](k = 1, 2, \ldots, m)$ and $[0, t_1]$. On the other hand, the mild solutions of IFDEs under generalized Caputo's derivative were expressed as integral over $[0, t]$ (see [14, 15, 28, 29]). Moreover generalized Caputo's derivative is more reasonable since under generalized Caputo's derivative, the obtained solution satisfies the given IFDEs. For more details see [14, 15, 7]. In [17], the authors have investigated the well posed-ness for solutions of Caputo's SFDE in $L^p(p \geq 2)$ space:

$$ {}^cD_t^\alpha X(t) = b(t, X(t)) + \sigma(t, X(t))\frac{dW_t}{dt}, \qquad (*) $$

where $\alpha \in (\frac{1}{2}, 1)$, $b, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion.

We have motivated by the work of Huong et al. [17, 14, 15, 28, 29] and [14, 15, 28, 29] respectively, we have introduced the impulsive function and sectorial operator A with generalized Caputo's derivative in $(*)$, which is not study yet. The purpose of this paper is to study the existence, uniqueness and continuous dependence of

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solution for ISFDE of the form

$$ {}^{C}D_t^{\alpha} X_t = AX_t + \mu(X_t, t) + \sigma(X_t, t)\frac{dB_t}{dt}, \quad t \in J = (0, T],\ t \neq t_k, \tag{1} $$

$$ \Delta X_{t_k} = I_k(X_{t_k^-}), \quad k = 1, 2, ...m, \tag{2} $$

$$ X(0) = X_0 \tag{3} $$

where X_t is stochastic process and ${}^{C}D_t^{\alpha}$ is generalized Caputo's fractional derivative of order $\alpha \in (1/2, 1)$. Linear operator A, defined from the domain $D(A) \subset \mathbb{R}^n$ into $\mathbb{R}^n$, is such that A generates $\alpha-$ resolvent family $\{S_\alpha(t) : t \geq 0\}$ of bounded linear operators in $\mathbb{R}^n$. The functions $\mu : \mathbb{R}^n \times J \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times J \to \mathbb{R}^n$ are given and satisfy some assumptions and $(B_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion defined on complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. Here $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, and the functions $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, ...m$, are bounded and the notation $\triangle X_{t_k} = X_{t_k^+} - X_{t_k^-}$ where $X_{t_k^+}$ and $X_{t_k^-}$ represent the right and left- hand limits of X_t at $t = t_k$ respectively, also we take $X_{t_i^-} = X_{t_i}$.

The work of this paper is based on [17, 14, 7]. This paper is concerned with ISFDE with generalized Caputo's derivative in L^p space with $p \geq 2$. The rest of this article is organized as follows: In section 2, we provide some basic definitions and essential preliminary results that will be used in the subsequent sections. Section 3 is devoted to the main results. In section 4, one example is given for validation of results. Finally we give conclusions in section 5.

2 Preliminaries

In the present section, we review some basic definitions, properties and lemmas which are required for establishing our results.

Definition 2.1 The fractional integral of order $\alpha > 0$ for a function $f \in L^1_{loc}(\mathbb{R}^+, X)$ is given as

$$ J_t^0 f(t) = f(t),\ J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t > 0, \tag{4} $$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 The Caputo's fractional derivative of order α for a function $f \in$

$C^n(\mathbb{R}^+, X)$ can be defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = J^{n-\alpha} f^{(n)}(t), \tag{5}$$

for $n-1 < \alpha < n,\ n \in N$. If $0 < \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds. \tag{6}$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3 A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi\iota} \int_c \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu,\ \alpha,\ \beta > 0,\ z \in \mathbb{C},$$

where c is a contour which starts and ends at ∞ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag-Lefller functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0.$$

For more details one can see the monographs of I. Podlubny [20].

Definition 2.4 ([4], Definition 2.3) Let A be a closed and linear operator with domain $D(A)$ defined on X and α>0. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α- order resolvent operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha x dt,\ Re\lambda > \omega,\ x \in X.$$

In this case $\{S_\alpha(t)\}_{t\geq 0}$ is called α-order resolvent operator generated by A.

Lemma 2.5 [7] Suppose A is a sectorial operator and $\alpha \in (0,1)$, then

$$^{c}D_t^{\alpha}[T_\alpha(t)x_0] = A[T_\alpha(t)x_0]$$

and

$$\begin{aligned} &^{c}D_t^{\alpha}\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] \\ &= A\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] + \mu(X_t,t)+\sigma(X_t,t)\frac{dB_t}{dt}, \end{aligned}$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$.

