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Discrete Half Range Finite Fourier Series for 2^k With Single Parameter

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Abstract

In this paper, we use the difference operator ℓ to produce discrete half-range finite fourier series (DHFFS) for 2^k. The appropriate examples are provided to highlight the findings.

Key Words: Difference Equation, Generalized Difference Operator, Difference Operator,Fourier Series, Polynomial

AMS classification: 39A70, 39A10, 47B39, 80A20

1 Introduction

The Discrete Fourier Series (DFS) is a mathematical tool used to analyze and represent periodic signals in the frequency domain. It is discrete counterpart of the continuous Fourier series, which is used for continuous time signals [1, 2]. The DFS allows us to decompose a periodic signal into a sum of complex exponential functions. It is based on the fundamental idea that any periodic signal can be represented as a combination of sinusoidal components at different frequencies. Let's consider a discrete time signal $x(n)$ with period N. [7] The DFS representation of $x(n)$ is given by $x(n) = \sum_{k=0}^{N-1} x(k)e^{j2\pi nk/N}$. Where $x(n)$ is the periodic signal of length N, $x(k)$ represents the complex amplitude of the k th harmonic component of $x(n)$, j is the imaginary unit and n is the discrete time index ranging from 0 to $N-1$. Since $x(k)$ can be calculated using the formula $x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{j2\pi nk/N}$ represents a complex exponential with frequency k/N and the sum calculates the contribution of each sample $x(n)$ to the k th harmonic component. By calculating the values of $x(k)$ for different values of k, we obtain the frequency spectrum of the periodic signal $x(n)$. The magnitude of $x(k)$ gives the amplitude of the corresponding frequency

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component, and the phase angle represents the phase shift[7, 9]. The DFS widely used in various fields, including signal processing , telecommunications, image processing , an audio analysis. It allows us to analyze the frequency content of signal and extract relevant information, such as dominant frequencies or harmonic relationships[10].

2 Discrete Half range Fourier series

A Half Range Series would be one in which the sine or cosine term of the series is eliminated because of specific characteristics of the functions themselves [5, 6]. In the range $(-\wp, \wp)$, the generalized form of the FS is given as,

$$f(t) = \frac{a_0}{2} + \sum_{w=1}^{\infty}(a_w cos\frac{w\pi t}{\wp} + b_w sin\frac{w\pi t}{\wp})$$

and the Fourier coefficients evaluated as,

$$a_0 = \frac{1}{\wp}\int_{-\wp}^{\wp} f(t)dt, a_w = \frac{1}{\wp}\int_{-\wp}^{\wp} f(t)cos\frac{w\pi t}{\wp}dt$$

and

$$b_w = \frac{1}{\wp}\int_{-\wp}^{\wp} f(t)sin\frac{w\pi t}{\wp}dt.$$

The coefficient b_w will be zero if the function $f(x)$ is even since it will be an integral of the product of an odd and an even function, which will ultimately be an odd function within the bounds $(-\wp, \wp)$. Consequently, the series would be shown as

$$f(x) = \frac{a_0}{2} + \sum_{w=1}^{\infty} a_w cos\frac{w\pi x}{\wp}.$$

This series is called the HRCS, the equation for the series no longer contains any sine terms. therefore all sine terms are eliminated and only the cosine terms are present. The coefficients a_0 and a_w also turn out to be zero if the function $f(x)$ is an odd function since a_0 denotes an odd function within the limits $(-\wp, \wp)$ and a_w denotes the product of an odd and an even function, which will eventually be an odd function within the limits $(-\wp, \wp)$. Consequently, the series would be shown as,

$$\sum_{w=1}^{\infty} b_w sin \frac{w\pi x}{\wp}.$$

As a result, the series don't have cosine terms in, and as a result, it is now known as the HRSS because it only have sine terms and does away with all cosine terms.

3 Discrete Finite Fourier Series

The Discrete Finite Fourier Series(DFFS) allows to analysis and representation of finite length, non periodic signals. Unlike the DFS, which is defined for periodic signals of arbitrary length and does not require periodicity [3]. In general, any periodic function can be expressed in terms of the Fourier series, which is an infinite sum of sine and cosine functions. The coefficients of the sine and cosine functions and maybe the number of terms in the Finite Fourier Series (FFS) are the only differences between two different functions' Fourier representations.
In this chapter we introduce FFS for arbitrary functions using inverse of generalized difference operator Δ_ℓ. A periodic sequence x with period N can also be represented using the Discrete Fourier Series (DFS). The sum of N complex exponential with frequency of $k2\pi N$ can be used to represent the periodic sequence x.

