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Some Properties of Block Adjacency Energy

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Abstract

Block adjacency energy $E_{BA}(G)$ is the sum of eigenvalues of block adjacency matrix $BA(G)$. The present research work intends to study some properties of block adjacency energy such as block adjacency equienergetic, hyperenergetic, nonhyperenergetic, borderenergetic, hypoenergetic and blockenergetic graphs using block adjacency matrix. It is also verified the existence of smith graph.

Key words: Block adjacency matrix, block adjacency energy, block adjacency equienergetic, block adjacency hyperenergetic, block adjacency hypoenergetic.

AMS classification: 05C76, 05C50.

1 Introduction

Block is maximal connected graph with no cutvertices. The helm graph H_t [7], where $t \geq 3$ indicates the number of pendent edges, is the graph obtained from a n-wheel graph W_n by joining a pendent edge at each vertex of the cycle. The number of blocks in H_t are $(t+1)$. Smith graph [2] is a graph whose adjacency spectrum lies in [-2,2]. If two blocks are incident with a common cutpoint, then they are called as mutually adjacent.

The line graph $L(G)$ is a graph whose vertex set corresponds to the edges of G such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent [3].

The block graph $B(G)$ is a graph whose vertex set corresponds to the blocks of G such that two vertices of $B(G)$ are adjacent whenever the corresponding blocks contain a common cutvertex of G [3]. Undefined graph terminologies are referred from [3].

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The study of the graph energy was stimulated from quantum chemistry where the outline of unsaturated hydrocarbon is represented by a graph. In a hydrocarbon system, vertices and edges of a molecular graph represent the carbon atoms and chemical bond between them respectively. Chemical application of graph theory in molecular orbital theory was put forth by E. H$\ddot{u}$ckel in 1930. The energy levels of electrons in such a molecule are eigenvalues of graph.

Since the graph energy appeared to be a fascinating concept in mathematics, during 1978 Gutman initiated the research on graph energy concept by considering above relationship between electron energy and eigenvalues of the corresponding graph. This resulted in establishment of voluminous graph energies from eigenvalues of different matrices of graph in the mathematical and mathematico-chemical literature.

The adjacency matrix $A(G) = (a_{ij})$ of order n whose (i, j)-entry is defined as [4]

$$A(G) = (a_{ij}) = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & otherwise. \end{cases}$$

If $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ of $A(G)$ are the eigenvalues of the graph G, then the energy $E_A(G)$ [4] is defined to be the sum of the absolute values of eigenvalues.

$$E_A(G) = \Sigma_i^n |\lambda_i|.$$

The spectrum of a graph G is the collection of eigenvalues of $A(G)$ along with their multiplicities [1].
Further Gutman [4] speculated that the complete graph K_n has the maximum energy among all the graphs. Also he proved that the hyperenergetic graphs with n vertices exist only for $n \geq 8$ [5]. Thus graph is referred to be as hyperenergetic if the energy of graph exceeds the energy of graph K_n, i.e.,$E_A(G) > 2(n-1)$ and is said to be non-hyperenergetic if $E_A(G) < 2(n-1)$. A noncomplete graph whose energy is equal to $2(n-1)$ is called borderenergetic [9]. A graph G is said to be hypoenergetic [6] if $E_A(G)$ is less than its order otherwise it is said to be non-hypoenergetic. Two non-isomorphic graphs G_1 and G_2 with the same spectra are said to be cospectral graphs otherwise non-cospectral graphs. The graphs G_1 and G_2 are referred to as equienergetic, if $E(G_1) = E(G_2)$ [10].

Let G be a simple, connected graph with n-vertices $v_1, v_2, ..., v_n$. $B = b_1, b_2, b_3, ...$ be the total number of blocks in G and $B \geq 2$. Then the block adjacency matrix [8] $BA(G) = [b_{ij}]$ is defined as

$$BA(G) = [b_{ij}] = \begin{cases} 1, & \text{if } b_i \text{ and } b_j \text{ are adjacent;} \\ 0, & otherwise. \end{cases}$$

The block adjacency matrix $BA(G)$ is a real symmetric matrix. If $\gamma_1, \gamma_2, \gamma_3 ..., \gamma_B$ are eigenvalues of $BA(G)$, then the block adjacency energy of a graph $E_{BA}(G)$ is defined as

$$E_{BA}(G) = \sum_{i=1}^{B} |\gamma_i|$$

With the objective of obtaining properties analogous to the energy properties of adjacency energy, this study is initiated and we define the following.
The graphs G_1 and G_2 with B blocks are referred to as block adjacency equienergetic, if $E_{BA}(G_1) = E_{BA}(G_2)$.
A graph with B blocks is said to be block adjacency hyperenergetic if $E_{BA}(G) > 2(B-1)$.
A graph with B blocks is said to be block adjacency non-hyperenergetic if $E_{BA}(G) < 2(B-1)$.
A graph with B blocks is said to be block adjacency borderenergetic if $E_{BA}(G) = 2(B-1)$.
A graph with B blocks is said to be block adjacency hypoenergetic if $E_{BA}(G) < B$.
A graph with B blocks is said to be blockenergetic if $E_{BA}(G) = B$.
These parameters are studied for certain class of graphs.