Lemma 2.6 If the functions $\mu : \mathbb{R}^d \times J \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times J \to \mathbb{R}^d$ are $\mathcal{F}_t-$ adapted process and A is a sectorial operator, then a piecewise $\mathcal{F}_t-$ adapted stochastic process X_t is called a solution of (1)-(3) if $X(0) = X_0$ and the following equality holds on $\mathbb{X}_t^p$ for $t \in J = [0, T,]$

$$X_t = \begin{cases} T_\alpha(t)X_0 + \sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds \\ + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s \end{cases} \tag{7}$$

Proof: If $t \in [0, t_1]$, then

$$^{c}D_t^{\alpha}X_t = AX_t + \mu(X_t,t) + \sigma(X_t,t)\frac{dB_t}{dt}, \tag{8}$$

$$X(0) = X_0. \tag{9}$$

Now applying the Riemann-Liouville fractional integral operator on both side, we have

$$\begin{aligned} X_t + c_1 &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mu(X_s,s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(X_s,s)dB_s, \end{aligned}$$

using initial condition, we get $c_1 = -X_0$, then

$$\begin{aligned} X_t &= X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned}$$

Now, if $t \in (t_1, t_2]$, then

$$\begin{aligned} {}^c D_t^\alpha X_t &= AX_t + \mu(X_t, t) + \sigma(X_t, t) \frac{dB_t}{dt}, \\ X_{t_1^+} &= X_{t_1^-} + I_1(X_{t_1^-}). \end{aligned}$$

Again applying the Riemann-Liouville fractional integral operator on both side, we have

$$\begin{aligned} X_t + c_2 &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s, \end{aligned}$$

using initial condition $X_{t_1^+} = X_{t_1^-} + I_1(X_{t_1^-})$, we get $c_2 = -X_0 - I_1(X_{t_1^-})$, then we have

$$\begin{aligned} X_t &= X_0 + I_1(X_{t_1^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned}$$

Similarly, if $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} X_t &= X_0 + \sum_{i=1}^{k} I_i(X_{t_i^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AX_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s. \end{aligned} \tag{10}$$

Now for $t \in [0,T]$, Eq. (10) can be written as

$$
\begin{aligned}
X_t &= X_0 + \sum_{i=1}^{m} \chi_i(t) I_i(X_{t_i^-}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A X_s ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(X_s, s) ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(X_s, s) dB_s, \qquad (11)
\end{aligned}
$$

where

$$
\chi_i(t) = \begin{cases} 0, \ t \leq t_i \\ 1, \ t > t_i. \end{cases}
$$

Now we adopt the idea used in [24] and taking the Laplace transform of the eq. (11) on both sides, we obtain

$$
\begin{aligned}
L\{X_t\} &= \frac{1}{\lambda}(X_0) + \sum_{i=1}^{m} \frac{e^{-\lambda t_i}}{\lambda} I_i(X_{t_i^-}) + \frac{1}{\lambda^\alpha} A L\{X_t\} + \frac{1}{\lambda^\alpha} L\{\mu(X_t, t)\} \\
&\quad + \frac{1}{\lambda^\alpha} L\{\sigma(X_t, t) \frac{dB_t}{dt}\} \qquad (12)
\end{aligned}
$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from eq. (12), we obtain

$$
\begin{aligned}
L\{X_t\} &= \frac{\lambda^{\alpha-1}}{\lambda^\alpha I - A}(X_0) + \sum_{i=1}^{m} \frac{\lambda^{\alpha-1}}{\lambda^\alpha I - A} e^{-\lambda t_i} I_i(X_{t_i^-}) + \frac{1}{\lambda^\alpha I - A} L\{\mu(X_t, t)\} \\
&\quad + \frac{1}{\lambda^\alpha I - A} L\{\sigma(X_t, t) \frac{dB_t}{dt}\}. \qquad (13)
\end{aligned}
$$

Taking the inverse Laplace transform on both sides of eq. (13), we get

$$
X_t = E_{\alpha,1}(At^\alpha) X_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \mu(X_s, s) ds
$$

$$
+ \sum_{i=1}^{m} E_{\alpha,1}(A(t-t_i)^\alpha) \chi_i(t) I_i(X_{t_i^-}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \sigma(X_s, s) dB_s. \qquad (14)
$$

Now we rewrite the eq. (14) as

$$\begin{aligned} X_t &= T_\alpha(t)X_0 + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}), \quad t \in [0,T], t \neq t_1,\ldots,t_m. \end{aligned} \tag{15}$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$. Now, we show that the solution (15) satisfies the system (1)-(3).

Step 1. We apply the Caputo's derivative on both side of eq. (15)

$$\begin{aligned} &{}^cD_t^\alpha X_t = \\ &{}^cD_t^\alpha [T_\alpha(t)X_0] +{}^c D_t^\alpha \left[\sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-})\right] \\ &+{}^cD_t^\alpha \left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] \\ &= AT_\alpha(t)X_0 + A\sum_{0<t_i<t} T_\alpha(t-t_i)I_i(X_{t_i^-}) \\ &+A\left[\int_0^t S_\alpha(t-s)\left(\mu(X_s,s)+\sigma(X_s,s)\frac{dB_s}{ds}\right)ds\right] + \mu(X_t,t) + \sigma(X_t,t)\frac{dB_t}{dt}. \end{aligned}$$