Lemma 3.1 [4] For $k \in [a, b]$ and if $\ell = \frac{b-a}{M}$, then we have

$$\Delta_\ell^{-1}\psi(k)\Big|_b^a = \sum_{r=1}^{M} \psi(b - r\ell) = \sum_{r=0}^{M-1} \psi(a + r\ell).$$

In general, we express

$$\Delta_\ell^{-1}\psi(k)\Big|_k^j = \sum_{r=1}^{[\frac{k}{\ell}]} \psi(k - r\ell) = \sum_{r=0}^{[\frac{k}{\ell}]-1} \psi(j + r\ell), k \in (\ell, \infty), j = k - [\frac{k}{\ell}]\ell.$$

Definition 3.2 [5] Let $\psi(k)$ and $\psi_1(k)$ be a complex valued functions defined on [a,b] and $\ell = \frac{b-a}{M}$. With regard to ℓ, the discrete inner product of u and v is defined as

$$(\psi, \psi_1)_\ell = \ell \Delta_\ell^{-1} \psi(k) \psi_1^*(k) \big|_a^b = \ell \sum_{r=0}^{M-1} \psi(a + r\ell) \psi_1^*(a + r\ell)$$

and the norm of u related to ℓ is defined as

$$\|\psi\|_{(\ell)} = (\psi, \psi)_\ell^{\frac{1}{2}} = \left[\ell \left(\Delta_\ell^{-1} |\psi(k)|^2 \right) \big|_a^b \right]^{\frac{1}{2}} = \left[\ell \sum_{r=0}^{M-1} \left| \psi(a + r\ell)^2 \right| \right]^{\frac{1}{2}}$$

4 Finite Fourier series and inverse of 2^k

Here, we discuss some fundamental definitions and findings related to finite Fourier series and introduce the inverse of 2^k with respect to the difference operator Δ_ℓ.

Definition 4.1 [8] Consider $\psi(k)$, $k \in [\ell, \infty)$, be a real or complex valued function. Then, the ℓ-difference operator Δ_ℓ on $\psi(k)$ is defined as $\Delta_\ell \psi(k) = \psi(k + \ell) - \psi(k)$ and its infinte ℓ-difference sum is defined by $\Delta_\ell^{-1} \psi(k) = \sum_{r=0}^{\infty} \psi(k + r\ell)$.

Definition 4.2 [6] If $\Delta_\ell \psi_1(k) = \psi(k)$, then the finite inverse principle law is provided by letting $\psi(k)$ and $\psi_1(k)$ be the two real valued functions defined on $(-\infty, \infty)$.

$$\psi_1(k) - \psi_1(k - \beta\ell) = \sum_{r=1}^{\beta} \psi(k - r\ell), \beta \in Z^+.$$

Applying definition 2.2.1 yields the following updated identities:
(i) $\Delta_\ell k_\ell^{(\beta)} = (\beta\ell) k_\ell^{(\beta-1)}$, (ii) $\Delta_\ell^{-1} k_\ell^{(\beta)} = \frac{k_\ell^{(\beta+1)}}{\ell(\beta+1)}$, (iii) $\Delta_\ell^{-1} k^\beta = \sum_{r=1}^{\beta} \frac{S_r^\beta \ell^{\beta-r} k_\ell^{(r+1)}}{(r+1)\ell}$.

Lemma 4.3 Let $\ell > 0$ and $\psi(k), \psi_1(k)$ are real valued bounded functions. Then

$$\Delta_\ell^{-1}(\psi(k)\psi_1(k)) = \psi(k) \Delta_\ell^{-1} \psi_1(k) - \Delta_\ell^{-1} \left(\Delta_\ell^{-1} \psi_1(k + \ell) \Delta_\ell \psi(k) \right).$$

Definition 4.4 In order to define the finite Fourier series,

$\psi(k) = \frac{a_0}{2} + \sum_{w=1}^{\eta-1} (a_w \cos wk + b_w \sin wk) + \frac{a_\eta}{2} \cos \eta k, k \in \{a + r\ell\}_{r=0}^{2\eta-1}$

where $a_w = \frac{\ell}{\pi}\Delta_\ell^{-1}\psi(k) \cos wk\Big|_a^{a+2\pi}, b_w = \frac{\ell}{\pi}\Delta_\ell^{-1}\psi(k) \sin wk\Big|_a^{a+2\pi}$

