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A Study on Fuzzy Simply Lindelof Spaces

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Abstract

In this paper, the concept of fuzzy simply Lindelof spaces is introduced. Several characterizations of fuzzy simply Lindelof spaces are given.

Key words: Fuzzy dense set, Fuzzy nowhere dense set, Fuzzy simply open set, fuzzy second category space, Fuzzy submaximal space, Strongly irresolvable space.

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1 Introduction

Many generalizations of Lindelof spaces have been introduced and studied by several authors. Among the various covering properties of topological spaces, a lot of attention has been made to those covers which involve open and regular open sets in classical topology. In 1982 G.Balasubramanian introduced and studied the notion of nearly Lindeof spaces. In 1984 S.Willard and U.N.B. Dissanayake gave the notion of almost Lindelof spaces and in 1996 F. Cammaroto and G. Santoro introduced the notion of weakly regular Lindelof spaces on using regular covers. In 1965, L.A.Zadeh introduced the concept of fuzzy sets as a new approach for modelling uncertainties. In 1968, C.L.Chang introduced the concept of fuzzy topological spaces. The paper of Chang proved the way for the subsequent tremendous growth of the numerous fuzzy topological spaces. In this paper by means of fuzzy simply open sets, the notion of fuzzy simply Lindelof spaces, is introduced and studied. Several examples are given to illustrate the concepts introduced in this paper.

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2 Preliminaries

Definition 2.1 Let (X,T) be a fuzzy topological space and λ be any fuzzy in (X,T). The interior and the closure of λ are defined as follows

(i) $\text{Int}(\lambda) = \vee\{\ \mu/\mu \leq \lambda, \mu \in T\}$
(ii) $Cl(\lambda) = \wedge\{\ \mu/\lambda \leq \mu, 1-\mu \in T\}$

Definition 2.2 A fuzzy set λ in a fuzzy topological space (X,T), is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$. That is, $\text{cl}(\lambda) = 1$, in (X,T).

Definition 2.3 A fuzzy set λ in a fuzzy topological space (X,T), is called fuzzy nowhere dense if there exists no non zero fuzzy open set μ in (X,T) such that $\mu <$ cl(λ). That is, $\text{intcl}(\lambda) = 0$, in (X,T).

Definition 2.4 A fuzzy set λ in a fuzzy topological space (X,T), is called a fuzzy simply open set if Bd(λ) is a fuzzy nowhere dense set in (X,T). That is, λ is a fuzzy simply open set in (X,T) if $\text{cl}\ (\lambda) \wedge \text{cl}(1-\lambda)$, is a fuzzy nowhere dense set in (X,T).

Definition 2.5 A fuzzy topological space (X,T) is said to be fuzzy Lindelof if every fuzzy open cover of X has a countable subcover. That is, for every fuzzy open cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of X, there exists $\{\lambda_{\alpha_n}\}_{n\in N}$ of fuzzy open sets in (X,T) such that $\bigvee_{n\in N} \{\lambda_{\alpha_n}\} = 1$.

Theorem 2.6 If λ is a fuzzy open and fuzzy dense set in a fuzzy topological space (X,T), then λ is a fuzzy simply open set in (X,T).

Definition 2.7 Let (X,T) be a fuzzy topological space. A fuzzy set λ in a (X,T) is called a fuzzy first category set if $\lambda = \bigvee_{\alpha=1}^{\infty} (\lambda_\alpha)$, where (λ_α) 's are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy second category.

Definition 2.8 A fuzzy topological space (X,T) is called fuzzy first category space if $1_x = V_{\alpha=1}^{\infty} (\lambda_\alpha)$, where (λ_α)'s are fuzzy nowhere dense sets in (X,T). A fuzzy topological space which is not of fuzzy first category is said to be of fuzzy second category.

Definition 2.9 A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T) such that cl $(\lambda) = 1$, then $\lambda \in T$

Theorem 2.10 If λ is a fuzzy closed set with $\text{int}(\lambda) = 0$, in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.11 If λ is a fuzzy simply open set in a fuzzy topological (X, T), then $\lambda \wedge (1 - \lambda)$ is a fuzzy nowhere dense set in (X, T).

Theorem 2.12 If λ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.13 If $\lambda = \mu \vee \delta$ where μ is a fuzzy open and fuzzy dense set and δ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy simply open set in (X, T).

Theorem 2.14 If int $(\lambda) = 0$ for a fuzzy set λ in a fuzzy strongly irresolvable space (X, T), then λ is a fuzzy simply open set in (X, T).