2 Preliminaries

The following are some of the important theorems used for obtaining further results.

Theorem 2.1 [8] If G be a graph with B mutually adjacent blocks, then the block adjacency spectrum and energy of G are

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$$
\begin{aligned}
Spec(BA)(G) &= \begin{pmatrix} -1 & (B-1) \\ (B-1) & 1 \end{pmatrix} \\
E_{BA}(G) &= 2(B-1)
\end{aligned}
$$

Theorem 2.2 [8] If H_t, $t \geq 3$ be a Helm graph with B blocks, then the block adjacency spectrum and energy of H_t are

$$
\begin{aligned}
Spec(BA)(H_t) &= \begin{pmatrix} \sqrt{B-1} & 0 & -\sqrt{B-1} \\ 1 & (B-2) & 1 \end{pmatrix} \\
E_{BA}(H_t) &= 2\sqrt{B-1}
\end{aligned}
$$

The following observations are of immediate use.

Observation 2.3 The block adjacency energy of graph G with B mutually adjacent blocks ($E_{BA}(G) = 2(B-1)$) is same as the adjacency energy of complete graph K_n and it has the maximum block adjacency energy .

Observation 2.4 If G is a graph with $B > 3$ blocks, then its $E_{BA}(G) > 4$.

Observation 2.5 If G is a graph with $B < 2$ blocks, the its $E_{BA}(G) = 0$.

Observation 2.6 If G is a graph with $B \geq 2$ blocks, then $E_{BA}(G) \geq 2$.

Observation 2.7 If G is a graph with B mutually adjacent blocks, then block adjacency energy of $B(G)$ is zero.

Observation 2.8 If G is a graph with $B \leq 2$ blocks, then block adjacency energy of $B(G)$ is zero.

3 Main results

3.1 Block adjacency equienergetic graphs

Theorem 3.1 The helm graph H_4 and graph with three mutually adjacent blocks are non-cospectral block adjacency equienergetic.
Proof: Let G_1 and G_2 be helm graph H_4 and graph with three mutually adjacent blocks respectively. Clearly G_1 and G_2 are non-isomorphic.
Consider the graph G_1, a helm graph H_4 $(B = 5)$.
From Theorem 2.2, the spectra of H_4 is

$$Spec(BA)(G_1) = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}. \quad (1)$$

and its block adjacency energy is

$$E_{BA}(G_1) = 2\sqrt{B-1} = 2\sqrt{5-1} = 4. \quad (2)$$

Next consider the graph G_2, a graph with three mutually adjacent blocks.
From Theorem 2.1, the spectra of G_2 is

$$Spec(BA)(G_2) = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}. \quad (3)$$

and thus, its block adjacency energy is

$$E_{BA}(G_2) = 2(B-1) = 2(3-1) = 4 \quad (4)$$

From Equations (1), (2), (3) and (4), G_1 and G_2 are non-isomorphic, non-cospectral block adjacency equienergetic.

Theorem 3.2 Block graph of helm graph H_t, $t \geq 3$ and graph with $B \geq 3$ mutually adjacent blocks are block adjacency equienergetic.
Proof: Let H_t, $t \geq 3$ be a helm graph. The block graph $B(H_t)$ is a graph in which all the blocks are mutually adjacent. From Theorem 2.1, block adjacency energy of graph with B mutually adjacent blocks is $2(B-1)$. Hence they are equienergetic.

Theorem 3.3 If G is a path graph P_n, $n \geq 2$, having any number of blocks at the terminal vertices, then $B(G)$ and $L(G)$ are block adjacency equienergetic.
Proof: Let G be a path graph P_n, $n \geq 2$, having any number of blocks at the terminal vertices.
We consider the following cases.
Case (1). Let G be a graph with number of blocks incident to the terminal vertices of P_2 (i.e., $n = 2$), then on constructing $B(G)$ or $L(G)$, it contains only one cutvertex with two blocks. Then the spectra of $B(G)$ or $L(G)$ is

$$Spec(BA)(B(G)) \quad = \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5}$$

thus, its block adjacency energy is

$$E_{BA}(B(G)) = 2(B-1) = 2(2-1) = 2 \tag{6}$$

$$i.e., E_{BA}(B(G)) = E_{BA}(L(G)) = 2.$$

Case (2). Let G be a graph with number of blocks incident to the terminal vertices of P_3 (i.e., $n = 3$), then on obtaining $B(G)$ or $L(G)$ of G it consists of two cutvertices with three blocks. Then the spectra of G is

