



Stability of Cubic-Quartic Functional Equation in Multi-Fuzzy Banach Spaces

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Abstract

In this paper we establish the Stability of Cubic-Quartic Functional Equations in Multi-Fuzzy Banach Spaces. $g(2a + b) + g(2a - b) = 3g(a + b) + g(-a - b) + 3g(a - b) + g(b - a) + 18g(a) + 6g(-a) - 3g(b) - 3g(-b)$.

Key words: Hyers-Ulam stability, Multi-Fuzzy-Banach Spaces, Cubic-Quartic Functional Equation, Fixed Point Method.

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1. Introduction and Preliminaries

The first stability problem of functional equation was raised by S.M. Ulam [18] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. The affirmative solution to this question was given by D.H. Hyers [8] in 1941. In the year 1950, T. Aoki [2] generalized the Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [7]. After that, the stability problem of several functional equations have been extensively investigated by a number of authors [3, 17] on various spaces like, normed spaces, Banach spaces, Fuzzy normed spaces, Non-Archimedean space and etc. Sun-Young Jung, Choonkil Park and Dong Yun Shin [11] proved the Fuzzy stability of the Cubic-Quartic functional equation by using fixed point method. The multi-Banach space was first investigated by Dales and Polyakov [4]. Theory of multi-Banach spaces is similar to operator sequence space and has some connections with operator spaces and Banach spaces. In 2007 H.G. Dales and M.S. Moslehian [5] first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The stability of functional equations on multi-normed spaces was proved by many mathematicians (see, [6, 10, 12]).

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Definition 1.1 [9] Let (E, N) be a fuzzy normed space. A multi-fuzzy norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence N_k such that N_k is a fuzzy norm on $E^k (k \in \mathbb{N})$, $N_1(x, t) = N(x, t)$ for each $x \in E$ and $t \in \mathbb{R}$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(1). for each $\sigma \in, x \in E^k$ and $t \in \mathbb{R}$,

$$N_k(A_\sigma(x), t) = N_k(x, t);$$

(2). for each $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k, x \in E^k$ and $t \in \mathbb{R}$,

$$N_k(M_\alpha(x), t) \geq N_k(\max_{i \in \mathbb{N}_k} |\alpha_i| x, t);$$

(3). for each $x_1, \dots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}((x_1, \dots, x_k, 0), t) = N_k((x_1, \dots, x_k), t);$$

(4). for each $x_1, \dots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}((x_1, \dots, x_k, x_k), t) = N_k((x_1, \dots, x_k), t)$$

In such a case $\{(E^k, N_k), k \in \mathbb{N}\}$ is called a muti-fuzzy normed space.

$$Dg(a, b) = g(2a + b) + g(2a - b) - 3g(a + b) - g(-a - b) - 3g(a - b) - g(b - a) - 18g(a) - 6g(-a) + 3g(b) + 3g(-b) \quad (1)$$

Throughout this paper, assume that \mathcal{X} be a linear space and let $(\mathcal{Y}^n, \|\cdot\|_n)$ be a multi-Banach space.

Theorem 1.2 Let $\phi : \mathcal{X}^{2k} \rightarrow [0, \infty)$ be a function such that there exists an $\mathcal{L} < 1$ with

$$\phi(a_1, b_1, \dots, a_k, b_k) \leq \frac{\mathcal{L}}{8} \phi(2a_1, 2b_1, \dots, 2a_k, 2b_k)$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, \dots, k$. Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be an odd mapping satisfies

$$N_k((Dg(a_1, b_1), \dots, Dg(a_k, b_k)), t) \geq \frac{t}{t + \phi(a_1, b_1, \dots, a_k, b_k)} \quad (2)$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$. Then

$$\mathcal{C}(a) = N - \lim_{n \rightarrow \infty} 8^n g\left(\frac{a}{2^n}\right)$$

exists for each $a \in \mathcal{X}$ and defines a cubic mapping $\mathcal{C} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$N_k((g(a_1) - \mathcal{C}(a_1), \dots, g(a_k) - \mathcal{C}(a_k)), t) \geq \frac{(16 - 16\mathcal{L})t}{(16 - 16\mathcal{L})t + \mathcal{L}\phi(a_1, 0, \dots, a_k, 0)} \quad (3)$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$.

