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Some new oscillation criteria for certain class of fractional partial differential equations with damping

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Abstract

In this paper, we obtain some new oscillation criteria for certain class of fractional partial differential equations with damping. Using the generalized Riccati technique and integral averaging method, new oscillation criteria are established.

Key words: Fractional partial differential equations, oscillation, damping term.
AMS classification: 34A08, 35R11, 34K11.

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and have gained considerable importance due to their various field such as viscoelasticity, signal processing, rheology, control theory, probability, statistics and economics, robotics etc[7,10,11,17,19,22]. Recently, the theory of fractional differential equations and their applications have been attracting more and more attention in the literature[1,8,12,15,16].

Nowadays the interest in the study of oscillation theory for various equations like ordinary and partial differential equations, difference equations, dynamics equations on time scales and fractional differential equations is an interesting area of research and much effort has been made to establish oscillation criteria for these equations[4,13,18,20,21] and very few publications paid the attention to the oscillation of fractional differential equation, see for example[3,5,6,9,14].

In 2004, Abdullah[2], studied a note on the oscillation of second order differential

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equations with damping term of the form

$$y'' + p(x)y' + q(x)y = 0$$

where p(x) and q(x) are continuous functions on the interval $[\alpha, \infty)$. where α is real number.

Motivated by this paper, we proposed to study the Fractional partial analog. In this paper, we study the oscillatory behavior of solutions of Fractional partial differential equations with damping term of the form.

$$\begin{aligned} &\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + p(t)D^{\alpha}_{+,t}u(x,t) \\ &+\ q(t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right) = a(t)\Delta u(x,t) +\ F(x,t), \qquad (1) \\ &\qquad\qquad\qquad\qquad (x,t) \in G = \Omega \times R_+, \end{aligned}$$

with the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0, \quad (x,t) \in \partial\Omega \times R_+. \qquad (2)$$

where Ω is a bounded domain of R^N with piecewise smooth boundary $\partial\Omega$; $\alpha \in (0,1)$ is a constant; $G = \Omega \times R_+$, $R_+ = [0, \infty)$, $D^{\alpha}_{+,t}u$ is the Riemann- Liouville fractional derivative of order α of u with respect to t, Δ is the Laplacian operator and N is the unit exterior normal vector to $\partial\Omega$.

A solution u(x,t) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Throughout this paper, we assume that the following conditions hold:

(A_1) $r(t) \in C^{\alpha}\left((0,\infty);(0,\infty)\right), \int^{\infty} \dfrac{ds}{r(s)} = \infty.$

$p(t) \in C\left((0,\infty);R\right)$ and $q(t) \in C\left((0,\infty);R\right)$

(A_2) $a(t) \in C\left([0,\infty];R_+\right)$

(A_3) $f(u) \in C(R;R)$ is convex in R_+ such that $\dfrac{f(u)}{u} \geq \mu$ for certain constant $\mu > 0$ for all u$\neq$ 0.

(A_4) $F \in C\left(\overline{G}, R\right)$ is a continuous function such that $\int_{\Omega} F(x,t)dx \leq 0$.

By solution of equation(1). We mean a function $u(x,t) \in C^{(1+\alpha)}\left(\overline{\Omega} \times (0,\infty)\right)$ such that $f \int_0^t (t-s)^{-\alpha} u(x,s)ds \in C\left(\overline{G}, R\right), D^{\alpha}_{+,t} u(x,t) \in C^1\left(\overline{G}, R\right)$.

2 Preliminaries

The following notations will be used for our convenience

$$U(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x,t)dx, \quad where \quad |\Omega| = \int_{\Omega} dx.$$

Definition 2.1 The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$(D^{\alpha}_{+,t} u)(x,t) := \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(x,s)ds \tag{3}$$

provided the right hand side is pointwise defined on R_+, where Γ is the gamma function.

Definition 2.2 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$(I^{\alpha}_{+} y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds \quad \text{for} \quad t > 0 \tag{4}$$

provided the right hand side is pointwise defined on R_+.

Definition 2.3 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$(D^{\alpha}_{+} y)(t) := \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left(I^{\lceil \alpha \rceil - \alpha}_{+} y\right)(t) \quad \text{for} \quad t > 0 \tag{5}$$

provided the right hand side is pointwise defined on R_+, where $\lceil \alpha \rceil$ is the ceiling function of α.