3 Fuzzy Simply Lindelof Spaces

Definition 3.1 A fuzzy topological space (X,T) is said to be fuzzy simply Lindelof if each cover of X by fuzzy simply open sets has a countable subcover. That is, (X, T) is a fuzzy simply Lindelof space if $V_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ where intcl $[\text{bd}(\lambda_\alpha)] = 0$ in (X, T), then $V_{n\in\mathbb{N}} \{\lambda_{\alpha_n}\} = 1$ in (X, T).

Proposition 3.2 If (X,T) is a fuzzy simply Lindelof space and if $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ where $\{\lambda_\alpha\}$'s are fuzzy closed sets with int $(\lambda_\alpha) = 0$ in (X, T), then $\bigvee_{n\in N} \{\lambda_{\alpha_n}\} = 1$, in (X, T).

proof:

Let (X, T) be a fuzzy simply Lindelof space. By hypothesis, the fuzzy sets $\{\lambda_\alpha\}$'s are fuzzy closed sets with int $(\lambda_\alpha) = 0$ in (X, T). Then by Theorem 2.10, $\{\lambda_\alpha\}$'s

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are fuzzy simply open sets in (X,T). Now $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$ where $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T), implies that $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a fuzzy simply open cover of X. Since (X,T) is a fuzzy simply Lindelof space, there exist a countable subcover $\{\lambda_{\alpha_n}\}_{n\in\mathbb{N}}$ of fuzzy simply open sets, for X. Hence $\bigvee_{n\in\mathbb{N}}\{\lambda_{\alpha_n}\} = 1$, where $(1-\lambda_{\alpha_n}) \in T$ and int $\{\lambda_{\alpha_n}\} = 0$, in (X,T).

Proposition 3.3 If (X,T) is a fuzzy simply Lindelof space and if $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$ where $\lambda_\alpha \in T$ and $\mathrm{cl}\,(\lambda_\alpha) = 1$ in (X,T), then $\bigwedge_{n\in N}\{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

proof:

Let (X,T) be a fuzzy simply Lindelof space. By hypothesis, the fuzzy sets $\{\lambda_\alpha\}$'s are fuzzy open and fuzzy dense sets in (X,T).Then by Theorem 2.10, $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T). Now $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$ where $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T), implies that $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a fuzzy simply open cover of X. Since (X,T) is a fuzzy simply Lindelof space, there exist a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of fuzzy simply open sets for X and then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\} = 1$ in (X,T). Then $1-\bigvee_{n\in N}\{\lambda_{\alpha_n}\} = 0$ and hence $\bigwedge_{n\in N}(1-\lambda_{\alpha_n}) = 0$. Let $\mu_{\alpha_n} = 1-\lambda_{\alpha_n}$. Now intcl $(1-\lambda_{\alpha_n}) = 1 - cl$ int $(\lambda_{\alpha_n}) = 1-\mathrm{cl}\,(\lambda_{\alpha_n}) = 1-1 = 0$, and thus $(1-\lambda_{\alpha_n})$'s are fuzzy nowhere dense set in (X,T).Hence $\bigwedge_{n\in N}\{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X,T).

Proposition 3.4 If (X,T) is a fuzzy simply Lindelof space, then (X,T) is a fuzzy second category space.

proof:

Let (X,T) be a fuzzy simply Lindelof space and $\{\lambda_\alpha\}_{\alpha\in\Delta}$ be a cover of X by fuzzy simply open sets in (X,T). By Theorem 2.11, $\{\lambda_{\alpha_n}\wedge((1-\lambda_{\alpha_n})\}$'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X. Then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\} = 1$ in (X,T). Now $[\lambda_{\alpha_n}\wedge(1-\ \lambda_{\alpha_n})] \le \lambda_{\alpha_n}$ in (X,T) implies that $\bigvee_{n\in N}[\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})] \le \bigvee_{n\in N}\{\lambda_{\alpha_n}\}$ and then $\bigvee_{n\in N}\ [\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})] \le 1$. Then $\bigvee_{n\in N}[\lambda_{\alpha_n}\wedge((1-\lambda_{\alpha_n})] \ne 1$, where $\{\lambda_{\alpha_n}\wedge(1-\lambda_{\alpha_n})\}$'s are fuzzy nowhere dense sets, implies that (X,T) is not a fuzzy first category space and hence (X,T) is a fuzzy second category space.