Definition 4.5 If $\psi(k)$ is an even function, then the DHFFS of odd and even functions is defined as follows: $\psi(k) = \frac{a_0}{2} + \sum_{w=1}^{\eta-1} a_w coswk + \frac{a_\eta}{2} cos\eta k$

where $a_w = \frac{\ell}{\pi}\Delta_\ell^{-1}\psi(k)coswk\Big|_0^{\pi}$
If $\psi(k)$ is odd function, then $\psi(k) = \sum_{w=1}^{\eta-1} b_w \sin wk$

where $b_w = \frac{2\ell}{\pi}\Delta_\ell^{-1}\psi(k) \sin wk\Big|_0^{\pi}$

Theorem 4.6 Let $\ell > 0$, $k \in [\ell, \infty)$. Then the inverse of 2^k is

$$\Delta_\ell^{-1}2^k = \frac{2^k}{2^\ell - 1} = \sum_{r=0}^{\left[\frac{k}{\ell}\right]} 2^{k-r\ell}. \tag{1}$$

Proof: From 4.1, we get $\Delta_\ell 2^k = 2^k(2^\ell - 1)$
Which completes the proof of the theorem.

Example 4.7 Taking $k = 100$ and $\ell = 50$ in (1), we get

$\sum_{r=0}^{\left[\frac{k}{\ell}\right]} 2^{k-r\ell} = \sum_{r=0}^{2} 2^{100-r50} = 1.2676506 \times 10^{30} + 1.12589990 \times 10^{15} + 1 = 1.2676506 \times 10^{30}$

$\Delta_\ell^{-1}2^k = \frac{2^k}{2^\ell - 1} = \frac{2^{100}}{2^{50} - 1} = \frac{1.427247693 \times 10^{45}}{1.125899907 \times 10^{15} - 1} = 1.2676506 \times 10^{30}.$

Example 4.8 Taking $k = 20$ and $\ell = 4$ in (1), we get

$\sum_{r=0}^{\left[\frac{k}{\ell}\right]} 2^{k-r\ell} = \sum_{r=0}^{\left[\frac{20}{4}\right]} 2^{20-r4} = \sum_{r=0}^{5} 2^{20-r4} = 2^{20} + 2^{20-4} + 2^{20-8} + 2^{20-12} + 2^{20-16} + 2^{20-20} =$
$2^{20} + 2^{16} + 2^{12} + 2^8 + 2^4 + 2^0 = 1048576 + 65536 + 4096 + 256 + 16 + 1 = 1118481$

$\Delta_\ell^{-1}2^k = \frac{2^k}{2^\ell - 1} = \frac{2^{20}}{2^4 - 1} = 1118481.$

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5 Discrete Finite Half Range Fourier Series for 2^k

The Discrete Finite Half Range Fourier Series (DHFFS) for 2^k is introduced in this section.

Theorem 5.1 Let $\ell = \frac{\pi}{\eta} > 0$, $k \in [\ell, \infty)$ then the DHFFS for 2^k is

$$2^k = \sum_{w=0}^{\eta-1} b_w \sin wk \tag{2}$$

where $b_w = \Delta_\ell^{-1} 2^k \sin wk$.

Corollary 5.2 Taking $\eta = 2$ and $\ell = \frac{\pi}{2}$ in (2), we arrive $2^k = 2.970686424 \sin k$.

Proof: From (2), we get $2^k = \sum_{w=1}^{1} b_w \sin wk = b_1 \sin k$

Where $b_1 = \frac{2\ell}{\pi} \Delta_\ell^{-1} 2^k \sin k \Big|_0^\pi = \frac{2(\frac{\pi}{2})}{\pi} \Delta_\ell^{-1} 2^k \sin k \Big|_0^\pi = \Delta_\ell^{-1} 2^k \sin k \Big|_0^\pi$

$= \sum_{r=0}^{[\frac{k}{\ell}]} 2^{(k-r\ell)} \sin(k - r\ell)\Big|_0^\pi = \sum_{r=0}^{[\frac{k}{\frac{\pi}{2}}]} 2^{(k-r\frac{\pi}{2})} \sin(k - r\frac{\pi}{2})\Big|_0^\pi = \sum_{r=0}^{2} 2^{(\pi-r\frac{\pi}{2})} \sin(\pi - r\frac{\pi}{2}) - 0$

$= 2^{(\pi-0)} \sin(\pi - 0) + 2^{(\pi-\frac{\pi}{2})} \sin(\pi - \frac{\pi}{2}) + 2^{(\pi-2\frac{\pi}{2})} \sin(\pi - 2\frac{\pi}{2})$

$= 2^\pi \sin \pi + 2^{\frac{\pi}{2}} \sin \frac{\pi}{2} + 2^0 \sin 0 = 2.970686424.$

Which completes the proof of 5.2.