Therefore

$${}^cD_t^\alpha X_t = AX_t + \mu(X_t,t) + \sigma(X_t,t)\frac{dB_t}{dt}.$$

Step 2. From solution (15), we have

$$\begin{aligned} X_{t_k^+} &= T_\alpha(t_k)X_0 + \int_0^{t_k} S_\alpha(t_k-s)\mu(X_s,s)ds + \int_0^{t_k} S_\alpha(t_k-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i\leq t_k} T_\alpha(t_k-t_i)I_i(X_{t_i^-}), \end{aligned}$$

and

$$\begin{aligned} X_{t_k^-} &= T_\alpha(t_k)X_0 + \int_0^{t_k} S_\alpha(t_k-s)\mu(X_s,s)ds + \int_0^{t_k} S_\alpha(t_k-s)\sigma(X_s,s)dB_s \\ &\quad + \sum_{0<t_i\leq t_{k-1}} T_\alpha(t_k-t_i)I_i(X_{t_i^-}), \end{aligned}$$

So

$$\begin{aligned} X_{t_k^+} - X_{t_k^-} &= T_\alpha(t_k - t_k) I_k(x(t_k^-)) \\ &= T_\alpha(0) I_k(x(t_k^-)) \\ &= I_k(x(t_k^-)), \quad k = 1, 2, \dots m. \end{aligned}$$

Thus we observe that the solution (15) satisfies the system (1)-(3). □

We impose the following assumptions to developed our results.

(H1) The functions $\mu : \mathbb{R}^n \times J \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times J \to \mathbb{R}^n$ are Lipschitz continuous and for all $X_t, Y_t \in \mathbb{R}^n, t \in [0, T]$, there exists constants $L_\mu > 0, L_\sigma > 0$ such that
$|\mu(X_t, t) - \mu(Y_t, t)|_p \leq L_\mu |X_t - Y_t|_p$ and $|\sigma(X_t, t) - \sigma(Y_t, t)|_p \leq L_\sigma |X_t - Y_t|_p$.

(H2) The functions μ and σ are essentially bounded, i.e. there exists constant $M > 0$, such that
$esssup_{s \in [0,T]} |\mu(0, s)|_p < M$ and $esssup_{s \in [0,T]} |\sigma(0, s)|_p < M$.

(H3) There exist some positive constants $d_i (i = 1, 2, \dots, m)$ such that

$$|I_i(X_t) - I_i(Y_t)|_p \leq d_i |X_t - Y_t|_p.$$

If $\alpha \in (1/2, 1)$, $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $t > 0$, we have $\|E_{\alpha,1}(At^\alpha)\|_{L(X)} \leq Me^{\omega t}$ and $\|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_{L(X)} \leq ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Let $\widetilde{M}_T = \sup_{t \in [0,T]} \|E_{\alpha,1}(At^\alpha)\|_{L(X)}$, $\widetilde{M}_S = \sup_{t \in [0,T]} ce^{\omega t}(1 + t^{1-\alpha})$, where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So we have $\|E_{\alpha,1}(At^\alpha)\|_{L(X)} \leq \widetilde{M}_T$ and $\|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_{L(X)} \leq t^{\alpha-1} \widetilde{M}_S$. For details see [10] and references therein.

3 The main results

For $p \geq 2, t \in [0, \infty)$, we denote $\mathbb{X}_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, the space of all $\mathcal{F}_t-$measurable and p^{th} integrable functions $X_t = ((X_1)_t, (X_2)_t, \dots, (X_n)_t) : \Omega \to \mathbb{R}^n$, endowed with the norm

$$\|X_t\|_{L^p} = \left(\sum_{i=1}^n E|(X_i)_t|^p \right)^{1/p}$$

and $J = [0, T]$. Thus $(\mathbb{X}_t^p, \|.\|_{L^p})$ is a Banach space. Let $PC(J, \mathbb{X}_t^p)$ be the Banach space of the peicewise continuous mapping from J to $\mathbb{X}_t^p$, satisfying the condition

$$esssup_{t\in[0,T]}\|X_t\|_{L^p} < \infty$$

and $\mathbb{H}_p$ be the closed subspace of the $\mathcal{F}_t-$ measurable peicewise continous processes X in $PC(J, \mathbb{X}_t^p)$ such that X_t is It$\widehat{o}$ process and $X(0) = X_0$ is $\mathcal{F}_0-$ measurable, endowed with the norm

$$\|X\|_{\mathbb{H}_p} = (esssup_{t\in[0,T]}\|X_t\|_{L^p})^{1/p} < \infty.$$

Obviously, $(\mathbb{H}_p, \|.\|_{\mathbb{H}_p})$ is also a Banach space. However, in several estimates below, we use $\mathbb{R}^n$ with the $p-$norm, i.e. for any vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the p norm of x is defined by $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and for It$\widehat{o}$ process $X_t \in \mathbb{R}^n$ is defined as $|X_t|_p = (\sum_{i=1}^n |(X_i)_t|^p)^{1/p}$. Now we define relation between L^p-norm and $p-$norm as follows:

$$\|X_t\|_{L^p}^p = \sum_{i=1}^n E(|(X_i)_t|^p) = E\left(\sum_{i=1}^n |(X_i)_t|^p\right),$$

which gives

$$\|X_t\|_{L^p}^p = E(|X_t|_p^p). \tag{16}$$

In the proof of our results, we use the following elementary inequality

$$|x+y|_p^p \leq 2^{p-1}(|x|_p^p + |y|_p^p)$$

for all $x, y \in \mathbb{R}^n$.