Lemma 2.4 Let y be the solution of (1) and

$$K(t) := \int_0^t (t-s)^{-\alpha} y(s)ds \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad t > 0. \tag{6}$$

Then $K'(t) = \Gamma(1-\alpha)(D_+^\alpha y)(t)$.

3 Main Results

Finally, we give our main results. In this section, we establish some sufficient conditions for oscillation behavior of (1) and (2).

Theorem 3.1 If the functional differential inequality

$$\frac{d}{dt}\left(r(t)D_+^\alpha U(t)\right) + p(t)D_+^\alpha U(t) + q(t)f(K(t)) \leq 0 \tag{7}$$

has no eventually positive solution, then every solution of (1) and (2) is oscillatory in G.

Proof: Assume to the contrary that there is a nonoscillatory solution $u(x,t)$ to the problem (1) and (2). Without loss of generality we may assume that $u(x,t) > 0 \in G \times [t_0, \infty); t_0 \geq 0$.

Integrating (1) with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left(r(t)\int_\Omega D_{+,t}^\alpha u(x,t)dx\right) + p(t)\int_\Omega D_{+,t}^\alpha u(x,t)dx +$$

$$\int_\Omega q(t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx = a(t)\int_\Omega \Delta u(x,t)dx + \int_\Omega F(x,t)dx \tag{8}$$

Using Green's formula, it is obvious that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}dS = 0, \quad t \geq t_1 \tag{9}$$

where dS is surface element on $\partial\Omega$.

Moreover, using Jensen's inequality and from (A_3), it follows that

$$\int_\Omega q(t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx = q(t)\int_\Omega f\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx$$

$$\geq q(t)f\left(\int_0^t (t-s)^{-\alpha}\int_\Omega u(x,s)dx\right)ds$$

$$\geq q(t)f\left(\int_0^t (t-s)^{-\alpha} \mid \Omega \mid U(s)ds\right)$$

$$= \mid \Omega \mid q(t) f(K(t)) \quad . \tag{10}$$

Combining $(8)-(10)$ and using (A_4), we have

$$\frac{d}{dt}\left(r(t) D_+^\alpha U(t)\right) + p(t) D_+^\alpha U(t) + q(t) f(K(t)) \leq 0.$$

Therefore U(t) is an eventually positive solution of (7). This contradicts the hypothesis and complete the proof.

Theorem 3.2 If $p(t) < 0 \in [t_0, \infty)$ is such that

$$\lim_{t \to \infty}\left[\frac{1}{4}\int_{t_0}^{t}\left(4\mu q(s) - \frac{(p(s))^2}{r(s)\Gamma(1-\alpha)}\right) ds\right] = \infty \tag{11}$$

then any solution of the differential equation (7) is oscillatory on $[t_0, \infty)$
Proof:Suppose that the differential equation (7) is nonoscillatory then there exists a nontrivial solution of (7) that has no zero on (T, ∞) for $T > t_0$.

Let W(t) be the function defined by $W(t) = -\dfrac{r(t) D_+^\alpha U(t)}{K(t)}$ for $t \in [T, \infty)$.
Then W(t) is well defined function and satisfies the Riccati equation

$$\begin{aligned}
W'(t) &= \frac{p(t) r(t) D_+^\alpha U(t)}{r(t) K(t)} + \frac{q(t) f(K(t))}{K(t)} + \frac{\Gamma(1-\alpha)(r(t) D_+^\alpha U(t))^2}{r(t) K^2(t)} \\
&\leq \frac{p(t)(-W(t))}{r(t)} + \mu q(t) + \frac{\Gamma(1-\alpha) W^2(t)}{r(t)} \\
&= \mu q(t) - \frac{p(t) W(t)}{r(t)} + \frac{\Gamma(1-\alpha) W^2(t)}{r(t)} \\
&= \mu q(t) + \frac{1}{r(t)}[\Gamma(1-\alpha) W^2(t) - p(t) W(t)] \\
&= \mu q(t) + \frac{\Gamma(1-\alpha)}{r(t)}\left(W(t) - \frac{p(t)}{2\Gamma(1-\alpha)}\right)^2 - \frac{(p(t))^2}{4 r(t)\Gamma(1-\alpha)}
\end{aligned}$$

