2456-8686, vii(i), 2023 : 035-045
https://doi.org/10.26524/cm161

ORIGINAL RESEARCH **Open Access**

Discrete Laplace Transform Arrising From q-Difference Operator

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Received: 24 March 2023 / Accepted: 10 May 2023/ Published online: 19 June 2023

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Abstract

In this paper, we obtain discrete Laplace transform and it's properties for certain functions using the inverse of q-difference operator, suitable examples are inserted to illustrate the main results.

Key words: Laplace Transform, Linear, Difference Operator

AMS classification: 44A10, 26A33

1 Introduction

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering. This subject originated from the operational methods applied by the English engineer Oliver-Heaviside (1850-1925), unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916-17. It was found that Heavisides operational calculus is best introduced by means of a particular type of definite integrals called Laplace Transforms.

The method of Laplace Transform has the advantage of directly giving the solution of differential equations with given the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants.

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2 Preliminaries

Definition 2.1 Let $f : I_q \to R$ and $1 \neq q \in [0,\infty)$ be fixed. The q-Difference operator denoted as Δ_q on $f(t)$ is defined as

$$\Delta_q f(t) = f(tq) - f(t), \ \ t \in I_q \tag{1}$$

Definition 2.2 If $g : I_q \to R$ such that $\Delta_q g(t) = f(t)$ then it's inverse q-difference operator denoted as Δ_q^{-1} is defined as

$$g(t) = \Delta_q^{-1} f(t) + c \tag{2}$$

where C is a constant.

Lemma 2.3 For any positive integer $n > 1$ and $q \neq 1$ then the difference operator Δ_q for the polynomial factorial function $t_q^{(n)}$ is given by

$$\Delta_q t_q^{(n)} = (q^n - 1)t_q^{(n)}, t \in (R) \tag{3}$$

Lemma 2.4 Let $n \in N(1)$ and $q \neq 1$. Then the inverse difference operator Δ_q^{-1} for the polynomial factorial function $t_q^{(n)}$ is

$$\Delta_q^{-1} t_q^{(n)} = \frac{t_q^{(n)}}{q^n - 1} + c, \ \ t \in R \tag{4}$$

Remark 2.5 Throught this paper, we 0represent the inverse difference operator as Δ_q^{-1}. Where Δ_q^{-1} is nothing but the reciprocal of the q-Difference operator.

Remark 2.6 Let $n \in N$ and $t \in R$. The q-difference operator for the polynomial function is given by

$$\Delta_q(t^n) = t^n(q^n - 1) \tag{5}$$

and it's inverse difference operator for the polynomial function is given by

$$\Delta_q^{-1}(t^n) = \frac{t^n}{q^n - 1} \tag{6}$$

Lemma 2.7 Let $f, g \in I_q \to R$ and $q \neq 1$. The product rule of the q-difference operator is defined by

$$\Delta_q^{-1}\{f(t).g(t)\} = f(t)\Delta_q^{-1}g(t) - \Delta_q^{-1}(\Delta_q^{-1}g(tq).\Delta_q f(t)) \tag{7}$$

Corollary 2.8 Let $t \in R$ and $q \in \{0,1\}$ be any real number. Then $\Delta_q^{-1}(0) = 1$ and

$$\Delta_q^{-1}(1) = \frac{\log(t)}{\log(q)} \tag{8}$$

3 Discrete Laplace Transform and It's Properties

In this section we define the discrete Laplace Transform, properties and obtain discrete Laplace transforms of certain functions.

Definition 3.1 The discrete Laplace Transform of $u(k)$ is defined as

$$L_q(u(k)) = \overline{u}_q(s) = (q-1)\Delta_q^{-1}k.u(k)e^{-sk}\Big|_0^\infty \tag{9}$$

Proposition 3.2 ***Linear Property:***

If $L_q(u(k)) = \overline{u}_q(s)$ and $L_q(v(k)) = \overline{u}_q(s)$ then

$$L_q(au(k) + bv(k)) = a\overline{u}_q(s) + b\overline{u}_q(s) \tag{10}$$

where a & b are constants.

Change of Scale Property:

$L_q(au(k) + bv(k)) = \overline{u}_q(s)$ then

$$L_q(u(ak)) = \frac{1}{a}\overline{u}_q\left(\frac{s}{a}\right) \tag{11}$$

Proof: Linear Property:

$$\begin{aligned} L_q(au(k) + bv(k)) &= (q-1)\Delta_q^{-1}k(au(k) + bv(k))e^{-sk}\Big|_0^\infty \\ &= a(q-1)\Delta_q^{-1}ku(k)e^{-sk}\Big|_0^\infty + b(q-1)\Delta_q^{-1}kv(k)e^{-sk}\Big|_0^\infty \\ &= a\overline{u}_q(s) + b\overline{v}_q(s) \end{aligned}$$

Hence proved.