Proof: Letting $b_i = 0$ where $i = 1, \dots, k$ in (2), we get

$$N_k((2g(2a_1) - 16g(a_1), \dots, 2g(2a_k) - 16g(a_k)), t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \quad (4)$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$. Consider the set $\mathcal{S} := \{h : \mathcal{X} \rightarrow \mathcal{Y}\}$ and introduce the generalized metric on \mathcal{S} :

$$d(h, l) = \inf \left\{ \mu \in \mathbb{R}_+ : N(h(a_1) - l(a_1), \dots, h(a_k) - l(a_k), \mu t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \right\}$$

where as usual $\inf \phi = +\infty$. It is easy to prove that (\mathcal{S}, d) is complete. See [13]. Now, we consider the linear mapping $\mathcal{J} : \mathcal{S} \rightarrow \mathcal{S}$ such that

$$\mathcal{J}h(a) = 8h\left(\frac{a}{2}\right)$$

for all $a \in \mathcal{X}$. Let $h, l \in \mathcal{S}$ be given such that $d(h, l) = \epsilon$. Then

$$N_k((h(a_1) - l(a_1), \dots, h(a_k) - l(a_k)), \epsilon t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$. Hence

$$\begin{aligned} & N_k((\mathcal{J}h(a_1) - \mathcal{J}l(a_1), \dots, \mathcal{J}h(a_k) - \mathcal{J}l(a_k)), \mathcal{L}\epsilon t) \\ &= N_k\left(\left(8h\left(\frac{a_1}{2}\right) - 8l\left(\frac{a_1}{2}\right), \dots, 8h\left(\frac{a_k}{2}\right) - 8l\left(\frac{a_k}{2}\right)\right), \mathcal{L}\epsilon t\right) \\ &= N_k\left(\left(h\left(\frac{a_1}{2}\right) - l\left(\frac{a_1}{2}\right), \dots, h\left(\frac{a_k}{2}\right) - l\left(\frac{a_k}{2}\right)\right), \frac{\mathcal{L}}{8}\epsilon t\right) \\ &\geq \frac{\frac{\mathcal{L}t}{8}}{\frac{\mathcal{L}t}{8} + \frac{\mathcal{L}}{8}\phi(a_1, 0, \dots, a_k, 0)} \\ &= \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \end{aligned}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$. So $d(h, l) = \epsilon$ implies that $d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}\epsilon$. This means that

$$d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}d(h, l)$$

for all $h, l \in \mathcal{S}$. It follows from (4) that

$$N_k\left(\left(g(a_1) - 8\left(\frac{a_1}{2}\right), \dots, g(a_k) - 8\left(\frac{a_k}{2}\right)\right), \frac{\mathcal{L}}{16}t\right) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$. So $d(g, \mathcal{J}g) \leq \frac{\mathcal{L}}{16}$.

By Theorem 2.2 [14], there exists a cubic mapping $\mathcal{C} : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following:

(1). \mathcal{C} is a fixed point of \mathcal{J} , i.e

$$\mathcal{C}\left(\frac{a}{2}\right) = \frac{1}{8}\mathcal{C}(a) \quad (5)$$

for all $a \in \mathcal{X}$. The mapping \mathcal{C} is a unique fixed point of \mathcal{J} in the set $\mathcal{M} = \{h \in \mathcal{S} : d(g, h) < \infty\}$. This implies that \mathcal{C} is a unique mapping satisfying (5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N_k((g(a_1) - \mathcal{C}(a_1), \dots, g(a_k) - \mathcal{C}(a_k)), \mu t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$.

(2). $d(\mathcal{J}^n g, \mathcal{C}) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N - \lim_{n \rightarrow \infty} 8^n g\left(\frac{a}{2^n}\right) = \mathcal{C}(a) \quad \text{for all } a \in \mathcal{X}.$$

(3). $d(g, \mathcal{C}) \leq \frac{1}{1 - \mathcal{L}}d(g, \mathcal{J}g)$, which implies the inequality

$$d(g, \mathcal{C}) \leq \frac{\mathcal{L}}{16 - 16\mathcal{L}}.$$

This implies that the inequality (3) holds. By (2),

$$N_k\left(\left(8^n Dg\left(\frac{a_1}{2}, \frac{b_1}{2}\right), \dots, 8^n Dg\left(\frac{a_k}{2}, \frac{b_k}{2}\right)\right), 8^n t\right) \geq \frac{t}{t + \phi\left(\frac{a_1}{2^n}, \frac{b_1}{2^n}, \dots, \frac{a_k}{2^n}, \frac{b_k}{2^n}\right)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$ and all $n \in \mathbb{N}$. So

$$N_k\left(\left(8^n Dg\left(\frac{a_1}{2}, \frac{b_1}{2}\right), \dots, 8^n Dg\left(\frac{a_k}{2}, \frac{b_k}{2}\right)\right), t\right) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{\mathcal{L}^n}{8^n}\phi(a_1, b_1, \dots, a_k, b_k)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{\mathcal{L}^n}{8^n}\phi(a_1, b_1, \dots, a_k, b_k)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$.