Integrating $t \to \infty$, we get

$$\int_T^t W'(s) ds \leq \int_T^t \left[\mu q(s) + \frac{\Gamma(1-\alpha)}{r(s)}\left(W(s) - \frac{p(s)}{2\Gamma(1-\alpha)}\right)^2 - \frac{(p(s))^2}{4 r(s)\Gamma(1-\alpha)}\right] ds$$

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$$W(t) - W(T) = \int_T^t \left(\frac{\Gamma(1-\alpha)}{r(s)} \left[W(s) - \frac{p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds + \frac{1}{4} \int_T^t \left(4\mu q(s) - \frac{(p(s))^2}{4r(s)\Gamma(1-\alpha)} \right) ds$$

$$= W(T) + \int_T^t \left(\frac{\Gamma(1-\alpha)}{r(s)} \left[W(s) - \frac{p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds + \frac{1}{4} \int_T^t \left(4\mu q(s) - \frac{(p(s))^2}{4r(s)\Gamma(1-\alpha)} \right) ds.$$

Now the equation (11) implies that there exists $t_* > T$ such that

$$W(t) > \int_{t_*}^t \left(\frac{\Gamma(1-\alpha)}{r(s)} \left[W(s) - \frac{p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds \quad on\ [t_*, \infty)$$

Define

$$Q(t) = \int_{t_*}^t \left(\frac{\Gamma(1-\alpha)}{r(s)} \left[W(s) - \frac{p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds \quad on\ [t_*, \infty) \tag{12}$$

then $W(t) > Q(t) >$ on $[t_*, \infty)$
Differentiating (12), we get

$$\begin{aligned} Q'(t) &= \frac{\Gamma(1-\alpha)}{r(t)} \left[W(t) - \frac{p(t)}{2\Gamma(1-\alpha)} \right]^2 \\ &> \frac{\Gamma(1-\alpha)}{r(t)} [Q(t)]^2 \end{aligned}$$

Since $p(t) < 0$,

$$\frac{\Gamma(1-\alpha)}{r(t)} < \frac{Q'(t)}{Q(t)^2},$$

Integrating both sides of this inequality from t_* to t, we get

$$\int_{t_*}^t \frac{\Gamma(1-\alpha)}{r(s)} ds < \frac{1}{Q(t_*)} - \frac{1}{Q(t)}.$$

Therefore since $Q(t) > 0$, we conclude that

$$\lim_{t\to\infty} \int_{t_*}^t \frac{\Gamma(1-\alpha)}{r(s)} ds < \frac{1}{Q(t_*)}.$$

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But this is not true. Thus the differential equation (7) is oscillatory, and this completes the proof.

Theorem 3.3 If $p(t) < 0$ on $[t_0, \infty)$ and there exists a non vanishing function $c(t) \in C'[t_0, \infty); c(t) > 0$, such that

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{c(s)} ds = \infty \tag{13}$$

and

$$\lim_{t\to\infty} \left[-\frac{1}{4\Gamma(1-\alpha)} \int_{t_0}^{t} \left(\frac{r(s)c'(s)^2}{c(s)} - 2c'(s)p(s) + \frac{c(s)p^2(s)}{r(s)} - 4\mu q(s)c(s)\Gamma(1-\alpha) \right) ds + \frac{c'(t)r(t)}{2\Gamma(1-\alpha)} \right] = \infty \tag{14}$$

then any solution of the differential equation (7) is oscillatory.
Proof: Suppose that there exists a nonoscillatory solution U(t) of the differential equation (7). Define for $t > t_0$

$$W(t) = -\frac{c(t)r(t))D_+^{\alpha}U(t)}{K(t)} \tag{15}$$

where c(t) is a non vanishing function belonging to $C'[t_0, \infty)$ and $\dfrac{1}{c(t)} > 0$ on $[t_0, \infty)$. Differentiating (15) with respect to t, on the interval $[t_0, \infty)$.