Example 5.3 Particularly taking $k = \dfrac{\pi}{2}$ in 5.2, we get

$2^{\frac{\pi}{2}} = 2.970686424 \sin \frac{\pi}{2} = 2.970686424.$

Corollary 5.4 Taking $\eta = 10$ and $\ell = \dfrac{\pi}{10}$ in (2), we obtain $2^k = 4.173500633 \sin k$

$-2.141387815 \sin 2k + 1.822992214 \sin 3k - 1.041149877 \sin 4k + 0.959654883 \sin 5k$

$-0.558364363 \sin 6k + 0.493213624 \sin 7k - 0.250946954 \sin 8k + 0.153736744 \sin 9k.$

Proof: From (2), we get $2^k = b_1 \sin k + b_2 \sin 2k + b_3 \sin 3k + b_4 \sin 4k + b_5 \sin 5k$

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$$+b_6 \sin 6k + b_7 \sin 7k + b_8 \sin 8k + b_9 \sin 9k.$$

Where, $b_1 = \frac{2\ell}{\pi}\Delta_\ell^{-1} 2^k \sin k\Big|_0^\pi = \frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1} 2^k \sin k\Big|_0^\pi = \frac{2}{10}\sum_{r=0}^{[\frac{k}{\ell}]} 2^{(k-r\ell)} \sin(k-r\ell)\Big|_0^\pi$

$$= \frac{2}{10}\sum_{r=0}^{[\frac{k}{\frac{\pi}{10}}]} 2^{(k-r\frac{\pi}{10})} \sin(k - r\tfrac{\pi}{10})\Big|_0^\pi = \frac{2}{10}\sum_{r=0}^{10} 2^{(\pi - r\frac{\pi}{10})} \sin(\pi - r\tfrac{\pi}{10}) - 0$$

$$= \frac{2}{10}\Big[2^\pi \sin\pi + 2^{\frac{9\pi}{10}} \sin\tfrac{9\pi}{10} + 2^{\frac{4\pi}{5}} \sin\tfrac{4\pi}{5} + 2^{\frac{7\pi}{10}} \sin\tfrac{7\pi}{10} + 2^{\frac{3\pi}{5}} \sin\tfrac{3\pi}{5} + 2^{\frac{\pi}{2}} \sin(\tfrac{\pi}{2})$$

$$+2^{\frac{2\pi}{5}} \sin\tfrac{2\pi}{5} + 2^{\frac{3\pi}{10}} \sin\tfrac{3\pi}{10} + 2^{\frac{\pi}{5}} \sin\tfrac{\pi}{5} + 2^{\frac{\pi}{10}} \sin\tfrac{\pi}{10} + 2^0 \sin 0\Big]$$

$$= \frac{1}{5}[0 + 2.193434271 + 3.355749882 + 3.714985993 + 3.51264693$$

$$+2.970686424 + 2.272436596 + 1.554789974 + 0.908576312 + 0.384196785]$$

$$= \frac{1}{5}(20.86750317) = 4.173500633$$

$$b_2 = \frac{2\ell}{\pi}\Delta_\ell^{-1} 2^k \sin 2k\Big|_0^\pi = \frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1} 2^k \sin 2k\Big|_0^\pi = \frac{1}{5}\sum_{r=0}^{[\frac{k}{\ell}]} 2^{(k-r\ell)} \sin 2(k-r\ell)\Big|_0^\pi$$

$$= \frac{1}{5}\sum_{r=0}^{[\frac{k}{\frac{\pi}{10}}]} 2^{(k-r\frac{\pi}{10})} \sin 2(k - r\tfrac{\pi}{10})\Big|_0^\pi = \frac{1}{5}\sum_{r=0}^{10} 2^{(\pi - r\frac{\pi}{10})} \sin 2(\pi - r\tfrac{\pi}{10}) - 0$$

$$= \frac{1}{5}\Big[2^\pi \sin 2\pi + 2^{\frac{9\pi}{10}} \sin(\tfrac{9\pi}{5}) + 2^{\frac{4\pi}{5}} \sin(\tfrac{8\pi}{5}) + 2^{\frac{7\pi}{10}} \sin(\tfrac{7\pi}{5}) + 2^{\frac{3\pi}{5}} \sin(\tfrac{6\pi}{5}) + 2^{\frac{\pi}{2}} \sin(\pi)$$