$$N_k((DC(a_1, b_1), \dots, DC(a_k, b_k)), t) = 1$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$. Thus the mapping $\mathcal{C} : \mathcal{X} \rightarrow \mathcal{Y}$ is cubic, as desired.

Corollary 1.3 Let $\vartheta \geq 0$ and let γ be a real number and $\gamma > 3$. Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be an odd mapping satisfying

$$N_k((Dg(a_1, b_1), \dots, Dg(a_k, b_k)), t) \geq \frac{t}{t + \vartheta (\|a_1\|^\gamma + \|b_1\|^\gamma, \dots, \|a_k\|^\gamma + \|b_k\|^\gamma)} \quad (6)$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$. Then $\mathcal{C}(a) = N - \lim_{n \rightarrow \infty} 8^n g(\frac{a}{2^n})$ exists, and defines a cubic mapping $\mathcal{C} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$N_k((g(a_1) - \mathcal{C}(a_1), \dots, g(a_k) - \mathcal{C}(a_k)), t) \geq \frac{2(2^\gamma - 8)t}{2(2^\gamma - 8)t + \vartheta \|a_1\|^\gamma, \dots, \|a_k\|^\gamma}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$.

Proof: The proof follows from Theorem (1.2) by taking

$$\phi(a_1, b_1, \dots, a_k, b_k) = \vartheta (\|a_1\|^\gamma + \|b_1\|^\gamma, \dots, \|a_k\|^\gamma + \|b_k\|^\gamma)$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$. Then we can choose $L = 2^{3-\gamma}$ and we get the desired result.

Theorem 1.4 Let $\phi : \mathcal{X}^{2k} \rightarrow [0, \infty)$ be a function such that there exists an $\mathcal{L} < 1$ with

$$\phi(a_1, b_1, \dots, a_k, b_k) \leq \frac{\mathcal{L}}{16} \phi(2a_1, 2b_1, \dots, 2a_k, 2b_k)$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, \dots, k$. Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be an even mapping satisfying the inequality (2). Then

$$\mathcal{Q}(a) = N - \lim_{n \rightarrow \infty} 16^n g\left(\frac{a}{2^n}\right)$$

exists for each $a \in \mathcal{X}$ and defines a quartic function $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$N_k((g(a_1) - \mathcal{Q}(a_1), \dots, g(a_k) - \mathcal{Q}(a_k)), t) \geq \frac{(32 - 32\mathcal{L})t}{(32 - 32\mathcal{L})t + \mathcal{L}\phi(a_1, 0, \dots, a_k, 0)} \quad (7)$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$.

Proof: Letting $b_i = 0$ where $i = 1, \dots, k$ in (2), we get

$$N_k((2g(2a_1) - 32g(a_1), \dots, 2g(2a_k) - 32g(a_k)), t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \quad (8)$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$. Consider the set $\mathcal{S} := \{h : \mathcal{X} \rightarrow \mathcal{Y}\}$ and introduce the generalized metric on \mathcal{S} :

$$d(h, l) = \inf \left\{ \mu \in \mathbb{R}_+ : N_k(h(a_1) - l(a_1), \dots, h(a_k) - l(a_k), \mu t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \right\}$$

where as usual $\inf \phi = +\infty$. It is easy to show that (\mathcal{S}, d) is complete. see [13]. Now, we consider the linear mapping $\mathcal{J} : \mathcal{S} \rightarrow \mathcal{S}$ such that

$$\mathcal{J}h(a) = 16h\left(\frac{a}{2}\right)$$

for all $a \in \mathcal{X}$. Let $h, l \in \mathcal{S}$ be given such that $d(h, l) = \epsilon$. Then

$$N_k((h(a_1) - l(a_1), \dots, h(a_k) - l(a_k)), \epsilon t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$ where $i = 1, \dots, k$ and all $t > 0$. Hence