$$\begin{aligned} W'(t) &= \frac{c'(t)}{c(t)} W(t) + \frac{p(t)c(t)D_+^{\alpha}U(t)}{K(t)} + \frac{c(t)q(t)f(K(t))}{K(t)} + \frac{c(t)r(t)D_+^{\alpha}U(t)\Gamma(1-\alpha)}{K^2(t)} \\ &\le \frac{c'(t)}{c(t)} W(t) + \frac{p(t)}{K(t)}(-W(t)) + c(t)q(t)\mu + \frac{\Gamma(1-\alpha)}{c(t)r(t)} W^2(t) \\ &\le \frac{\Gamma(1-\alpha)}{c(t)r(t)} \left[W(t) + \frac{c'(t)r(t)}{2\Gamma(1-\alpha)} - \frac{c(t)p(t)}{2\Gamma(1-\alpha)} \right]^2 \\ &\quad -\frac{1}{4\Gamma(1-\alpha)} \left[\frac{r(t)c'^2(t)}{c(t)} - 2c'(t)p(t) + \frac{c(t)p^2(t)}{r(t)} - 4\mu q(t)c(t)\Gamma(1-\alpha) \right] \end{aligned}$$

Now for $t \in [t_0, \infty)$, defining

$$H(t) = W(t) + \frac{c'(t)r(t)}{2\Gamma(1-\alpha} \tag{16}$$

$$\begin{aligned} W'(t) \leq & \frac{\Gamma(1-\alpha)}{c(t)r(t)} \left[H(t) - \frac{c(t)p(t)}{2\Gamma(1-\alpha)} \right]^2 \\ & - \frac{1}{4\Gamma(1-\alpha)} \left[\frac{r(t)c'^2(t)}{c(t)} - 2c'(t)p(t) + \frac{c(t)p^2(t)}{r(t)} - 4\mu q(t)c(t)\Gamma(1-\alpha) \right] \end{aligned}$$

Integrating with $t_0 \to t$, we get

$$\begin{aligned} W(t) \leq & W(t_0) + \int_{t_0}^{t} \left(\frac{\Gamma(1-\alpha)}{c(s)r(s)} \left[H(s) - \frac{c(s)p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds \\ & - \frac{1}{4\Gamma(1-\alpha)} \int_{t_0}^{t} \left[\frac{r(s)c'^2(s)}{c(s)} - 2c'(s)p(s) + \frac{c(s)p^2(s)}{r(s)} - 4\mu q(s)c(s)\Gamma(1-\alpha) \right] ds \end{aligned}$$

By using equation (16) we have

$$\begin{aligned} H(t) \leq & W(t_0) + \int_{t_0}^{t} \left(\frac{\Gamma(1-\alpha)}{c(s)r(s)} \left[H(s) - \frac{c(s)p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds \\ & - \frac{1}{4\Gamma(1-\alpha)} \int_{t_0}^{t} \left[\frac{r(s)c'^2(s)}{c(s)} - 2c'(s)p(s) + \frac{c(s)p^2(s)}{r(s)} - 4\mu q(s)c(s)\Gamma(1-\alpha) \right] ds + \frac{c'(t)r(t)}{2\Gamma(1-\alpha)} \end{aligned}$$

Now the equation (14) implies that there exists $T > t_0$ such that

$$H(t) > \int_{t_0}^{t} \left(\frac{\Gamma(1-\alpha)}{c(s)r(s)} \left[H(s) - \frac{c(s)p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds$$

holds $t > T$. Define a function Q(t) for $t > T$ by

$$Q(t) = \int_{t_0}^{t} \left(\frac{\Gamma(1-\alpha)}{c(s)r(s)} \left[H(s) - \frac{c(s)p(s)}{2\Gamma(1-\alpha)} \right]^2 \right) ds \tag{17}$$

Since $p(t) < 0$, then $H(t) > Q(t) >$ on $[t_*, \infty)$

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Differentiating (17), we get

$$\begin{aligned} Q'(t) =& \frac{\Gamma(1-\alpha)}{c(t)r(t)} \left[H(t) - \frac{c(t)p(t)}{2\Gamma(1-\alpha)} \right]^2 \\ Q'(t) >& \frac{\Gamma(1-\alpha)}{c(t)r(t)} Q^2(t) \end{aligned}$$

Therefore,

$$\frac{\Gamma(1-\alpha)}{c(t)r(t)} < \frac{Q'(t)}{Q^2(t)}$$

Integrating with T to t. we get

$$\int_T^t \frac{\Gamma(1-\alpha)}{c(s)r(s)} ds < \int_T^t \frac{Q'(s)}{Q^2(s)} ds$$

Since $Q(t) < 0$,therefore

$$\int_T^t \frac{\Gamma(1-\alpha)}{c(s)r(s)} ds < \frac{1}{Q(T)}$$

But this is not true. Thus the differential equation (7) is oscillatory and this completes the proof.

