2456-8686, vi(ii), 2022: 135-141
https://doi.org/10.26524/cm156

ORIGINAL RESEARCH **Open Access**

On Connectivity of Zero-Divisor Graphs and Complement Graphs of some Semi-Local Rings

Subhash Mallinath Gaded[1] and Nithya Sai Narayana[2]

Received: 31 October 2022/ Accepted: 23 November 2022/ Published online: 09 December 2022

Abstract

Zero-divisor graphs have been a key area of focus for many researchers. For the semi local ring R of finite cartesian product of finite fields, we consider the zero divisor graph of R denoted by $\Gamma(R)$ with vertex set as the non-zero zero-divisors of R where two vertices u and v are adjacent if and only if the product of u and v is the additive identity of the Ring R. The objective of this paper is to determine the number of cut vertices and cut edges, vertex connectivity and edge connectivity of the zero divisor graph $\Gamma(R)$ and complement graph $\overline{\Gamma(R)}$.

Key words: Zero-divisor graphs, Cut vertices, Cut edges, Vertex connectivity, Edge connectivity.

AMS classification: 13A70, 05C25, 05C40.

1 Introduction

Zero divisors graphs have been studied on general rings. The idea of Zero Divisor graphs of a commutative ring R has been introduced by I. Beck [3]. Originally all the elements of the ring R has been considered as vertices of this graph and two vertices u and v are adjacent if and only if $u \cdot v = 0$. This definition is modified by Anderson and Livingston [2] wherein the vertex set is reduced to only the set of non-zero zero divisors of R. Let $n \geq 2$ and $F_1, F_2. \cdots, F_n$ be finite fields then $R = F_1 \times F_2 \times \cdots F_n$ is a semi local ring. Let $Z^*(R)$ denote the set of non-zero, zero-divisors of R and let $\Gamma(R)$ denote the graph with vertex set as $Z^*(R)$ and edge set as $\{xy : x \cdot y = 0, x, y \in Z^*(R)\}$ [4]. Since $Z(R)$ is closed under multiplication, the complement graph $\overline{\Gamma(R)}$ of the zero-divisor graph $\Gamma(R)$ satisfies the property: $rs \in E(\overline{\Gamma(R)})$ if and only if $r \cdot s \in Z^*(R)$.[1]

For basic graph theoretical terminologies we adopt the definitions of [6]. A vertex v is called as cut vertex of a graph G if number of components of $G - v$ is greater than

[1] Department of Mathematics, University of Mumbai, Mumbai, Maharashtra, India.
Email: subhash.gaded@ssrkt.edu.in
[2] Department of Mathematics, University of Mumbai, Mumbai, Maharashtra, India.
Email: narayana_nithya@yahoo.com

the number of components of G and an edge e of a graph G is said to be a cut edge if number of components of $G-e$ is greater than the number of components of G[6]. A subset $S \subset V(G)$ is said to be a vertex cut if $G-S$ is disconnected or has only one vertex and the cardinality of a minimum vertex cut is called as vertex connectivity of a graph G, denoted by $\kappa(G)$[6]. Simlarly $F \subset E(G)$ is said to be an edge cut if $G-F$ is disconnected and minimum size of an edge cut is called the edge connectivity of a graph G, denoted by $\lambda(G)$. [7] In this paper, we determine the number of cut vertices, cut edges, vertex connectivity, edge connectivity of zero divisor graph over the ring $R = F_1 \times F_2 \times \cdots \times F_n, (n \geqslant 2)$, where F_1, F_2, $\cdots$, F_n are finite fields.