To prove the existence and uniqueness of solution of the equation (1)-(3), we define an operator $N : \mathbb{H}_p \to \mathbb{H}_p$ by

$$\begin{aligned} N(X_t) &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-}) \\ &\quad + \int_0^t S_\alpha(t-s)\mu(X_s, s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s, s)dB_s. \end{aligned} \tag{17}$$

The following lemma is devoted to show the operator N is well defined.

Lemma 3.1 Suppose that the conditions $(H1), (H2)$ and $(H3)$ are true, then for

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$t \in [0,T]$, the operator N is well-defined.

Proof: Let $X \in \mathbb{H}_p([0,T])$ be arbitrary, for all $t \in [0,T]$, we have

$$\|N(X_t)\|_{L^p}^p \leq 2^{2p-2}\widetilde{M}_T^p\|X_0\|_{L^p}^p + 2^{2p-2}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-})\right\|_{L^p}^p$$
$$+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|_{L^p}^p + 2^{2p-2}\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L^p}^p. \quad (18)$$

Now we find individually the norms of each terms of eq. (18).

Step 1 : From (H3), we have

$$\begin{aligned}\|I_i(X_{t_i^-})\|_{L^p}^p = \sum_{j=1}^n E|I_i(X_j)_{t_i}|^p &\leq d_i^p \sum_{j=1}^n E|(X_j)_{t_i}|^p = d_i^p\|X_{t_i}\|_{L^p}^p \\ . &\leq d_i^p esssup_{t\in[0,T]}\|X_t\|_{L^p}^p \\ &\leq d_i^p\|X\|_{\mathbb{H}_p}^p.\end{aligned}$$

So,

$$\begin{aligned}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-})\right\|_{L^p}^p &\leq (2^{p-1})^{\lceil\frac{m}{2}\rceil}\widetilde{M}_T^p\sum_{0<t_i<t}\|I_i(X_{t_i})\|_{L^p}^p \\ &\leq (2^{p-1})^{\lceil\frac{m}{2}\rceil}\widetilde{M}_T^p D\|X\|_{\mathbb{H}_p}^p \quad (19)\end{aligned}$$

where $D = \sum_{i=1}^m d_i^p$ and $\lceil x\rceil$ is least integer function.

Step 2 : By the Holder's inequality, with constants $P = \frac{p}{p-1}, Q = p$ such that $\frac{1}{P} + \frac{1}{Q} = 1$, we obtain that

$$\begin{aligned}\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|_{L^p}^p &\leq \widetilde{M}_s^p\sum_{i=1}^n E\left(\int_0^t (t-s)^{\alpha-1}|\mu_i(X_s,s)|ds\right)^p \\ &\leq \widetilde{M}_s^p\sum_{i=1}^n E\left(\left(\int_0^t (t-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{p-1}\left(\int_0^t |\mu_i(X_s,s)|^p ds\right)\right) \\ &\leq \frac{\widetilde{M}_s^p T^{(\alpha p-1)}(p-1)^{p-1}}{(\alpha p-1)^{p-1}}\int_0^t\sum_{i=1}^n E(|\mu_i(X_s,s)|^p)ds \\ &\leq \frac{\widetilde{M}_s^p T^{(\alpha p-1)}(p-1)^{p-1}}{(\alpha p-1)^{p-1}}\int_0^t \|\mu(X_s,s)\|_{L^p}^p ds.\end{aligned}$$

Now, using result (16) and hypothesis (H1), we obtain

$$\begin{aligned}\|\mu(X_s,s)\|_{L^p}^p &= E(|\mu(X_s,s)|_p^p)\\ &\leq 2^{p-1}E\left(|\mu(X_s,s)-\mu(0,s)|_p^p+|\mu(0,s)|_p^p\right)\\ &\leq 2^{p-1}E\left(L_\mu^p|X_s|_p^p+|\mu(0,s)|_p^p\right)\\ &\leq 2^{p-1}L_\mu^p\|X_s\|_{L^p}^p+2^{p-1}|\mu(0,s)|_p^p.\end{aligned}$$

So from (H2), we have

$$\begin{aligned}\|\mu(X_s,s)\|_{L^p}^p &\leq 2^{p-1}L_\mu^p\|X\|_{\mathbb{H}_p}^p+2^{p-1}esssup_{s\in[0,T]}|\mu(0,s)|_p^p\\ &\leq 2^{p-1}L_\mu^p\|X\|_{\mathbb{H}_p}^p+2^{p-1}M^p.\end{aligned}$$