$$+2^{\frac{2\pi}{5}} \sin(\tfrac{4\pi}{5}) + 2^{\frac{3\pi}{10}} \sin(\tfrac{3\pi}{5}) + 2^{\frac{\pi}{5}} \sin(2\tfrac{\pi}{5}) + 2^{\frac{\pi}{10}} \sin(\tfrac{\pi}{5}) + 2^0 \sin 2(0)\Big]$$

$$= \frac{1}{5}[0 - 4.172159912 - 5.429717367 - 4.367227958 - 2.170935193 + 0$$

$$+1.404443054 + 1.827765234 + 1.470107354 + 0.730785711 + 0]$$

$$b_2 = -2.141387815$$

$$b_3 = \frac{2\ell}{\pi}\Delta_\ell^{-1} 2^k \sin 3k\Big|_0^\pi = \frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1} 2^k \sin 3k\Big|_0^\pi = \frac{1}{5}\sum_{r=0}^{[\frac{k}{\ell}]} 2^{(k-r\ell)} \sin 3(k-r\ell)\Big|_0^\pi$$

$$= \frac{1}{5}\sum_{r=0}^{[\frac{k}{\frac{\pi}{10}}]} 2^{(k-r\frac{\pi}{10})} \sin 3(k - r\tfrac{\pi}{10})\Big|_0^\pi = \frac{1}{5}\sum_{r=0}^{10} 2^{(\pi - r\frac{\pi}{10})} \sin 3(\pi - r\tfrac{\pi}{10}) - 0$$

$$= \frac{1}{5}\Big[2^\pi \sin 3\pi + 2^{\frac{9\pi}{10}} \sin(\tfrac{27\pi}{10}) + 2^{\frac{4\pi}{5}} \sin(\tfrac{12\pi}{5}) + 2^{\frac{7\pi}{10}} \sin(\tfrac{21\pi}{10}) + 2^{\frac{3\pi}{5}} \sin(\tfrac{9\pi}{5}) +$$

$2^{\frac{\pi}{2}}\sin(\frac{3\pi}{2})+2^{\frac{2\pi}{5}}\sin(\frac{6\pi}{5})+2^{\frac{3\pi}{10}}\sin(\frac{9\pi}{10})+2^{\frac{\pi}{5}}\sin(3\frac{\pi}{5})+2^{\frac{\pi}{10}}\sin(\frac{3\pi}{10})+2^0\sin 0\Big]$

$=\frac{1}{5}\,[0+5.742485473+5.429717367+1.418998382-2.170935193$

$-2.970686424-1.404443054+0.593876924+1.470107354+1.005840242]$

$=\frac{1}{5}[9.11496107]=1.822992214$

Applying the above procedure we get

$b_4=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 4k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 4k\Big|_0^\pi=-1.041149877$

$b_5=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 5k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 5k\Big|_0^\pi=0.959654883$

$b_6=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 6k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 6k\Big|_0^\pi=-0.558364363$

$b_7=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 7k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 7k\Big|_0^\pi=0.493213624$

$b_8=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 8k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 8k\Big|_0^\pi=-0.250946954$

$b_9=\frac{2\ell}{\pi}\Delta_\ell^{-1}2^k\sin 9k\Big|_0^\pi=\frac{2(\frac{\pi}{10})}{\pi}\Delta_\ell^{-1}2^k\sin 9k\Big|_0^\pi=0.153736744$
From this we can get the proof of 5.4

Example 5.5 Particularly taking $k=\frac{\pi}{10}$ in 5.4, we have

$2^{\frac{\pi}{10}}=4.173500633\sin\frac{\pi}{10}-2.141387815\sin\frac{2\pi}{10}+1.822992214\sin\frac{3\pi}{10}$

-1.041149877
$sin\frac{4\pi}{10}+0.959654883\sin\frac{5\pi}{10}-0.558364363\sin\frac{6\pi}{10}$

$+0.493213624\sin\frac{7\pi}{10}-0.250946954\sin\frac{8\pi}{10}+0.153736744\sin\frac{9\pi}{10}$

$1.243286913=1.243286913$

6 Conclusion

The DHFFS allows us to analyze the frequency content of a finite-length signal by calculating the complex amplitudes 2^k for different values of k. The magnitude of 2^k gives the amplitude of the corresponding frequency component, and the phase angle represents the phase shift. compared to the DFS, the DHFFS is more flexible as it can handle non-periodic signals. It is commonly used in areas such as signal processing,

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data analysis, and digital communications to analyze and manipulate finite-length signals in the frequency.

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