2 Cut vertices and Cut edges, Vertex connectivity and Edge connectivity

Remark 2.1 If $R = F_1 \times F_2$, and $|F_1| = |F_2| = 2$, then $\Gamma(R)$ is K_2 and does not contain cut vertex. Further, if $R = F_1 \times F_2$ with $|F_1| \geqslant 2$, $|F_2| \geqslant 2$, then it is easy to check $\Gamma(R)$ is a complete bipartite graph $K_{1,|F_2|-1}\ or\ K_{|F_1|-1,1}$ which is also a tree. Hence, the number of cut vertices in this case is 1. [6]

Theorem 2.2 (i) Let $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. If $n \geqslant 3$ then the Zero-divisor graph $\Gamma(R)$ has exactly n cut vertices.
(ii) Let $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$. If $n \geq 2$ and $|F_1| \geqslant 3$ or if $n \geqslant 3$ and $|F_2| \geq 3$ then the Zero-divisor graph $\Gamma(R)$ has no cut vertex.
Proof: (i) Let $n \geqslant 3$ and $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. There are $C(n,r)$ vertices of degree $2^r - 1, (1 \leqslant r \leqslant n-1)$ in $\Gamma(R)$. Let $V_r = \{v \in V(\Gamma(R)) : v$ has exactly r number of zero entries in its coordinates $\}, (1 \leqslant r \leqslant n-1)$. A vertex in the set V_{n-1} has unique $'1'$ say at i^{th} position and all the $n-1$ positions have $'0'$ entries. A vertex in the set V_1 has $'0'$ in i^{th} position and $'1'$ in all other positions. Thus elements of V_1 are of degree $'1'$. A vertex in V_{n-1} is adjacent to an unique vertex in V_1 and thus every vertex of V_{n-1} is a cut vertex. Further V_1 has no cut vertex and $< V_1 \cup V_{n-1} >$ is connected. We will prove that for every $z \in V(\Gamma(R)) \setminus \{V_1 \cup V_{n-1}\}$ is not a cut vertex. Suppose, $z \in V_r, (2 \leqslant r \leqslant n-2)$. We will prove that $\Gamma(R) - z$ is connected. Now $z \in V_r \implies z$ contain non-zero entries, that is $'1'$ in $n-r$ positions, t_1, $\cdots$, t_{n-r}, $(2 \leqslant r \leqslant n-2)$ with $(1 \leqslant t_1 < \cdots < t_{n-r} \leqslant n)$. Let y_1, $y_2 \in \{V_2 \cup \cdots \cup V_{n-2}\} - z$. Since, y_1 and y_2 contains at least two $0's$, both are adjacent to at least two vertices in V_{n-1}. If y_1 and y_2 are adjacent to same vertex, say, x in V_{n-1}, then y_1, x, y_2 is a path connecting y_1 and y_2. If y_1 and y_2 are adjacent to different vertices, say, x_1 and x_2 in V_{n-1}, then

since, x_1 and x_2 are adjacent in V_{n-1}, we have a path y_1, x_1, x_2, y_2 connecting y_1 and y_2 and this path does not contain z. Thus every pair of vertices
(a) $y_1, y_2 \in \{V_1 \cup V_{n-1}\}$
(b) $y_1, y_2 \in \{V_2 \cup \cdots \cup V_{n-2}\} - z$
(c) one of $y_1, y_2 \in \{V_1 \cup V_{n-1}\}$ and the other $\in \{V_2 \cup \cdots \cup V_{n-2}\} - z$
are connected in a path not containing z. Therefore, $\Gamma(R) - z$ is connected. Thus, z is not a cut vertex. Hence, the zero-divisor graph $\Gamma(R)$ of semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| = |F_2| = \cdots = |F_n| = 2, (n \geqslant 3)$, has exactly $|V_{n-1}| = n$ cut vertices.