Therefore

$$\left\|\int_0^t S_\alpha(t-s)\mu(X_s,s)ds\right\|_{L^p}^p \leq \frac{\widetilde{M}_s^p T^{\alpha p}(2p-2)^{p-1}}{(\alpha p-1)^{p-1}}\left(L_\mu^p\|X\|_{\mathbb{H}_p}^p+M^p\right), \tag{20}$$

Step 3 : In this step we use the Burkholder-Davis-Gundy and the Holder's inequalities, with holder constants $P=\frac{p}{p-2}, Q=\frac{p}{2}$, s.t. $\frac{1}{P}+\frac{1}{Q}=1$, then we obtain

$$\begin{aligned}\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L^p}^p &= \sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\sigma_i(X_s,s)dB_s)\right|^p\right)\\ &\leq \sum_{i=1}^n C_pE\left(\left|\int_0^t |S_\alpha(t-s)|^2|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right)\\ &\leq C_p\widetilde{M}_s^p\sum_{i=1}^n E\left(\left|\int_0^t (t-s)^{2\alpha-2}|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right)\\ &\leq C_p\widetilde{M}_s^p\sum_{i=1}^n E\left(\left|\int_0^t \left((t-s)^{2\alpha-2}\right)^{\frac{p-2}{p}}\left((t-s)^{2\alpha-2}\right)^{\frac{2}{p}}|\sigma_i(X_s,s)|^2ds)\right|^{\frac{p}{2}}\right),\end{aligned}$$

where $C_p = \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}$. Now, by Holder's inequality, we have

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L^p}^p \\
&\leq C_p\widetilde{M}_S^p\sum_{i=1}^n\left(\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{\frac{p-2}{2}}\left(\int_0^t(t-s)^{2\alpha-2}|\sigma_i(X_s,s)|^p ds\right)\right) \\
&\leq C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t(t-s)^{2\alpha-2}\left(\sum_{i=1}^n E|\sigma_i(X_s,s)|^p\right)ds \\
&\leq C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t(t-s)^{2\alpha-2}\|\sigma(X_s,s)\|_{L^p}^p ds
\end{aligned}$$

Now from result (16) and assumptions (H1) and (H2), we obtain

$$\begin{aligned}
\|\sigma(X_s,s)\|_{L^p}^p &\leq 2^{p-1}E\left(|\sigma(X_s,s)-\sigma(0,s)|_p^p+|\sigma(0,s)|_p^p\right) \\
&\leq 2^{p-1}(L_\sigma^p\|X_s\|_{L_p}^p+|\sigma(0,s)|_P^p) \\
&\leq 2^{p-1}(L_\sigma^p esssup_{s\in[0,T]}\|X_s\|_{L^p}^p+esssup_{s\in[0,T]}|\sigma(0,s)|_p^p \\
&\leq\ \leq 2^{p-1}(L_\sigma^p\|X\|_{\mathbb{H}_p}^p+M^p)
\end{aligned}$$

and

$$\left\|\int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s\right\|_{L_p}^p \leq 2^{p-1}C_p\widetilde{M}_S^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}(L_\sigma^p\|X\|_{\mathbb{H}_p}^p+M^p). \qquad (21)$$

Now, from inequalities (19), (20) and (21), we get

$$\|N(X)\|_{\mathbb{H}_p}<\infty.$$

Which implies that the operator N is well defined.

Our next result is based on the Banach contraction principle.
Theorem 3.2 Assume that the assumptions (H1), and (H3) are satisfied. If $A \in \mathcal{A}^\alpha(\theta_0,\omega_0)$, then the system (1)-(3) has a unique solution in J if $2^{\frac{2p-2}{p}}\mathcal{V}<1$, where

$$\mathcal{V}=\left[(2^{p-1})^{\lceil\frac{m}{2}\rceil-1}\widetilde{M}_T^pD+\widetilde{M}_S^p\left(L_\mu^pT^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1}+c_pL_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}.$$

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Proof: Consider the operator $N : \mathbb{H}_p \to \mathbb{H}_p$ defined by

$$\begin{aligned} N(X_t) &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)I_i(X_{t_i^-}) \\ &\quad + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds + \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s. \end{aligned} \tag{22}$$

To prove that N has a unique fixed point. Let $X, Y \in \mathbb{H}_p$, then for all $t \in [0,T]$, we have

$$\begin{aligned} \|N(X_t)-N(Y_t)\|_{L_p}^p &\leq 2^{p-1}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-})-I_i(Y_{t_i^-}))\right\|_{L_p}^p \\ &\quad +2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L_p}^p \\ &\quad +2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L_p}^p \end{aligned} \tag{23}$$