Proposition 3.5 If $\{\lambda_a\}_{\alpha\in\Delta}$ is a cover of X by fuzzy nowhere dense sets in a fuzzy simply Lindelof space (X,T), then there is a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X.

proof:

If $\{\lambda_a\}_{\alpha\in\Delta}$ is a cover of X by fuzzy nowhere dense sets in (X,T), then $\bigvee_{\alpha\epsilon\Delta}$ $\{\lambda_\alpha\}=1$, where $\operatorname{intcl}(\lambda_\alpha)=0$ in (X,T). By theorem 2.12, the fuzzy nowhere dense sets $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T) and thus $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a cove of X by fuzzy simply open sets. Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of X. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$ where intcl $(\lambda_{\alpha_n})=0$ in (X,T).

Proposition 3.6 If $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\lambda_\alpha=\mu_\alpha\vee\delta_\alpha$ and $\{\mu_\alpha\}$'s are fuzzy open and fuzzy dense sets and $\{\delta_\alpha\}$'s are fuzzy nowhere dense sets in a fuzzy simply Lindelof space (X,T), then $\eta\vee\delta=1$, where $\eta\in T$ and $cl(\eta)=1$ and δ is a fuzzy first category set in (X,T).

Proof:

Let (X,T) be a fuzzy simply Lindelof space such that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ and $\lambda_\alpha=\mu_\alpha\vee\delta_\alpha$, where $\mu_\alpha\in T$ and $cl\,(\mu_\alpha)=1$ and intcl $(\delta_\alpha)=0$ in (X,T) By theorem 2.13, $\{\lambda_\alpha\}$'s are fuzzy simply open sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ of X. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$. Then, $\bigvee_{n\in N}\ [\mu_{\alpha_n}\vee\delta_{\alpha_n}]=\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\bigvee\left[\bigvee_{n\in N}(\delta_{\alpha_n})\right]$ implies that $\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\bigvee\left[\bigvee_{n\in N}(\delta_{\alpha_n})\right]=1$. Since $\{\delta_\alpha\}$'s are fuzzy nowhere dense sets in $(X,T),\bigvee_{n\in\mathbb{N}}\{\delta_{\alpha_n}\}=\delta$, implies that δ is a fuzzy first category set,in (X,T). Since $(\mu_{\alpha_n})\in T,\bigvee_{n\in N}(\mu_{\alpha_n})\in T$. Also $cl\left[\bigvee_n\in\mathbb{N}\ \ (\mu_{\alpha_n})\right]\geq\bigvee_{n\in N}cl\left[(\mu_{\alpha_n})\right]$ implies that $cl\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]\geq\bigvee 1$ and hence $cl\left[\bigvee_{n\in N}(\mu_{\alpha_n})\right]=1$. Let $\eta=\bigvee_{n\in N}(\mu_{\alpha_n})$. Then η is a fuzzy open and fuzzy dense set in (X,T). Thus $\delta\vee\eta=1$ in (X,T).

Proposition 3.7 If $\{\lambda_\alpha\}_{\alpha\in\Delta}$ is a cover of X by fuzzy sets with $\operatorname{int}(\lambda_\alpha)=0$, in a fuzzy strongly irresolvable and fuzzy simply Lindelof space (X,T), then $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$.

Proof:

Suppose that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$, where $\operatorname{int}(\lambda_\alpha)=0$ in (X,T).Since (X,T) is a fuzzy strongly irresolvable space, by Theorem 2.14, $\{\lambda_{\alpha_n}\}$'s are fuzzy simply open sets in (X,T). Since (X,T) is a fuzzy simply Lindelof space, $V_{\alpha\in\Delta}\{\lambda_\alpha\}=1$, where $\{\lambda_\alpha\}$'s are fuzzy simply open sets implies that there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X by fuzzy simply open sets. That is, $\bigvee_{n\in N}\{\lambda_{\alpha_n}\}=1$, where $\operatorname{int}(\lambda_{\alpha_n})=0$ in (X,T).

Proposition 3.8 If (X,T) is a fuzzy simply Lindelof space and fuzzy submaximal space and if $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\}=1$ where $\{\lambda_\alpha\}$'s are fuzzy dense sets in (X,T), then $\bigwedge_{n\in N}$

$\{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X, T).

proof:

Let (X, T) be a fuzzy simply Lindelof and fuzzy submaximal space. By hypothesis $\bigvee_{\alpha\in\Delta} \{\lambda_a\} = 1$, where $\mathrm{cl}(\lambda_\alpha) = 1$ in (X, T). Since (X, T) is a fuzzy submaximal space, the fuzzy dense sets $\{\lambda_a\}$'s are fuzzy open sets in (X, T). Hence $\{\lambda_\alpha\}$'s are fuzzy open sets and fuzzy dense sets in (X, T). Then, by proposition 3.3, $\bigwedge_{n\in N} \{\mu_{\alpha_n}\} = 0$, where (μ_{α_n})'s are fuzzy nowhere dense sets in (X, T).