Change of Scale Property:

$L_q(u(ak)) = (q-1)\Delta_q^{-1} k u(ak) e^{-sk}\Big|_0^\infty$

Here $ak = p$

$$k = \frac{p}{a}$$

$$L_q(u(ak)) = (q-1)\Delta_q^{-1} \frac{p}{a} u(p) e^{\frac{-s}{a}.p}\Big|_0^\infty$$

$$= \frac{1}{a}(q-1)\Delta_q^{-1} p.u(p) e^{\frac{-s}{a}.p}\Big|_0^\infty$$

$$= \frac{1}{a}.\overline{u}_q\left(\frac{s}{a}\right)$$

Hence proved.

Example 3.3 Taking $u(k) = \dfrac{1}{k}$ in equation (9), we get

$$L_q\left(\frac{1}{k}\right) = (q-1)\Delta_q^{-1} k.\frac{1}{k} e^{-sk}\Big|_0^\infty$$

$$= (q-1)\Delta_q^{-1} e^{-sk}\Big|_0^\infty$$

$$= (q-1)\sum_{r=1}^{\infty} e^{\frac{-sk}{q^r}}$$

Here $k = 1$ and $q = 2$. Then

$$L_q(1) = (2-1)\sum_{r=1}^{\infty} e^{\frac{-s1}{2^r}}$$

$$= \sum_{r=1}^{\infty} e^{\frac{-s}{2^r}}$$

Proposition 3.4 *Shifting Property:*

If $L_q(u(k)) = \overline{u}_q(s)$ then

$$L_q(e^{-ak}u(k)) = \overline{u}_q(s+a) \tag{12}$$

Proof: $L_q(e^{-ak}u(k)) = (q-1)\Delta_q^{-1} k.e^{-ak}u(k).e{-sk}\Big|_0^\infty$

$$= (q-1)\Delta_q^{-1} k u(k) e^{-(s+a)k}\Big|_0^\infty$$

$$= \overline{u}_q(s+a)$$

Hence proved.

Example 3.5 Taking $u(k) = \log k$ in equation (9)

$$L_q(u(k)) = (q-1)\Delta_q^{-1}k.u(k)e^{-sk}\Big|_0^{\infty}$$

$$\begin{aligned} L_q(\log k) &= (q-1)\Delta_q^{-1}k.\log ke^{-sk}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.e^{ak}.e^{-sk}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.e^{-(s-a)k}\Big|_0^{\infty} \\ &= \overline{u}_q(s-a) \end{aligned}$$

which gives

$$\begin{aligned} L_q(e^{-ak}\log k) &= (q-1)\Delta_q^{-1}k.e^{ak}\log ke^{-sk}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.\log k.e^{-(s+a)k}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.e^{ak}.e^{-(s+a)k}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.e^{sk}\Big|_0^{\infty} \\ &= \overline{u}_q(s) \end{aligned}$$

Theorem 3.6 Let $k \in (0,\infty)$, $q > 0$, $s > 0$ and $L_q(u(k)) = \overline{u}q(s)$ Then

$$L_q(\Delta_q u(k)) = (q-1)\Delta_q^{-1}k.\Delta_q u(k)e^{-sk}\Big|_0^{\infty}$$

Proof:

$$\begin{aligned} &= (q-1)\Delta_q^{-1}k.u(kq) - u(k)e^{-sk}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.u(kq)e^{-sk} - \Delta_q^{-1}k.u(k).e^{-sk}\Big|_0^{\infty} \\ &= (q-1)\Delta_q^{-1}k.u(kq)e^{-sk} - (q-1)\Delta_q^{-1}.u(k)e^{-sk}\Big|_0^{\infty} \\ &= (q-1)\sum_{r=0}^{\infty} g\left(\frac{k}{q^r}\right) - (q-1)\sum_{r=0}^{\infty} h\left(\frac{k}{q^2}\right) \\ &= (q-1)\frac{k}{q^r}.u\left(\frac{k}{q^r}\right)e^{-\frac{sk}{q^r}} - (q-1)\frac{k}{q^r}.u\left(\frac{k}{q^r}\right)e^{-\frac{sk}{q^r}} \end{aligned}$$

$$= (q-1)\frac{k}{q^r}.u\left(\frac{k}{q^{r-1}}\right)e^{-\frac{sk}{q^r}} - (q-1)\frac{k}{q^r}.u\left(\frac{k}{q^r}\right)e^{-\frac{sk}{q^r}}$$

$$= (q-1)\frac{k}{q^r}.e^{-\frac{sk}{q^r}}\left[u\left(\frac{k}{q^{r-1}}\right) - u\left(\frac{k}{q^r}\right)\right]$$

$$= (q-1)\frac{k}{q^r}.e^{-\frac{sk}{q^r}}$$

Theorem 3.7 Let $k \in (0,\infty)$, $q > 0$ and $s > 0$, then

$L_q(\sin ak) = (q-1)\sum_{r=0}^{\infty}\frac{k}{q^r}.\sin a\frac{k}{q^r}e^{-\frac{sk}{q^r}}$ and