$$\begin{aligned} & N_k((\mathcal{J}h(a_1) - \mathcal{J}l(a_1), \dots, \mathcal{J}h(a_k) - \mathcal{J}l(a_k)), \mathcal{L}\epsilon t) \\ &= N_k\left(\left(16h\left(\frac{a_1}{2}\right) - 16l\left(\frac{a_1}{2}\right), \dots, 16h\left(\frac{a_k}{2}\right) - 16l\left(\frac{a_k}{2}\right)\right), \mathcal{L}\epsilon t\right) \\ &= N_k\left(\left(h\left(\frac{a_1}{2}\right) - l\left(\frac{a_1}{2}\right), \dots, h\left(\frac{a_k}{2}\right) - l\left(\frac{a_k}{2}\right)\right), \frac{\mathcal{L}}{16}\epsilon t\right) \\ &\geq \frac{\frac{\mathcal{L}t}{16}}{\frac{\mathcal{L}t}{16} + \frac{\mathcal{L}}{16}\phi(a_1, 0, \dots, a_k, 0)} \\ &= \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)} \end{aligned}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$. So $d(h, l) = \epsilon$ implies that $d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}\epsilon$. This means that

$$d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}d(h, l)$$

for all $h, l \in \mathcal{S}$. It follows from (8) that

$$N_k\left(\left(g(a_1) - 16g\left(\frac{a_1}{2}\right), \dots, g(a_k) - 16g\left(\frac{a_k}{2}\right)\right), \frac{\mathcal{L}}{32}t\right) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$. So $d(g, \mathcal{J}g) \leq \frac{\mathcal{L}}{32}$.

By Theorem 2.2 [14], there exists a quartic mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following:

(1). \mathcal{Q} is a fixed point of \mathcal{J} , i.e

$$\mathcal{Q}\left(\frac{a}{2}\right) = \frac{1}{16}\mathcal{Q}(a) \quad (9)$$

for all $a \in \mathcal{X}$. The mapping \mathcal{Q} is a unique fixed point of \mathcal{J} in the set $\mathcal{M} = \{h \in \mathcal{S} : d(g, h) < \infty\}$. This implies that \mathcal{Q} is a unique mapping satisfying (9) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N_k((g(a_1) - \mathcal{Q}(a_1), \dots, g(a_k) - \mathcal{Q}(a_k)), \mu t) \geq \frac{t}{t + \phi(a_1, 0, \dots, a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, \dots, k$ and all $t > 0$.

(2). $d(\mathcal{J}^n g, \mathcal{Q}) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N - \lim_{n \rightarrow \infty} 16^n g\left(\frac{a}{2^n}\right) = \mathcal{Q}(a)$$

for all $a \in \mathcal{X}$.

(3). $d(g, \mathcal{Q}) \leq \frac{1}{1-L}d(g, \mathcal{J}g)$, which implies the inequality

$$d(g, \mathcal{Q}) \leq \frac{\mathcal{L}}{32 - 32\mathcal{L}}.$$

This implies that the inequality (7) holds. The rest of the proof is similar to the proof of Theorem (1.2).

Corollary 1.5 Let $\vartheta \geq 0$ and let γ be a real number and $\gamma > 4$. Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be an even mapping satisfying

$$N_k((Dg(a_1, b_1), \dots, Dg(a_k, b_k)), t) \geq \frac{t}{t + \vartheta(\|a_1\|^\gamma + \|b_1\|^\gamma, \dots, \|a_k\|^\gamma + \|b_k\|^\gamma)} \quad (10)$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$. Then $\mathcal{Q}(a) = N - \lim_{n \rightarrow \infty} 16^n g(\frac{a}{2^n})$ exists, and defines a quartic mapping $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$N_k((g(a_1) - \mathcal{Q}(a_1), \dots, g(a_k) - \mathcal{Q}(a_k)), t) \geq \frac{2(2^\gamma - 16)t}{2(2^\gamma - 16)t + \vartheta\|a_1\|^\gamma, \dots, \|a_k\|^\gamma}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, \dots, k$ all $t > 0$.

The proof follows from Theorem (1.4) by taking

$$\phi(a_1, b_1, \dots, a_k, b_k) = \vartheta(\|a_1\|^\gamma + \|b_1\|^\gamma, \dots, \|a_k\|^\gamma + \|b_k\|^\gamma)$$

for all $a_i, b_i \in a$, where $i = 1, \dots, k$ all $t > 0$. Then we can choose $\mathcal{L} = 2^{4-\gamma}$ and we get the desired result.

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