Proposition 3.9 If $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$, where intcl $(\lambda_\alpha) = 0$ in a fuzzy simply Lindelof space (X, T), then there exist a fuzzy first category set λ in (X, T) such that cl $(\lambda) = 1$ in (X, T).

proof:

Let (X, T) be a fuzzy simply Lindelof space such that $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ and $\mathrm{intcl}(\lambda_\alpha) = 0$. Now $\mathrm{int\,cl}(\lambda_\alpha) = 0$ in (X, T) implies that (λ_a)'s are fuzzy nowhere dense sets in (X,T). Then, by theorem 2.7, $\{\mathrm{cl}(\lambda_\alpha)\}$'s are fuzzy simply open sets in (X,T). Now $\lambda_\alpha \leq \mathrm{cl}(\lambda_\alpha)$ implies that $\bigvee_\alpha \in \Delta \{\lambda_\alpha\} \leq \bigvee_\alpha \in \Delta\, \mathrm{cl}(\lambda_\alpha)$ and then $1 \leq \bigvee_{\alpha\in\Delta} \mathrm{cl}(\lambda_\alpha)$. That is $\bigvee_{\alpha\in\Delta} \mathrm{cl}(\lambda_\alpha) = 1$ where $\{\mathrm{cl}(\lambda_\alpha)\}$'s are fuzzy simply open sets in the fuzzy Lindelof space (X,T). Then there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X by fuzzy simply open sets in (X, T). That is, $\bigvee_{n\in N} \mathrm{cl}(\lambda_{\alpha_n}) = 1$. Now, by lemma, $\bigvee_{n\in N} cl(\lambda_{\alpha_n}) \leq cl\left(\bigvee_{n\in N} \{\lambda_{\alpha_n}\}\right)$ implies that $1 \leq cl\left(\bigvee_{n\in N} \{\lambda_{\alpha_n}\}\right)$. That is, $cl\left(\bigvee_{n\in N} \{\lambda_{\alpha_n}\}\right) = 1$. Let $\bigvee_{n\in N} \{\lambda_{\alpha_n}\} = \lambda$. Then λ is a fuzzy first category set in (X, T) such that $\mathrm{cl}(\lambda) = 1$, in (X, T).

Proposition 3.10 If $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$ where (λ_α)'s are fuzzy nowhere dense sets in a fuzzy simply Lindelof and fuzzy Baire space (X, T) then (X, T) is a fuzzy resolvable space.

Proof:

Let (X, T) be a fuzzy simply Lindelof space such that $\bigvee_\alpha \epsilon\Delta \{\lambda_\alpha\} = 1$ where $\mathrm{intcl}(\lambda_\alpha) = 0$ in (X, T). Then, by proposition 3.9, there exist a fuzzy first category set λ in (X, T) such that $\mathrm{cl}(\lambda) = 1$. Since (X, T) is a fuzzy Baire space, by theorem int $(\lambda) = 0$ in (X, T). Now $\mathrm{cl}(1 - \lambda) = 1 - \mathrm{int}(\lambda) = 1 - 0 = 1$.Thus $\mathrm{cl}(\lambda) = 1$ and $\mathrm{cl}(1 - \lambda) = 1$ in (X, T). Hence (X, T) is a fuzzy resolvable space.

Proposition 3.11 If $\bigvee_{\alpha\in\Delta} \{\lambda_\alpha\} = 1$, where (λ_α)'s are fuzzy open sets in a fuzzy simply Lindelof and fuzzy hyper connected space (X, T) then there exists a countable subcover $\{\lambda_{\alpha_n}\}_{\mathrm{n}\in\mathrm{N}}$ for X.

Proof:

Let (X,T) be a fuzzy simply Lindelof and fuzzy hyper connected space such that $\bigvee_{\alpha\in\Delta}\{\lambda_\alpha\} = 1$, where $\lambda_\alpha \in T$. Since (X, T) is a fuzzy hyper connected space, the fuzzy open sets (λ_α)'s are fuzzy dense sets in (X, T). Then (λ_α)'s are fuzzy open and fuzzy dense sets in (X, T). Then, by theorem 2.6, (λ_a)'s are fuzzy simply open sets in (X, T). Since (X, T) is a fuzzy simply Lindelof space, there exists a countable subcover $\{\lambda_{\alpha_n}\}_{n\in N}$ for X.

4 Conclusion

In this paper we have presented a Fuzzy Simply Lindelof Spaces through definitions and also stated important theorems. Using that we have proved some propositions.

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