Further proof is given in the following steps
Step 1. From assumption (H3), we have

$$\begin{aligned} \|I_i(X_{t_i^-})-I_i(Y_{t_i^-}))\|_{L^p}^p &\leq d_i^p\|X_{t_i^-}-Y_{t_i^-}\|_{L^p}^p \\ &\leq d_i^p \; esssup_{t_i\in[0,T]}\|X_{t_i^-}-Y_{t_i^-}\|_{L^p}^p \\ &\leq d_i^p\|X-Y\|_{\mathbb{H}_p}^p. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-})-I_i(Y_{t_i^-}))\right\|_{L_p}^p \\ &\qquad \leq (2^{p-1})^{\lceil \frac{m}{2}\rceil}\widetilde{M}_T^p D\|X-Y\|_{\mathbb{H}_p}^p, \end{aligned} \tag{24}$$

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Step 2. By the Holder's inequality, we obtain

$$\begin{aligned}&\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L^p}^p\\&=\sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\{\mu_i(X_s,s)-\mu_i(Y_s,s)\}ds\right|^p\right)\\&\leq \widetilde{M}_S^p\sum_{i=1}^n E\left(\int_0^t (t-s)^{\alpha-1}|\mu_i(X_s,s)-\mu_i(Y_s,s)|\right)^p.\end{aligned}$$

Using Holder's inequality, we get

$$\begin{aligned}&\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L^p}^p\\&\leq \widetilde{M}_S^p\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{p}{p-1}}ds\right)^{p-1}\times\int_0^t\sum_{i=1}^n E|\mu_i(X_s,s)-\mu_i(Y_s,s)|^p ds\\&\leq \widetilde{M}_S^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}T^{\alpha p-1}\int_0^t\|\mu(X_s,s)-\mu(Y_s,s)\|_{L^p}^p ds\\&\leq \widetilde{M}_S^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}T^{\alpha p-1}L_\mu^p\int_0^t\|X_s-Y_s\|_{L^p}^p ds\end{aligned}$$

So

$$\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s)-\mu(Y_s,s))ds\right\|_{L_p}^p\leq \widetilde{M}_S^pT^{\alpha p}L_\mu^p\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\|X-Y\|_{\mathbb{H}^p}. \quad (25)$$

Step 3.

$$\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p=\sum_{i=1}^n E\left(\left|\int_0^t S_\alpha(t-s)\{\sigma_i(X_s,s)-\sigma_i(Y_s,s)\}dB_s\right|^p\right).$$

Using Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p \\
&\leq \sum_{i=1}^{n} C_p E\left(\int_0^t |S_\alpha(t-s)|^2|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}} \\
&\leq C_p\widetilde{M}_S^p \sum_{i=1}^{n} E\left(\int_0^t (t-s)^{2\alpha-2}|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}} \\
&\leq C_p\widetilde{M}_S^p \sum_{i=1}^{n} E\left(\int_0^t \{(t-s)^{2\alpha-2}\}^{\frac{p-2}{p}}\{(t-s)^{2\alpha-2}\}^{\frac{2}{p}}|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^2 ds\right)^{\frac{p}{2}}.
\end{aligned}$$

Now by Holder's inequality, we get

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L^p}^p \\
&\leq C_p\widetilde{M}_S^p \left(\int_0^t (t-s)^{2\alpha-2}ds\right)^{\frac{p-2}{2}}\left(\int_0^t (t-s)^{2\alpha-2}\sum_{i=1}^{n} E|\sigma_i(X_s,s)-\sigma_i(Y_s,s)|^p ds\right) \\
&\leq C_p\widetilde{M}_S^p \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}} L_\sigma^p \int_0^t (t-s)^{2\alpha-2}\|X_s-Y_s\|_{L^p}^p ds.
\end{aligned}$$

So

$$\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s)-\sigma(Y_s,s))dB_s\right\|_{L_p}^p \leq C_p\widetilde{M}_S^p \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}} L_\sigma^p\|X-Y\|_{\mathbb{H}_p}^p. \quad (26)$$

Now from inequalities (24), (25) and (26), we have

$$\|N(X)-N(Y)\|_{\mathbb{H}_p} \leq 2^{\frac{2p-2}{p}}\mathcal{V}\|X-Y\|_{\mathcal{H}_p}, \quad (27)$$

where

$$\mathcal{V} = \left[\left(2^{p-1}\right)^{\lceil\frac{m}{2}\rceil-1}\widetilde{M}_T^p D + \widetilde{M}_S^p\left(L_\mu^p T^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1} + (C_p)L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}.$$

Since $2^{\frac{2p-2}{p}}\mathcal{V} < 1$, implies that the map N is a contraction map.Therefore the map

N has a unique fixed point $X_t \in \mathbb{H}_p$, that is a solution of the system (1)-(3) on $[0,T]$. Hence the proof of Theorem is completed.
In the next result we prove the continuous dependence of solutions on the initial values.