$$Spec(BA)(G) \quad = \quad \begin{pmatrix} -1.4142 & 0 & 1.4142 \\ 1 & 0 & 1 \end{pmatrix}. \tag{7}$$

thus, its block adjacency energy is

$$E_{BA}(B(G)) = 2.8284 \tag{8}$$

$$i.e., E_{BA}(B(G)) = E_{BA}(L(G)) = 2.8284.$$

In general, let G be a graph with number of blocks incident to the terminal vertices of P_n, $n = 4, 5, 6, ...$, then block adjacency energy of $B(G)$ and $L(G)$ are equal. Thus,

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they equienergetic.

3.2 Block adjacency hyperenergetic, Non-hyperenergetic and Borderenergetic graphs

Theorem 3.4 For any graph G with B blocks, block adjacency hyperenergetic graph does not exists.
Proof: Let G be a graph with B blocks. From the Observation 2.3, it is known that among all the graphs with B blocks, graph with mutually adjacent blocks have the highest block adjacency energy.

$$i.e., E_{BA}(G) = 2(B-1).$$

Performing any graph transformations such as line graph, block graph, compliment graph etc. on G results into decrease in the number of blocks. As the number of blocks decrease correspondingly there is decrease in the block adjacency energy which is clearly $< 2(B-1)$. i.e., there does not exist any graph whose block adjacency energy is $> 2(B-1)$. By definition G is block adjacency hyperenergetic.

Theorem 3.5 The graph G with mutually adjacent $B \geq 2$ blocks is block adjacency borderenergetic.
Proof: Suppose G is graph with mutually adjacent $B \geq 2$ blocks.
The graph G is a block adjacency borderenergetic if $E_{BA}(G) = 2(B-1)$.
From Theorem 2.1, the block adjacency energy of G with mutually adjacent blocks is $2(B-1)$.
Hence the result.

Theorem 3.6 If G is a graph with $B \geq 2$ mutually adjacent blocks, then $\overline{G}$ is block adjacency non-hyperenergetic.
Proof: By definition, graph G is said to be block adjacency non-hyperenergetic if $E_{BA}(G) < 2(B-1)$.
Suppose G is a graph with $B \geq 2$ mutually adjacent blocks, from the Observation 2.6, $E_{BA}(G) \geq 2$.
Construction of $\overline{G}$ results into either disconnected graph or $\overline{G}$ itself is a block (B=1).
From the Observation 2.5, its block adjacency energy is zero. Clearly which is $<$ $2(B-1)$.

Theorem 3.7 If G is a graph with $B \leq 2$ blocks, then $B(G)$ or $L(G)$ is block

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adjacency non-hyperenergetic.
Proof: The graph G is said to be block adjacency non-hyperenergetic if $E_{BA}(G) < 2(B-1)$.
Let G be a graph with $B \leq 2$ blocks.
Case (1). Suppose G be a graph with $B \geq 3$, then its $B(G)$ or $L(G)$ results into the graph with two or more blocks and accordingly from the Observation 2.6, its block energy is ≥ 2.
Case (2). Suppose G is with $B \leq 2$, then its $B(G)$ or $L(G)$ is either trivial or contains only one block. From the Observation 2.8, its block adjacency energy is zero.
From the above two cases it is obvious that $B(G)$ or $L(G)$ is non-hyperenergetic only if G is with $B \leq 2$ blocks.

3.3 Block adjacency Hypoenergetic, Non-hypoenergetic and Blockenergetic

Theorem 3.8 If G is a graph with $B \geq 2$ mutually adjacent blocks, then $\overline{G}$ is block adjacency hypoenergetic.
Proof: Let G be a graph with B mutually adjacent blocks.
If $\overline{G}$ is obtained, then it results into either disconnected graph or $\overline{G}$ is a block and obviously block adjacency energy is zero.
i.e.,$E_{BA}(\overline{G}) < B$ and hence by definition of block adjacency hypoenergetic $\overline{G}$ is block adjacency hypoenergetic.

Theorem 3.9 If G is a graph with $B \geq 2$ mutually adjacent blocks, then $B(G)$ is block adjacency hypoenergetic.
Proof: The graph G is block adjacency hypoenergetic if $E_{BA}(G) < B$.
Let G be a graph with $B \geq 2$ mutually adjacent blocks and $B(G)$ be its block graph which is complete $(B = 1)$. From the Observation 2.7, its block adjacency energy is zero. i.e., $E_{BA}(B(G)) = 0 < B$. Hence by definition, $B(G)$ is block adjacency hypoenergetic.

