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On Paired Domination of Some Graphs

Rakhimol Isaac[1] and Parashree Pandya[2]

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Abstract

For a graph a subset of the vertex set is called a dominating set if every vertex in is adjacent to some vertex in D. The domination number is the minimum cardinality of a dominating set of a graph G. The paired dominating set of a graph is a dominating set and the subgraph induced by it contains a perfect matching. The paired domination number is the minimum cardinality of a paired dominating set in G. In this paper, we discuss the paired domination number of the graphs obtained by the k^{th} power of path and cycle and degree splitting graphs of some standard graphs.

Key words: Paired domination, Paired domination number, k^{th} Power of a graph, Degree splitting of a graph.

AMS classification: 05C38, 05C76, 05C90.

1 Introduction

We begin with finite, connected and undirected graph $G = (V, E)$ with vertex set V and edge set E, without loops and multiple edges. The notations and terminology used here in the sense of Clark and Holtan [2]. The subset $D \subseteq V$ is called dominating set if every vertex of V is either an element of D or is adjacent to an element of D. A dominating set D of a graph G is minimal if no proper subset of D is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a minimal dominating set of graph G.

In graph theory the study of dominating sets was introduced by Ore [6] in 1962 and Berge [1] in 1958. They introduced the term dominating set and domination number of a graph. The concept of domination in graphs and its many variations are studied by many researchers and the literature on this subject has been surveyed in the books by T. Haynes, S. Hedetniemi and P. Slater [3] [4]. In this paper we focus on

[1]Department of Mathematics, Atmiya University, Rajkot, Gujarat, India.
Email:rakhiisaac@yahoo.com, rakhimol.isaac@atmiyauni.ac.in
[2]Department of Mathematics, Atmiya University, Rajkot, Gujarat, India.
Email:parashree.pandya99@gmail.com

a variation named paired domination of graphs, obtained by some graph operations.

The degree of vertex v, denoted by $d(v)$, is the number of edges incident to v with counting loop twice. The set of edges with no common end vertex is called a matching. A matching of a graph G is perfect if every vertex of the graph is incident to an edge of the matching. The paired dominating set D of graph G is the subset of vertex set V of graph G such that the subset D is dominating set and the subgraph induced by D contains a perfect matching. A paired dominating set with minimum cardinality is called minimal paired dominating set. The paired domination number $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set D of graph G. Paired domination was introduced by Haynes and P. Slater [3]. Paired domination numbers of complete graph, star graph, wheel graph and complete bipartite graph are studied by Sangeetha and Swarnamalya [8]. They also studied paired domination of the corona graphs obtained by the standard graphs. Locating and paired dominating sets in graphs are studied by J. McCoy and Henning in [5] while the upper paired domination numbers of graphs are done by Ulatowski [9].

Definition 1.1 The distance between two vertices is the number of edges in a shortest path connecting them.

Definition 1.2 The k^{th} power of a graph G, denoted by G^k, is a graph having same vertex set as of G and two vertices are adjacent if and only if distance between them is at most k.

Definition 1.3 Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of all vertices of the same degree, with at least two elements and $T = V - \cup_{i=1}^{t} S_i$. The degree splitting graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i; $1 \leqslant i \leqslant t$.

Proposition 1.4 [3] If P_n, C_n, K_n, $K_{m,n}$ and $W_n : C_n + K_1$ are respectively path, cycle, complete graph, complete bipartite graph and wheel graph then,
i. $\gamma(P_n) = \left\lfloor \dfrac{n+2}{3} \right\rfloor$.
ii. $\gamma(C_n) = \left\lfloor \dfrac{n+2}{3} \right\rfloor$.
iii. $\gamma(K_n) = 1$.

iv. $\gamma(K_{m,n}) = 2$.
v. $\gamma(W_n) = 1$.

Proposition 1.5 [8]
i. $\gamma_{pr}(P_n) = 2 \cdot \left\lceil \frac{n}{4} \right\rceil$.
ii. $\gamma_{pr}(C_n) = 2 \cdot \left\lceil \frac{n}{4} \right\rceil$.
iii. $\gamma_{pr}(K_n) = 2$.
iv. $\gamma_{pr}(K_{m,n}) = 2$.
v. $\gamma_{pr}(W_n) = 2$.

Proposition 1.6 [7] If G is $k-$regular graph then $DS(G) \cong G + K_1$.