Definition 3.3 If $X_t, \widehat{X}_t$ be different solutions of the problem (1)-(3) with initial values $X_0, \widehat{X}_0$ respectively and for all $\epsilon > 0$, there exist $\delta > 0$ such that $\|X_t - \widehat{X}_t\| \leq \epsilon$ when $\|X_0 - \widehat{X}_0\| < \delta$ for all $t \in [0,T]$, then X_t is said to be continuous with respect to initial values (see definition 4.1 in [3]).

Theorem 3.4 Assume that the assumptions (H1), (H 2) are satisfied and

$$\left[(2^{p-1})^{\lceil \frac{m}{2} \rceil + 2} \widetilde{M}_T^p D\right] < 1.$$

Then the solution of the system (1)-(3) depends continuously on initial values.

Proof: Let for each initial values X_0, Y_0, there exist corresponding solutions X_t and Y_t of the system (1)-(3). Then for $t \in [0,T]$, we have

$$\begin{aligned} X_t &= E_{\alpha,1}(At^\alpha)X_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha) I_i(X_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(X_s,s)ds \\ &+ \int_0^t S_\alpha(t-s)\sigma(X_s,s)dB_s \end{aligned}$$

and

$$\begin{aligned} Y_t &= E_{\alpha,1}(At^\alpha)Y_0 + \sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha) I_i(Y_{t_i^-}) + \int_0^t S_\alpha(t-s)\mu(Y_s,s)ds \\ &+ \int_0^t S_\alpha(t-s)\sigma(Y_s,s)dB_s \end{aligned}$$

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Now it follows that

$$\begin{aligned}
&\|X_t - Y_t\|_{L_p}^p \\
&\leq 2^{2p-2}\widetilde{M}_T^p\|X_0 - Y_0\|_{L_p}^p + 2^{2p-2}\left\|\sum_{0<t_i<t} E_{\alpha,1}(A(t-t_i)^\alpha)(I_i(X_{t_i^-}) - I_i(Y_{t_i^-}))\right\|_{L_p}^p \\
&+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\mu(X_s,s) - \mu(Y_s,s))ds\right\|_{L_p}^p \\
&+2^{2p-2}\left\|\int_0^t S_\alpha(t-s)(\sigma(X_s,s) - \sigma(Y_s,s))dB_s\right\|_{L_p}^p.
\end{aligned}$$

By using Holder and the Burkholder - Davis - Gundy inequalities and assumption (H1), we get

$$\begin{aligned}
&\|X_t - Y_t\|_{L_p}^p \\
&\leq 2^{2p-2}\widetilde{M}_T^p\|X_0 - Y_0\|_{L_p}^p + (2^{p-1})^{\lceil\frac{m}{2}\rceil+2}\widetilde{M}_T^p D\|X_t - Y_t\|_{L_p}^p \\
&+\frac{2^{2p-2}\widetilde{M}_S^p T^{(\alpha p-2\alpha+1)}(p-1)^{(p-1)}L_\mu^p}{(\alpha p - 2\alpha + 1)^{p-1}}\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds \\
&+2^{2p-2}C_p\widetilde{M}_S^p L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds.
\end{aligned}$$

(for detail see Theorem 3.2)
Thus

$$\|X_t - Y_t\|_{L_p}^p \leq \nu_1\|X_0 - Y_0\|_{L_p}^p + \nu_2\int_0^t (t-s)^{2\alpha-2}\|X_s - Y_s\|_{L_p}^p ds \tag{28}$$

where

$$\begin{aligned}
\nu_1 &= \frac{2^{2p-2}\widetilde{M}_T^p}{\left[1 - (2^{p-1})^{\lceil\frac{m}{2}\rceil+2}\widetilde{M}_T^p D\right]} \\
\nu_2 &= \frac{2^{2p-2}\widetilde{M}_S^p}{\left[1 - (2^{p-1})^{\lceil\frac{m}{2}\rceil+2}\widetilde{M}_T^p D\right]}\left\{\frac{L_\mu^p T^{(\alpha p-2\alpha+1)}(p-1)^{p-1}}{(\alpha p - 2\alpha + 1)} + C_p L_\sigma^p\left(\frac{T^{\alpha p-1}}{2\alpha-1}\right)^{\frac{p-2}{2}}\right\}
\end{aligned}$$

Now applying the Gronwall inequality on equation (28) (see [25], corollary 2), we obtain

$$\|X_t - Y_t\|_{L_p}^p \leq E_{2\alpha-1}(\nu_2\Gamma(2\alpha-1)t^{2\alpha-1})\nu_1\|X_0 - Y_0\|_{L_p}^p.$$

Hence,

$$lim_{X_0 \to Y_0}\|X_t - Y_t\|_{L_p} = 0.$$

The theorem is proved.