$L_q(\cos ak) = (q-1)\sum_{r=0}^{\infty}\frac{k}{q^r}.\cos a\frac{k}{q^r}e^{-\frac{sk}{q^r}}$

Proof: $L_q(\sin ak) = (q-1)\Delta_q^{-1}k.\sin ak.e^{-sk}\Big|_0^{\infty}$

$$= (q-1)\Delta_q^{-1}g(k)\Big|_0^{\infty}$$

where $g(k) = k\sin ak.e^{-sk}$

$$= (q-1)\sum_{r=0}^{\infty} g\left(\frac{k}{q^r}\right)\Bigg|_0^{\infty}$$

$$= (q-1)\sum_{r=0}^{\infty}\frac{k}{q^r}.\sin a\frac{k}{q^r}.e^{-\frac{k\,'}{q^r}}$$

4 Fundamental Theorems of Quantum Calculus

4.1 Fundamental theorems Related to the q-Difference Operator

This section contains theorems and corollaries for the higher order inverse difference principle for the q-difference operator defined in equation (2.1). Additionally suitable numerical examples are provided to verify and validate the results.

Theorem 4.1 Let $q > 0, f(k)$ be a real valued function on $[0,\infty)$ Then

$$\sum_{r=1}^{m} f\left(\frac{k}{q^r}\right) = \Delta_q^{-1}f(k)\Big|_{\frac{k}{q^m}}^{k}$$

Proof: $\Delta_q^{-1} f(k) = g(k),\ f(k) = \Delta_q g(k),\ f(k) = g(kq) - q(k)$

$$f(k) + g(k) = g(kq) \tag{13}$$

Replace k by $\frac{k}{q}$ in equation (13)

$$f\left(\frac{k}{q}\right) + g\left(\frac{k}{q}\right) = g(k) \tag{14}$$

Replace k by $\frac{k}{q}$ in equation (14)

$$f\left(\frac{k}{q^2}\right) + g\left(\frac{k}{q^2}\right) = g\left(\frac{k}{q}\right) \tag{15}$$

Substitute equation (15) in equation (14)

$$(14) \Rightarrow f\left(\frac{k}{q}\right) + f\left(\frac{k}{q^2}\right) + g\left(\frac{k}{q^2}\right) = g(k) \tag{16}$$

Replace k by $\frac{k}{q}$ in equation (15)

$$(15) \Rightarrow f\left(\frac{k}{q^3}\right) + g\left(\frac{k}{q^3}\right) = g\left(\frac{k}{q^2}\right) \tag{17}$$

Substitute equation (17) in equation (16)

$$(16) \Rightarrow f\left(\frac{k}{q}\right) + f\left(\frac{k}{q^2}\right) + f\left(\frac{k}{q^3}\right) + g\left(\frac{k}{q^3}\right) = g(k) \tag{18}$$

Replace k by $\frac{k}{q}$ in equation (17)

$$(17) \Rightarrow f\left(\frac{k}{q^4}\right) + g\left(\frac{k}{q^4}\right) = g\left(\frac{k}{q^3}\right) \tag{19}$$

Substitute equation (19) in equation (18)

$$(18) \Rightarrow f\left(\frac{k}{q}\right)+f\left(\frac{k}{q^2}\right)+f\left(\frac{k}{q^3}\right)+f\left(\frac{k}{q^4}\right)+g\left(\frac{k}{q^4}\right)=g(k)$$

Continuing this process, we get

$\Rightarrow f\left(\frac{k}{q}\right)+f\left(\frac{k}{q^2}\right)+f\left(\frac{k}{q^3}\right)+\cdots+f\left(\frac{k}{q^m}\right)+g\left(\frac{k}{q^m}\right)=g(k)$

$\Rightarrow f\left(\frac{k}{q}\right)+f\left(\frac{k}{q^2}\right)+f\left(\frac{k}{q^3}\right)+\cdots+f\left(\frac{k}{q^m}\right)=g(k)-g\left(\frac{k}{q^m}\right)$

$\sum\limits_{r=1}^{m} f\left(\frac{k}{q^r}\right)=\Delta_q^{-1}f(k)=\Delta_q^{-1}f\left(\frac{k}{q^m}\right)$

$$\therefore \sum_{r=1}^{m} f\left(\frac{k}{q^r}\right)=\Delta_q^{-1}f(k)\Big|_{\frac{k}{q^m}}^{k} \tag{20}$$