2 Main results

Theorem 2.1 The paired domination number $\gamma_{pr}(P_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Proof: Let $V = \{v_1, v_2, v_3, \cdots, v_n\}$ be the set of vertices in P_n^k. Consider any two vertices v_i and v_j such that $v_iv_j \in E(P_n^k)$. Clearly, distance between v_i & v_j is atmost k in P_n. It is also clear that the vertex v_i can dominate atmost $2k$ vertices and similarly the vertex v_j can dominate $2k$ vertices among which at most $k-1$ vertices are same. Thus any two vertices of the set V can dominate at most $3k+1$ vertices.
Case 1: $n < 3k+1$
Consider the vertices v_i and v_j such that
(i) $d(v_i)$ is maximum,
(ii) $d(v_j)$ is at most one less than $d(v_i)$ and
(iii) $v_1v_i,\ v_jv_n$ and $v_iv_j \in E(P_n^k)$.
Clearly, the set $\{v_i, v_j\}$ can dominate V and thus, it forms a minimal paired dominating set.
Thus, $\gamma_{pr}(P_n^k) = 2 = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$ for $n < 3k+1$.
Case 2: $n \geqslant 3k+1$.
Consider two vertices v_{k+1} and v_{2k+1}. Clearly $v_{k+1}v_{2k+1} \in E(P_n^k)$ and the vertex v_{k+1} dominates set of $2k$ vertices $\{v_1, v_2, v_3, \cdots, v_k\} \cup \{v_{k+2}, v_{k+3}, \cdots, v_{2k+1}\}$. Similarly the vertex v_{2k+1} dominates $2k$ vertices among which $k-1$ vertices are dominated by v_{k+1}. Thus the set $D = \{v_{k+1}, v_{2k+1}\}$ dominates $2k + 2k - (k-1) = 3k+1$ vertices.
Subcase 1: $n \equiv 0(\text{mod } (3k+1))$

Take $n = t \cdot (3k+1)$.
As the vertices v_{k+1} and v_{2k+1} can dominate $3k+1$ vertices, the set V can be dominated by $2 \cdot t$ vertices which forms a minimal paired dominating set.

$$\begin{aligned} \therefore \gamma_{pr}(P_n^k) &= 2 \cdot t \\ &= 2 \cdot \frac{n}{3k+1} \quad \text{if } n = t \cdot (3k+1) \end{aligned}$$

Subcase 2: $n \equiv a(\text{mod } (3k+1)), a < 3k+1 \ \& \ a \neq 0$.
Take $n = a + t \cdot (3k+1)$.
We require $2 \cdot t$ vertices in order to dominate $t \cdot (3k+1)$ vertices as in subcase 1.
For the remaining vertices $a < 3k+1$, we require two vertices by case 1.

$$\begin{aligned} \therefore \gamma_{pr}(P_n^k) &= 2 + 2 \cdot t \\ &= 2 \cdot (t+1) \\ &= 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil \quad \text{for } n = a + t \cdot (3k+1) \text{ where } a < 3k+1 \end{aligned}$$

Thus, $\gamma_{pr}(P_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Theorem 2.2 The paired domination number $\gamma_{pr}(C_n^k) = 2 \cdot \left\lceil \frac{n}{3k+1} \right\rceil$.

Proof: The proof is similar to theorem 2.1.

Theorem 2.3 $\gamma_{pr}(DS(P_n)) = 4;\ n \geqslant 4$.

Proof: Consider path P_n with $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, e_3, \cdots, e_{n-1}\}$. There are two types of vertices in P_n: vertices v_1 & v_n with degree 1 and vertices $v_2, v_3, \cdots, v_{n-1}$ with degree 2. In order to get $DS(P_n)$, consider $S_1 = \{v_2, v_3, \cdots, v_{n-1}\}$, $S_2 = \{v_1, v_n\}$ and $T = \phi$. Take w_i corresponds to S_i and join w_i to S_i; $i = 1, 2$. If we consider $D = \{w_1, v_k, w_2, v_i;\ v_k \in S_1, v_i \in S_2\}$, we get D as a dominating set that induces a perfect matching. It is very clear to observe that by removing any pair of vertices from it, it will not remain as a dominating set. Hence D forms a minimal paired dominating set. Thus $\gamma_{pr}(DS(P_n)) = 4$.

Theorem 2.4 $\gamma_{pr}(DS(C_n)) = 2;\ n \geqslant 3$.

Proof: By proposition 1.6, $DS(C_n) \cong C_n + K_1 = W_n$.
$\therefore \gamma_{pr}(DS(C_n)) = 2$ by proposition 1.5.

3 Conclusion

The paired domination is mainly applied in networking and military surveillance. In this paper we have found the paired domination number of k^{th} power of path and cycle as well as that of degree splitting graph of path and cycle.

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