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Perfect Mean Cordial Labeling of $[P_n : S_k]$ Graphs

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Abstract

A vertex labeling $h : V(G) \rightarrow \{0,1,2,3\}$ is said to be perfect mean cordial labeling of a graph G if it induces an edge labeling h^* defined as follows :

$$h^*(uv) = \begin{cases} 1 & \text{if } \ 2|(h(u)+h(v)) \\ & \\ 0 & \text{otherwise} \qquad \text{for all } uv \in E(G) \end{cases}$$

with the condition that $|e_h(0) - e_h(1)| \leq 1$ and $|v_h(\alpha) - v_h(\beta)| \leq 1$ for all $\alpha, \beta \in \{0,1,2,3\}$,where $e_h(\delta)$ is number of edges label with $\delta(\delta = 0,1)$ and $v_h(\lambda)$ denote the number of vertices labeled with λ $(\lambda = 0,1,2,3)$. A graph G is said to be perfect mean cordial graph if it admits a perfect mean cordial labeling. In this paper, we investigate $[P_n : S_k]$ graphs are perfect mean cordial graphs.

Key words: Perfect mean cordial graph, perfect mean cordial labeling.

AMS classification: 05C78.

1 Introduction

In the present era, Graph Theory has become a highly challenging and interesting area for the study of numerous mathematicians and computer experts. Since it has many applications and scope for various researches, it has attracted the attention of the erudite scholars who have the overwhelming desire for updating the field of mathematics. Particularly graph labeling has become a widely popular and area of concern, since it offers wide range of applications. A graph labeling is an assignment of integers to the vertices or the edges, or both, subject to certain conditions. If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling.

In 1987, Cahit introduced the concept of cordial labeling as a weaker version

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of graceful and harmonious labeling. Various types of cordial labeling was studied by several authors. In [1], Perfect Mean Cordial labeling of graphs was introduced. Let $G = (V,E)$ be a graph. A vertex labeling $h : V(G) \to \{0,1\}$ with the induced edge labeling $h^* : E(G) \to \{0,1\}$ given by $h^*(e) = |h(w) - h(z)|$ is called binary vertex labeling. A binary vertex labeling of a graph G is called cordial labeling if $|v_h(0) - v_h(1)| \leq 1$ and $|e_h(0) - e_h(1)| \leq 1$, where $v_h(i)$ the number of vertices labeled with i $(i = 0,1)$ and $e_h(j)$ the number of edges labeled with j $(j = 0,1)$. A graph G is cordial if it admits cordial labeling.

Definition 1.1 [1] A vertex labeling $h : V(G) \to \{0,1,2,3\}$ with induced edge labeling $h^* : E(G) \to \{0,1\}$ defined by

$$h^*(uv) = \begin{cases} 1 & \text{if } 2|(h(u)+h(v)) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } uv \in E(G)$$

is called perfect mean cordial labeling of a graph G if $|e_h(0) - e_h(1)| \leq 1$ and $|v_h(\alpha) - v_h(\beta)| \leq 1$ for all $\alpha, \beta \in \{0,1,2,3\}$, where $v_h(\lambda)$ is the number of vertices labeled with λ $(\lambda = 0,1,2,3)$ and $e_h(\delta)$ is number of edges label with $\delta(\delta = 0,1)$. A graph G is said to be perfect mean cordial graph if it admits a perfect mean cordial labeling.

Example 1.2 The graph G which is shown in Figure 1.1 is a perfect mean cordial graph.

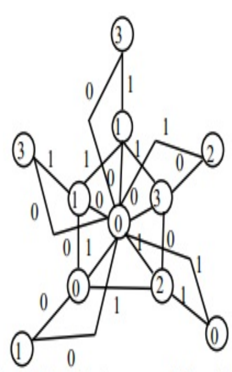

Figure 1.1.perfect mean cordial graph G

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Definition 1.3 [2] $[P_n : S_k]$ is a graph obtained from a path P_n by joining every vertex of P_n to a apex of a star S_k by an edge.

In this paper, we investigate $[P_n : S_k]$ graphs are perfect mean cordial graphs. Terms not defined are used in the sense of [3].

2 Main Results

In this section, we prove that graphs $[P_n : S_k]$ are perfect mean cordial graphs by considering all the values of n as modulo 4 and all the theorems are proved by splitting four cases in terms of k.

Theorem 2.1 $[P_n : S_k]$ is perfect mean cordial graph, when $n \equiv 0(mod4)$.
Proof: Let $V([P_n : S_k]) = \{u_i, v_i, v_{ij} : 1 \leq i \leq n, , 1 \leq j \leq k\}$
Let $E([P_n : S_k]) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup$
$\{v_i v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$

We construct vertex labeling $h : V([P_n : S_k]) \to \{0, 1, 2, 3\}$ as follows:

Case 1 : $k \not\equiv 3(mod4)$

$$h(u_i) = \begin{cases} 3 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 1 & i \equiv 2\,(mod4) \\ 2 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_i) = \begin{cases} 1 & i \equiv 0\,(mod4) \\ 2 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 3 & i \equiv 2\,(mod4) \\ 0 & i \equiv 3\,(mod4) \end{cases}$$

Subcase 1 : $k \equiv 0\,(mod4)$

$$\text{For all } i,\ h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

subcase 2 $: k \equiv 1(mod4)$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

subcase 3 $: k \equiv 2(mod4)$

If $i \not\equiv 3\,(mod4)$, we define $h(v_{ij})$ as in Subcase 2.