For the following theorem, we introduce a class of function R.
Let

$$D_0 = \{(t,s) : t > s \geq t_0\}$$

$$D = \{(t,s) : t \geq s \geq t_0\}$$

The function $H \in C(D, R)$ is said to belong to the class R, if
(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) = 0$ for $(t,s)t \in D_0$.
(ii) H has a continuous and non-positive partial derivative $\dfrac{\partial H(t,s)}{\partial s}$ on D_0 with respect to s.
We assume that $\phi(t)$ for $t \geq t_0$ are given continuous functions such that $\phi(t) \geq 0$ and differentiable and define.

$$\theta(t) = \frac{c'(t)}{c(t)} - \frac{p(t)}{r(t)} + 2\Gamma(1-\alpha)\phi(t)$$

$$\psi(t) = c(t)[(r(t)\phi(t))' + p(t)\phi(t) + \Gamma(1-\alpha)r(t)(\phi^2(t))]$$

Theorem 3.4 Suppose that the conditions $(A_1)-(A_5)$ and (11) hold. Furthermore assume that there exists $H \in R$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[(\mu c(s)q(s) - \psi(s))H(t,s) - \frac{1}{4}\frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)}\right] ds = \infty \quad (18)$$

Then every solution of (7) is oscillatory.
Proof:

Assume to the contrary that there is a nonoscillatory solution $U(t)$ to the problem (7). Without loss of generality we may assume that U(t) ia an eventually positive solution of (7).
Then there exists $t_1 \geq t_0$ such that $U(t) > 0$ and $K(t) > 0$ for $t \geq t_1$. We obtain $D_+^\alpha U(t) \geq 0$ for $t \geq t_1$.
Now we define the Riccati substitution W(t) by

$$W(t) = c(t)\left[\frac{r(t)D_{+,t}^\alpha U(t)}{K(t)} + r(t)\phi(t)\right] \quad (19)$$

Then we have

$$\begin{aligned} W'(t) &= c'(t)\left[\frac{r(t)D_{+,t}^\alpha U(t)}{K(t)} + r(t)\phi(t)\right] + c(t)\left[\frac{r(t)D_{+,t}^\alpha U(t)}{K(t)} + r(t)\phi(t)\right]' \\ &= \frac{c'(t)}{c(t)}W(t) + c(t)\left[\frac{r(t)(D_{+,t}^\alpha U(t))'}{K(t)} - \frac{K'(t)r(t)D_{+,t}^\alpha U(t)}{K^2(t)} + (r(t)\phi(t))'\right] \\ &\leq \frac{c'(t)}{c(t)}W(t) + c(t)\left[(r(t)\phi(t))' - \mu q(t) - \frac{p(t)}{r(t)}\left(\frac{W(t)}{c(t)} - r(t)\phi(t)\right)\right. \\ &\qquad \left. - \frac{\Gamma(1-\alpha)}{r(t)}\left(\frac{W(t)}{c(t)} - r(t)\phi(t)\right)^2\right] \end{aligned} \quad (20)$$

Let we see that

$$\left[\frac{W(t)}{c(t)} - r(t)\phi(t)\right]^2 = \left[\frac{W(t)}{c(t)}\right]^2 - 2\frac{W(t)r(t)\phi(t)}{c(t)} + (r(t)\phi(t))^2 \quad (21)$$