(ii) Case(1): Let $n \geq 2$, $|F_1| \geqslant 3$. We prove that $v \in V(\Gamma(R))$ is not a cut vertex. That is, we prove that $\Gamma(R) - v$ is connected. Fort $x, y \in \Gamma(R) - v$ if $xy \in E(\Gamma(R) - v)$, then nothing to prove. Suppose $xy \notin E(\Gamma(R) - v)$. Clearly x, y contains at least one entry as $'0'$ in its coordinates. Let x contains $'0'$ in i^{th} position and y contains $'0'$ in j^{th} position. It is possible that $i = j$.
Let $V_{n-1} = \{u \in V(\Gamma(R)) : u$ has exactly $n-1$ number of $'0'$ entries in its coordinates$\}$. Note that there exists $(|F_i| - 1)$ vertices in $V_{n-1} \subset V(\Gamma(R))$ containing non zero entry in i^{th} position. $\Rightarrow$ There exists at least $(|F_i| - 2)$ vertices in $V(\Gamma(R) - v)$ containing non zero entry in i^{th} position.
Since $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$ and $|F_1| \geqslant 3 \Rightarrow |F_i| - 2 \geq 1$, $(1 \leqslant i \leqslant n)$ and hence there is at least one vertex say $z \in V(\Gamma(R) - v)$ with one non-zero entry in i^{th} position and $'0'$ every where else. Clearly $xz \in E(\Gamma(R) - v)$. Similarly there is at least one vertex say $w \in V(\Gamma(R) - v)$ with one non-zero entry in j^{th} position and $'0'$ every where else with $yw \in E(\Gamma(R) - v)$. Further any two vertices in V_{n-1} with non-zero entries in different positions are adjacent. Thus, if $i = j$ then x and y are joined by the path $x - z - y$ and if $i \neq j$ then x and y are joined by the path $x - z - w - y$ in $\Gamma(R) - v$.
$\Rightarrow \Gamma(R) - v$ is connected $\Rightarrow v$ is not a cut vertex in $\Gamma(R)$.
Case(2): Let $n \geqslant 3$, $|F_2| \geq 3$. If $|F_1| \geq 3$, then by case (i) $\Gamma(R)$ has no cut vertex. Suppose $|F_1| = 2$, $|F_2| \geq 3$. For $v \in V(\Gamma(R))$ we prove that $\Gamma(R) - v$ is connected. For $x, y \in \Gamma(R) - v$ if $xy \in E(\Gamma(R) - v)$, we have nothing to prove. Let $xy \notin E(\Gamma(R) - v)$ and V_{n-1} be as defined in case (i). Note that x, y have $'0'$ entry in at least one position. Let x has $'0'$ entry at position i and y has $'0'$ entry in position j where $1 \leq i, j \leq n$. Further there exists a position say $k, 1 \leq k \leq n$ such that x, y both have non zero entry in position k.
If $k = 1$, then there exists at least two vertices in V_{n-1} with $'0'$ entry in position 1 and non zero entry in i and j position. This is possible as $n \geq 3$ and $|F_2| \geq 3$. Thus

there is a vertex $z \in \Gamma(R) - v$, such that xz, $yz \in E(\Gamma(R) - v)$ to get a path $x - z - y$ in $\Gamma(R) - v$.
If $k \neq 1$, then clearly there exists at least two vertices in V_{n-1} with $'0'$ entry in position 1 and non-zero entry in position i as $n \geq 3$ and hence there is a vertex $z_1 \in \Gamma(R) - v$ such that $xz_1 \in E(\Gamma(R) - v)$. Similarly there is a vertex $z_2 \in \Gamma(R) - v$ such that $yz_2 \in E(\Gamma(R) - v)$. If $i = j \implies z_1 = z_2$, then $x - z_1 - y$ is a path in $\Gamma(R) - v$. If $z_1 \neq z_2$, then since $z_1z_2 \in E(\Gamma(R) - v)$ we get, a $x - z_1 - z_2 - y$ path in $\Gamma(R) - v$ and hence v is not a cut vertex. Hence there is no cut vertex in $\Gamma(R)$ in this case also.