Example 4.2 Consider $f(k)=k, q=2, m=3$ in theorem (3.1)

Soln:

$$f(k)=k$$
$$\Delta_q f(k)=kq-k$$
$$\Delta_q f(k)=k(q-1)$$
$$\Delta_q^{-1}k=\frac{f(k)}{q-1}$$
$$\Delta_q^{-1}k=\frac{k}{q-1}$$

$\therefore (20) \Rightarrow \sum\limits_{r=1}^{3} f\left(\frac{k}{2^r}\right)=\Delta_q^{-1}f(k)\Big|_{\frac{k}{2^3}}^{k}$

$$\Rightarrow \sum_{r=1}^{3}\frac{k}{2^r}=\Delta_q^{-1}k\Big|_{\frac{k}{2^3}}^{k}$$
$$\Rightarrow \sum_{r=1}^{3}\frac{k}{2^r}=\frac{k}{q-1}\Big|_{\frac{k}{2^3}}^{k}$$
$$\Rightarrow \frac{k}{2}+\frac{k}{2^2}+\frac{k}{2^3}=\frac{k}{2-1}-\frac{\frac{k}{2^3}}{2-1}$$
$$\Rightarrow k\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right)=k\left(1-\frac{1}{2^3}\right)$$

$$\Rightarrow \frac{7}{8} = \frac{7}{8}$$

L.H.S=R.H.S

Remark 4.3 From the theorem $\lim\limits_{m\to\infty} \Delta_q^{-1} f\left(\frac{k}{q^m}\right) = 0$ then

$$\sum_{r=1}^{3} f\left(\frac{k}{q^r}\right) = \Delta_q^{-1} f(k) \tag{21}$$

4.2 Fundamental Theorems Related to $q(\infty)$-Difference Operator

Definitions, Lemmas, Carollaries and theorems for the higher order $q(\infty)$ inverse difference operator are found in this section. Suitable examples are provided for verification.

Definition 4.4 Let $I_q ß R$ for $a \in R$ and $q \in R - \{1\}$ the $\alpha(q)$ difference operator is defined by

$$\Delta_{q(\alpha)} f(t) = f(tq) - \alpha f(t) \tag{22}$$

If there exist a function $g : I_q \to R$ such that $\Delta_{q(\alpha)} g(t) = f(t)$ then the inverse $\alpha(q)$ difference operator is defined by

$$g(t) = \Delta_{q(\alpha)} f(t) + c \tag{23}$$

where c is a constant.

Theorem 4.5 Assume $\alpha 1 R - \{1\}, t \in [0,)$ and $1 \neq q > 0$, Then

$$\Delta_{q(\alpha)}^{-1}(1) = \frac{1}{1-\alpha} \tag{24}$$

and

$$\Delta_{q(\alpha)}^{-1} \log(t) = \frac{\log(t)}{1-\alpha} - \frac{\log(q)}{(1-\alpha)^2} \tag{25}$$

Proof: Since $\Delta_{q(\alpha)} f(t) = f(tq) - \alpha f(t)$

Hence $f(t) = 1$

$\therefore \Delta_{q(\alpha)} 1 = 1 - \alpha(1)$

$$= 1(1-\alpha)$$

$$\Delta_{q(\alpha)}1 = \frac{1}{1-\alpha}$$

Also Here $f(t) = \log t$

$$\Delta_{q(\alpha)}f(t) = f(tq) - \alpha f(t)$$

$$\Delta_{q(\alpha)}\log t = \log tq - \alpha \log t$$

$$= \log t + \log q - \alpha t$$

$$= \log t(1-\alpha) + \log q$$

$$\Delta_{q(\alpha)}\log t = (1-\alpha)\left(\log t \frac{\log q}{(1-\alpha)}\right)$$

$$\frac{\log q}{(1-\alpha)} = \Delta_{q(\alpha)}^{-1}\left(\log t \frac{\log q}{(1-\alpha)}\right)$$

$$= \Delta_{q(\alpha)}^{-1}\log t + \Delta_{q(\alpha)}^{-1}\frac{\log q}{(1-\alpha)}$$

Apply $\Delta_{q(\alpha)}^{-1}$ in $\frac{\log q}{(1-\alpha)}$ the value $\frac{\log q}{(1-\alpha)^2}$

$$\frac{\log q}{(1-\alpha)} = \Delta_{q(\alpha)}^{-1}\log t + \frac{\log q}{(1-\alpha)^2}$$

$$\therefore \Delta_{q(\alpha)}^{-1}\log t = \frac{\log q}{(1-\alpha)} - \frac{\log q}{(1-\alpha)^2}$$

Hence proved.

5 Conclusion

In this paper, we derived a number of properties in Laplace Transform using q-difference operator. We derived a number of fundamental theorems for q and $q(\alpha)$ difference operators. Suitable examples are provided for verification.

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