Substituting (16) into (15), we have

$$W'(t) \leq \left[\frac{c'(t)}{c(t)} - \frac{p(t)}{r(t)} + 2\Gamma(1-\alpha)\phi(t)\right] W(t) - \frac{\Gamma(1-\alpha)}{r(t)c(t)} W^2(t) +$$

$$c(t)\left[(r(t)\phi(t))' + p(t)\phi(t) + \Gamma(1-\alpha)r(t)\phi^2(t)\right] - \mu c(t)q(t)$$

$$\leq \theta(t)W(t) - \frac{\Gamma(1-\alpha)}{r(t)c(t)} W^2(t) + \psi(t) - \mu c(t)q(t)$$

Multiplying both sides by $H(t,s)$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} (\mu c(s)q(s) - \psi(s))H(t,s)ds \leq -\int_{t_1}^{t} W'(s)H(t,s)ds + \int_{t_1}^{t} \theta(s)W(s)H(t,s)ds$$

$$-\int_{t_1}^{t} \frac{\Gamma(1-\alpha)}{r(s)c(s)} W^2(s)H(t,s)ds \tag{22}$$

Using the integration by parts formula, we get

$$-\int_{t_1}^{t} W'(s)H(t,s)ds = -[H(t,s)W(s)]_{t_1}^{t} + \int_{t_1}^{t} H'_s(t,s)W(s)ds$$

$$< -H(t,t_1)W(t_1) + \int_{t_1}^{t} H'_s(t,s)W(s)ds$$

Substituting (18) into (17), we have

$$\int_{t_1}^{t} (\mu c(s)q(s) - \psi(s))H(t,s)ds$$

$$\leq H(t,t_1)W(t_1) + \int_{t_1}^{t} \left[[H'_s(t,s) + \theta(s)H(t,s)]W(s) - \frac{\Gamma(1-\alpha)H(t,s)}{r(s)c(s)} W^2(s)\right] ds$$

$$\leq H(t,t_1)W(t_1) + \int_{t_1}^{t} \left[h(t,s)W(s) - \frac{\Gamma(1-\alpha)H(t,s)}{r(s)c(s)} W^2(s)\right] ds$$

$$\leq H(t,t_1)W(t_1) + \int_{t_1}^{t} \left[\sqrt{\frac{\Gamma(1-\alpha)H(t,s)}{c(s)r(s)}} W(s) - \frac{1}{2}\sqrt{\frac{r(s)c(s)}{\Gamma(1-\alpha)H(t,s)}} h(t,s)\right]^2 ds +$$

$$\frac{1}{4}\int_{t_1}^{t} \frac{r(s)c(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} ds$$

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$$\leq H(t,t_1)W(t_1) + \frac{1}{4}\int_{t_1}^{t} \frac{r(s)c(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)}ds,$$

Which yields

$$\int_{t_1}^{t} \left[(\mu c(s)q(s) - \psi(s))H(t,s) - \frac{1}{4}\frac{r(s)c(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \leq H(t,t_1)W(t_1).$$

Since $0 < H(t,s) \leq H(t,t_1)$ for $t > s \leq t_1$,
we have $0 < \dfrac{H(t,s)}{H(t,t_1)} \leq 1$ for $t > s \leq t_1$.
Hence

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t} \left[(\mu c(s)q(s) - \psi(s))H(t,s) - \frac{1}{4}\frac{r(s)c(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \leq W(t_1).$$

Letting $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)}\int_{t_1}^{t} \left[(\mu c(s)q(s) - \psi(s))H(t,s) - \frac{1}{4}\frac{r(s)c(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \leq W(t_1).$$

which contradicts (13) and complete the proof.

In Theorem 3.4, if we choose $H(t,s) = (t-s)^\lambda$, $t \geq s \geq t_1$, where $\lambda > 1$ is a constant, then we obtain the following corollaries.

Corollary 3.5 Under the conditions of Theorem 3.4, if

$$\limsup_{t\to\infty} \frac{1}{(t-t_1)^\lambda}\int_{t_1}^{t} \left[(\mu c(s)q(s) - \psi(s))(t-s)^\lambda - \frac{1}{4}\frac{c(s)r(s)((t-s)\theta(s)-\lambda)}{\Gamma(1-\alpha)(t-s)} \right] ds < \infty \tag{23}$$

Then every solution of (7) is oscillatory.

4 Conclusion

In this paper, we establish some new oscillation criteria for certain class of fractional partial differential equations with damping. In future work, the obtained results will be extended to a higher order.

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