Theorem 2.3 (i) If $R = F_1 \times F_2$ with $|F_1| = 2$ then there are $|F_2| - 1$ cut edges in $\Gamma(R)$ and if $|F_2| = 2$ then there are $|F_1| - 1$ cut edges in $\Gamma(R)$.
(ii) If $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_i| = 2$, $(1 \leqslant i \leqslant n)$ and $n \geqslant 3$ then the $\Gamma(R)$ has exactly n cut edges.
(iii) If $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| \leqslant |F_2| \leqslant \cdots \leqslant |F_n|$ and if $n \geq 2$ with $|F_1| \geqslant 3$ or if $n \geqslant 3$ with $|F_2| \geq 3$ then $\Gamma(R)$ has no cut edges.
Proof: (i) If $|F_1| = 2$, then $\Gamma(R) = K_{1,|F_2|-1}$ which contains exactly $|F_2| - 1$ edges and each edge is a cut edge. If $|F_2| = 2$, then $\Gamma(R) = K_{|F_1|-1,1}$ which contains exactly $|F_1| - 1$ edges and each edge is a cut edge. (ii) Let $n \geqslant 3$ *and* $|F_i| = 2$, $(1 \leqslant i \leqslant n)$. A vertex with $'0'$ in one position and $'1'$ in remaining positions is a degree 1 vertex and hence the unique edge adjacent to this vertex is a cut edge. There are n vertices of degree 1 in $\Gamma(R)$ and hence $\Gamma(R)$ contains at least n cut edges. Let $E_C(\Gamma(R))$ be the set of all these n cut edges.

Now let $e = uv \in E(\Gamma(R)) \setminus E_C(\Gamma(R))$. Then both the endpoints u and v of edge e contain $'0'$ in at least two positions. Suppose, u contains $'0'$ in i^{th} and j^{th} position where $1 \leqslant i < j \leqslant n$ and v contains $'0'$ in l^{th} and m^{th} position where $1 \leqslant l < m \leqslant n$. Note that i, j may be equal to l, m. Further u is adjacent to vertex v_i containing $'1'$ in i^{th} position and $'0'$ in all other positions and v is adjacent to vertex v_l containing $'1'$ in l^{th} position and $'0'$ in all other positions.
If $i = l$ then uv_ivu is a cycle and hence uv is not a cut edge.
If $i \neq l$ then v_iv_l is an edge and uv_iv_lvu is a cycle and hence uv is not a cut edge.
Therefore, the zero divisor graph of Semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$ with $|F_1| = |F_2| = \cdots = |F_n| = 2, (n \geqslant 3)$ has exactly $|E_C(\Gamma(R))| = n$ cut edges.

(iii) By property of cut edge, $e = uv$ is a cut edge $\Rightarrow$ either u or v is a cut vertex. By theorem, 2.2, for $n \geq 2$ and $|F_1| \geqslant 3$ or $n \geqslant 3$, $|F_1| = 2$, $|F_2| \geqslant 3$, $\Gamma(R)$ has no cut vertex $\Rightarrow \Gamma(R)$ has no cut edge.

Theorem 2.4 $\overline{\Gamma(R)}$ has no cut vertex.

Proof: For $n=2$, $\overline{\Gamma(R)}$ is disconnected and hence has no cut vertex. If $n \geqslant 3$, we show that $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2. [5] Let u, $v \in V(\overline{\Gamma(R)})$. If $(u,v) \in E(\overline{\Gamma(R)})$, then $d(u,v)=1$. Suppose, $(u,v) \notin E(\overline{\Gamma(R)})$. Then u and v do not contain non-zero entries in the same position. Let u contains $t_i \in \{1,2,\cdots,m_i-1\}$ in the i^{th} position and v contains $t_j \in \{1,2,\cdots,m_j-1\}$ in the j^{th} position. Clearly, $i \neq j$. Let $z \in V(\overline{\Gamma(R)}) \setminus \{u,v\}$ contain non-zero entry in the i^{th} and j^{th} position and $'0'$ elsewhere. Then, clearly, $(u,z),(v,z) \in E(\overline{\Gamma(R)})$. Therefore, $u-z-v$ is a path connecting u and v. Thus, $d(u,v)=2$. Hence, $\overline{\Gamma(R)}$ is connected and diameter of $\overline{\Gamma(R)}$ is 2.

Let $v \in \overline{\Gamma(R)}$. We prove that v is not a cut vertex. Let $x,y \in \overline{\Gamma(R)}-v$. We claim that there exists an $x-y$ path in $\overline{\Gamma(R)}-v$.