4 Application

To illustrate our results, we consider the following ISFDE

$$\frac{\partial^{2/3}X(t,x)}{\partial t^{2/3}} = \frac{\partial^2 X(t,x)}{\partial x^2} + \frac{e^{t/2}X(t,x)}{13+X(t,x)} + e^t sin\left(\frac{x(t,x)}{21+X(t,x)}\right)\frac{dB_t}{dt}, \quad (29)$$

$$t \in [0,1], x \in (0,\pi)$$

$$X(t,0) = X(t,\pi) = 0, t \geq 0 \quad (30)$$

$$\Delta X(t,x)|_{t=\frac{1}{2}^-} = sin\left(\frac{1}{11}X\left(\frac{1}{2}^-, x\right)\right) \quad (31)$$

$$X(0,x) = X_0(x), \quad (32)$$

where $X_0 \in \mathbb{R}^n$ and B_t is a standard scalar Brownian motion. Let $\mathbb{X}_t^p = L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ and define the operator $A : D(A) \subset \mathbb{X}_t^p \to \mathbb{X}_t^p$ by $Au = u''$ with the domain $D(A) = \{u \in \mathbb{X}_t^p : u, u'$ are absolutely continuous,$u'' \in \mathbb{X}_t^p, u(0) = 0 = u(\pi)\}$. Then $Au = \sum_{n=1}^{\infty} n^2(u,u_n)u_n, u \in D(A)$, where $u_n(x) = \sqrt{\frac{2}{\pi}}sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A, it is well known from [31] A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}_t^p$ and given by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u,u_n)u_n$$

for all $u \in \mathbb{X}_t^p$ and every $t > 0$. From these expression it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda^\alpha, A) = (\lambda^\alpha I - A)^{-1}$ is a compact operator for all $\lambda^\alpha \in \rho(A)$ i.e. $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$. Therefore, from the subordination principal [[6] Theorems 3.1 and 3.3], we know that A generates α-order resolvent operator $S_\alpha(t)_{t\geq 0}$.

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Let $X(t,x) = X(t)(x) = X_t(x)$, where X_t is it$\widehat{o}$-process, and

$$\begin{aligned} \frac{e^{t/2}X_t(x)}{13+X_t(x)} &= \mu(X_t,t)(x) \\ e^t sin\left(\frac{X_t(x)}{21+X_t(x)}\right) &= \sigma(X_t,t)(x) \\ sin\left(\frac{1}{11}X_{1/2^-}(x)\right) &= I_k(X_{t_k^-})(x), t_k = 1/2, k = 1. \end{aligned}$$

Then with these settings problem (1)-(3) is an abstract version of problem (29)-(32). Now, for $t \in [0,1], X_t, Y_t \in \mathbb{X}_t^p$, we have

$$\begin{aligned} \|\mu(X_t,t)-\mu(Y_t,t)\|_{L^p} &= \left(\sum_{i=1}^n E\left|\frac{e^{t/2}(X_i)_t}{13+(X_i)_t} - \frac{e^{t/2}(Y_i)_t}{13+(Y_i)_t}\right|^p\right)^{1/p} \\ &\leq \frac{e^{t/2}}{13}\left(\sum_{i=1}^n |(X_i)_t-(Y_i)_t|^p\right)^{1/p} \\ &\leq \frac{e^{t/2}}{13}\|X_t-Y_t\|_{L^p} \end{aligned}$$

Similiarly,

$$\begin{aligned} \|\sigma(X_t,t)-\sigma(Y_t,t)\|_{L_p} &\leq \frac{e}{21}\|X_t-Y_t\|_{L_p} \\ \|I_k(X_{t_k^-})-I_k(Y_{t_k^-})\|_{L_p} &\leq \frac{1}{11}\|X_{t_k}-Y_{t_k}\|_{L_p}. \end{aligned}$$

Thus the functions μ, σ, and I_k, are satisfied the conditions (H1)-(H3) with $T = 1, L_\mu = \frac{e^{1/2}}{13}, L_\sigma = \frac{e}{21}$ and $D = \frac{1}{11}$. Now we take $\widehat{M_T} = 1, \widehat{M_S} = \frac{1}{\Gamma(\frac{2}{3})}$, and for $p = 2, C_p = 4$. Further

$$\begin{aligned} \mathcal{V} &= \left[\left(2^{p-1}\right)^{\lceil\frac{m}{2}\rceil-1}\widetilde{M}_T^pD + \widetilde{M}_S^p\left(L_\mu^pT^{\alpha p}\left(\frac{p-1}{2\alpha-1}\right)^{p-1} + (C_p)L_\sigma^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}}. \\ &= 0.47539. \end{aligned}$$

So, $2^{\frac{2p-2}{p}}\mathcal{V} = 0.9507 < 1$. Thus all conditions of Theorem 3.2 are fulfilled. So we deduce that problem (29) - (32) has a unique solution on $[0,1]$.

5 Conclusion

In this paper, we have firstly defined solution based on Laplace transform method in order to study the existence and uniqueness of ISFDE. Then, as a lemma, we proved that the operator, used in fixed point theorem, is well defined. In main results, by using Burkholder Davis Gundy and Holder's inequalities, we first proved the existence and uniqueness of solutions of ISFDE under Banach contraction theorem and then we showed Lipschitz continuity of solutions with respect to initial values. Finally we have given one example to illustrate our results.

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