Theorem 3.10 The graph with $B = 3$ which is not mutually adjacent blocks is block adjacency hypoenergetic.
Proof: Let G be a graph with $B = 3$ which is not mutually adjacent blocks.
Depending upon the number of blocks the following cases are considered.

Case (1). Suppose G is a graph with $B = 2$ blocks, then from Equation (6),

$$E_{BA}(G) = 2 = B.$$

Case (2). Suppose G is a graph with $B > 3$ blocks, from the Observation 2.4, $E_{BA}(G) > 4$.

$$i.e., E_{BA}(G) > B.$$

Case (3). Suppose G is with $B = 3$ blocks.
We consider the following sub cases.
Subcase (3.1). Suppose G is with $B = 3$ blocks which are mutually adjacent, then from Equation (4), $E_{BA}(G) = 4$.

$$i.e., E_{BA}(G) > B.$$

Subcase (3.2). Suppose G is a graph with $B = 3$ blocks which are not mutually adjacent, then from Equation (8),

$$E_{BA}(G) = 2.8284$$

On observing all the above cases, it is noted that $E_{BA}(G) > B$ only for the graph G with $B = 3$ blocks which are not mutually adjacent.

Theorem 3.11 The graph G is a blockenergetic if only if G has exactly two blocks.
Proof: Suppose the graph G is blockenergetic then by definition, $E_{BA}(G) = B$.
We consider the following cases.
Case (1) Suppose G is a graph with $B > 3$ blocks, then from the Observation 2.4, $E_{BA}(G) > 4$.
Case (2). If G be a graph with blocks $B = 3$ blocks, then from Theorem 2.1 and Equation (8), $E_{BA}(G) = 4$ and $E_{BA}(G) = 2.8284$ for the graph with mutually adjacent and not adjacent blocks respectively.
Case (3). Suppose G be a graph with blocks $B = 2$ blocks, then from Equation (6), $E_{BA}(G) = 2$

$$i.e., E_{BA}(G) = 2 = B.$$

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From the above cases it is clear that $E_{BA}(G) = B$ only for the graph G with $B = 2$ blocks.
Converse is obvious.

3.4 Existence of Smith graph

Theorem 3.12 For any graph G with B blocks, the Smith graph exists for block adjacency matrix $BA(G)$ if and only if
(i) G is exactly with 2 and 3 blocks which are mutually adjacent.
(ii) G is H_4.
Proof: Let G be a graph with B blocks.
(i). Suppose graph G is with B mutually adjacent blocks, then from Theorem 2.1, the eigenvalues G are -1 and $(B-1)$ occur with multiplicities $(B-1)$ and 1 respectively.
We consider the following cases.
Case(1). Suppose G is with $B \geq 4$ blocks, then from Theorem 2.1 the eigenvalues do not lie in [-2,2].
Case(2). Suppose G is with $B = 2$ blocks, then from Theorem 2.1 the eigenvalues are -1 and 1.
Case(3). Suppose G is with $B = 3$ blocks, then from Theorem 2.1 the eigenvalues are -1 and 2.
From above cases and definition of Smith graph, Smith graph exists only for G with $B = 2$ and 3 mutually blocks.
(ii). Suppose G is a helm graph H_t, $t \geq 3$, then from Theorem 2.2, the eigenvalues $-\sqrt{B-1}$, 0, $\sqrt{B-1}$ of $BA(H_t)$ occur with multiplicities 1, $(B-2)$ and 1 time respectively.
case(1). If G is H_3, then it contain $B = 4$ blocks. Clearly from Theorem 2.2, eigenvalues of G do not lie in the interval [-2,2].
Case(2). Suppose G is H_t, $t \geq 4$, then it contains $B \geq 6$ blocks. From Theorem 2.2, eigenvalues of G do not lie in the interval [-2,2].
Case(3). If G is H_4, then it contains 5 blocks. The eigenvalues are -2, 0 and 2.
Hence, by the above cases it follows that G is Smith graph only if $G = H_4$.
Conversely, suppose G is a graph with B blocks for which Smith graph exists.
From Theorems 2.1 and 2.2, it is clear that one of the eigenvalues of G lies in [-2,2] only for
i) G with $B = 2$ and 3 mutually adjacent blocks
ii) G is H_4.

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4 Conclusion

With the objective of studying some energy properties such as equienergeticity, hperenergeticity, non-hyperenergeticity, borderenergeticity and hypoenegeticity of block adjacency energy, the present research work is initiated. By performing graph tranformations on G with B blocks, the graphs are obtained for block adjacency equienergetic, non-hyperenergetic, borderenergetic, hypoenergetic and blockenergetic except for hyperenergetic. Further we have examined the existence of Smith graph for G with B blocks with respect to block adjacency matrix.

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