If $xy \in E(\overline{\Gamma(R)}-v)$ nothing to prove. If $xy \notin E(\overline{\Gamma(R)}-v)$ then x and y contain non-zero entries in different positions. That is, there exists $i \neq j$ such that x contains non-zero entry in i^{th} position and y contains non-zero entry in j^{th} position. Clearly, the vertex w with $'1'$ in i^{th} and j^{th} position and $'0'$ in other positions is such that xw, $wy \in E(\overline{\Gamma(R)})$. If $w \neq v$, then $x-w-y$ is a path connecting x and y in $\overline{\Gamma(R)}-v$. If $w=v$, then since $n \geqslant 3$ and $xy \notin E(\overline{\Gamma(R)}-v) \implies$ There exists a position k, $(1 \leqslant k \leqslant n)$ such that x,y have $'0'$ entry in position k. Let w_1 be the vertex with entry $'1'$ in i^{th} and k^{th} position and $'0'$ everywhere else and w_2 be the vertex with entry $'1'$ in j^{th} and k^{th} position and $'0'$ everywhere else. Clearly, $x-w_1-w_2-y$ is a path connecting x and y in $\overline{\Gamma(R)}-v \implies \overline{\Gamma(R)}-v$ is connected and v is not a cut vertex. Hence, $\overline{\Gamma(R)}$ has no cut vertex.

Corollary 2.5 $\overline{\Gamma(R)}$ has no cut edge.

Theorem 2.6 If $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$. then the vertex connectivity $\kappa(\Gamma(R))$ and edge connectivity $\lambda(\Gamma(R))$ of $\Gamma(R)$ is $\delta(\Gamma(R))$. In other words, $\kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i|-1 : 1 \leq i \leq n\}$.

Proof: If $|F_1| = |F_2| = \cdots = |F_n| = 2$, then by theorems 2.2 and 2.3, $\Gamma(R)$ contains cut vertex and cut edge and hence, $\kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = 1$. Let $|F_i| \geqslant 2$. Consider a vertex with a zero in the i^{th} position and non-zero entries in the remaining positions. The degree of such a vertex is $|F_i|-1$, $(1 \leqslant i \leqslant n)$. Therefore,

$\delta(\Gamma(R)) = min\{|F_i|-1 : 1 \leq i \leq n\}$. Let $S = \{v_1, \cdots, v_{\delta(\Gamma(R))-2}\} \subset V(\Gamma(R))$ with $|S| = \delta(\Gamma(R))-2$. We prove that $\Gamma(R)-S$ is connected.
For x, $y \in \Gamma(R)-S$ if $xy \in E(\Gamma(R)-S)$ we have nothing to prove.
Suppose $xy \notin E(\Gamma(R)-S)$. Note that x and y has at least one $'0'$ as its co-ordinates.
Let $T = \{v \in V(G) :\ v$ contains non-zero entry in exactly one position and '0' in remaining positions $\}$. There are $|F_i|-1$ vertices having one non-zero entry in i^{th} position, $(1 \leqslant i \leqslant n)$ and $'0'$ elsewhere and $|T| = \sum_{i=1}^{n}(|F_i|-1)$. Therefore (even if $S \subset T$), there exists a vertex in $T \setminus S$ with a non zero entry in i^{th} position and $'0'$ everywhere else for $i = 1, 2 \cdots, n$.
There exists two vertices z and w in T, such that,
(i) z, $w \notin S$
(ii) xz, $yw \in E(\Gamma(R)-S)$.
It is possible that $z = w$. Further, any two vertices in T with non-zero entries not in same position are adjacent. Thus, if $z = w$, then $x-z-y$ is a path in $\Gamma(R)-S$. If $z \neq w$, then $x-z-w-y$ is a path in $\Gamma(R)-S$ connecting x and y. $\implies$ $\Gamma(R)-S$ is connected. $\implies$ The minimum cardinality of a vertex cut in $\Gamma(R)$ is $min\{|F_i|-1 : 1 \leq i \leq n\}$.
$\implies$ $\kappa(\Gamma(R)) \geqslant min\{|F_i|-1 | 1 \leq i \leq n\}$.
But, $\delta(\Gamma(R)) = min\{|F_i|-1 | 1 \leq i \leq n\}$ and
$\kappa(\Gamma(R)) \leqslant \delta(\Gamma(R)) \implies \kappa(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i|-1 | 1 \leq i \leq n\}$.
Further, $\lambda(\Gamma(R))$- the edge connectivity of $\Gamma(R)$ satisfies,
$\kappa(\Gamma(R)) \leqslant \lambda(\Gamma(R)) \leqslant \delta(\Gamma(R))$[6].
$\implies \kappa(\Gamma(R)) = \lambda(\Gamma(R)) = \delta(\Gamma(R)) = min\{|F_i|-1 : 1 \leq i \leq n\}$.

Corollary 2.7 (i) If $n = 2$, then $\overline{\Gamma(R)}$ is disconnected, therefore, $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = 0$.
(ii) If $R = F_1 \times F_2 \times \cdots \times F_n$, $n \geqslant 3$ then $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$.
Proof: (i) Trivial.
(ii) $\delta(\overline{\Gamma(R)}) = (|V(\Gamma(R))|-1) - \Delta(\Gamma(R)) =$
$((\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1) - 1) - 1) - max\{(\frac{\prod_{i=1}^{n}|F_i|}{|F_j|}) - 1| \ (1 \leq j \leq n)\}$.
The maximum degree vertices in $(\overline{\Gamma(R)})$ contains $'0'$ in exactly one co-ordinate position and non-zero entries in remaining co-ordinate positions. And the number of such maximum degree vertices is
$((\prod_{i=1}^{n}|F_i| - \prod_{i=1}^{n}(|F_i|-1) - 1) - 1) - min\{|F_i|-1 | 1 \leq i \leq n\}$.
Therefore, if $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$ and $T \subset V(\overline{\Gamma(R)})$ is set of

maximum degree vertices then, clearly, for $n \geq 3$, $|T| - |S| \geqslant 2$. Consider the set $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$. Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in S$. Suppose $xy \notin E(\overline{\Gamma(R)})$. Then the non-zero entries of x and y are not in the common co-ordinate positions. That is, if x has non-zero entries in the co-ordinate positions $x_1, \cdots, x_k$ $(1 \leq k \leq n)$, then y contains $'0'$ entries in the co-ordinate positions $y_1, y_2, \cdots, y_k$. Since, $|T| - |S| \geqslant 2$, let $z \in T$, such that $z \notin S$. If $xz \in E(\overline{\Gamma(R)})$ then $yz \notin E(\overline{\Gamma(R)})$, since x and y have non-zero entries that are not in the common co-ordinate positions, and since $|T| - |S| \geqslant 2$, there exists $w \in T$, such that $yw \in E(\overline{\Gamma(R)})$. Since, T is a clique, $zw \in E(\overline{\Gamma(R)})$. Thus, $x - z - w - y$ is a path connecting x and y. Hence, $\overline{\Gamma(R)} \setminus \{S\}$ is connected if $S \subset V(\overline{\Gamma(R)})$ with $|S| = \delta(\overline{\Gamma(R)}) - 1$. Therefore, $\kappa(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$. Hence, $\kappa(\overline{\Gamma(R)}) = \lambda(\overline{\Gamma(R)}) = \delta(\overline{\Gamma(R)})$.

References

[1] Anderson DF, Asir T, Badawi A and Chelvam TT, “ Graphs from Rings”, Springer, 2021.

[2] Anderson DF, and Livingston PS, “The zero-divisor graph of a commutative ring”, journal of Algebra , 217(2), 434-477,1999.

[3] Beck I, “ Coloring of commutative rings”, Journal of Algebra,116(1), 208-226,1988.

[4] Birch LM, Thibodeaux JJ and Tucci RP, “ Zero divisor graphs of finite direct products of finite rings”, Communications in Algebra 9(42), 3852-3860,2014.

[5] Visweswaran S, “ Some results on the complement of the zero-divisor graph of a commutative ring”, Journal of Algebra and its Applications 10(03), 573-595, 2011.

[6] West DB, “ Introduction to graph theory”, Prentice hall Upper saddle River, Vol.2,2001.

[7] Wc R, Chen H and Deng H, “ On the monotonicity of topological indices and the connectivity of a graph”, Applied Mathematics and computation, 298, 188-200,2017.