If $i \equiv 3\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

Case 2 $: k \equiv 3(mod4)$

$$h(u_i) = \begin{cases} 2 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 2 & i \equiv 2\,(mod4) \\ 0 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_i) = \begin{cases} 3 & i \equiv 0\,(mod4) \\ 1 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 3 & i \equiv 2\,(mod4) \\ 1 & i \equiv 3\,(mod4) \end{cases}$$

If $i \not\equiv 2\,(mod4)$, we define $h(v_{ij})$ as in Subcase 2 of Case 1.

If $i \equiv 2\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

The induced edge labeling is,

$$h^*(uv) = \begin{cases} 1 & \text{if } \ 2|(h(u)+h(v)) \\ & \qquad \text{for all } uv \in E([P_n : S_k]) \\ 0 & \text{otherwise} \end{cases}$$

Examination of vertex and edge demands are illustrated below.

$n \equiv 0(mod4)$	Vertex values	Edge values
$k \equiv 0(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = \frac{(1+\frac{k}{2})n}{2}$	$e_h(0) = (1+\frac{k}{2})n - 1,$ $e_h(1) = (1+\frac{k}{2})n$
$k \equiv 1(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = \frac{(k+2)n}{4}$	$e_h(0) = \frac{(k+2)n}{2} - 1$ $e_h(1) = \frac{(k+2)n}{2}$
$k \equiv 2(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = (\frac{k-2}{4}+1)n$	$e_h(0) = (\frac{k-2}{2}+2)n - 1,$ $e_h(1) = (\frac{k-2}{2}+2)n$
$k \equiv 3(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = \frac{(k+2)n}{4}$	$e_h(0) = \frac{(k+2)n}{2} - 1$ $e_h(1) = \frac{(k+2)n}{2}$

Table 2.6 : vertex and edge demands of path $[P_n : S_k]$, when $n \equiv 0(mod4)$

2456-8686, vii(i), 2023 : 073-099
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Consequently, the graph $[P_n : S_k]$, where $n \equiv 0(mod4)$ fulfills the demands $|e_h(0) - e_h(1)| \leq 1$ and $|v_h(\alpha) - v_h(\beta)| \leq 1$ for all $\alpha, \beta \in \{0,1,2,3\}$.

Accordingly, $[P_n : S_k]$, where $n \equiv 0(mod4)$ is a perfect mean cordial graph.

Example 2.2 Illustration of the perfect mean cordial graph $[P_4 : S_5]$ is shown in the Figure 2.1.

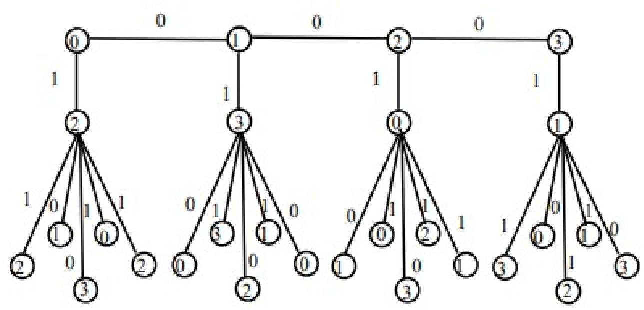

Figure .2.1. Perfect mean cordial labeling of $[P_4 : S_5]$

Theorem 2.3 $[P_n : S_k]$ is perfect mean cordial graph, when $n \equiv 1(mod4)$.

Proof: Let $V([P_n : S_k]) = \{u_i, v_i, v_{ij} : 1 \leq i \leq n, , 1 \leq j \leq k\}$

Let $E([P_n : S_k]) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup$
$\{v_i v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$

We construct vertex labeling $h : V([P_n : S_k]) \to \{0,1,2,3\}$ as follows:

Case 1 : $k \equiv 0(mod4)$

$$h(u_i) = \begin{cases} 3 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 1 & i \equiv 2\,(mod4) \\ 2 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_i) = \begin{cases} 1 & i \equiv 0\,(mod4) \\ 2 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 3 & i \equiv 2\,(mod4) \\ 0 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

Case 2 : $k \equiv 1(mod4)$

$$h(u_i) = \begin{cases} 1 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 2 & i \equiv 2\,(mod4) \\ 3 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_i) = \begin{cases} 0 & i \equiv 0\,(mod4) \\ 3 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 1 & i \equiv 2\,(mod4) \\ 2 & i \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod 4) \\ 2 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod 4) \\ 1 & j \equiv 3\,(mod 4) \end{cases}$$

If $h(v_{ij}),\ i \equiv 1\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod 4) \\ 1 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod 4) \\ 3 & j \equiv 3\,(mod 4) \end{cases}$$

If $h(v_{ij}),\ i \equiv 2\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod 4) \\ 3 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

If $h(v_{ij}),\ i \equiv 3\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod 4) \\ 0 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

Case 3 $k \equiv 2(mod 4)$

$$h(u_i) = \begin{cases} 2 & i \equiv 0\,(mod 4) \\ 0 & i \equiv 1\,(mod 4) \quad 1 \le i \le n \\ 2 & i \equiv 2\,(mod 4) \\ 0 & i \equiv 3\,(mod 4) \end{cases}$$

$$h(v_i) = \begin{cases} 3 & i \equiv 0\,(mod 4) \\ 1 & i \equiv 1\,(mod 4) \quad 1 \le i \le n \\ 3 & i \equiv 2\,(mod 4) \\ 1 & i \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 0\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod 4) \\ 3 & j \equiv 1\,(mod 4) \quad 1 \le j \le k \\ 2 & j \equiv 2\,(mod 4) \\ 0 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 1\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod 4) \\ 2 & j \equiv 1\,(mod 4) \quad 1 \le j \le k \\ 3 & j \equiv 2\,(mod 4) \\ 1 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 2\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod 4) \\ 1 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 3\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod 4) \\ 0 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod 4) \\ 3 & j \equiv 3\,(mod 4) \end{cases}$$

Case 4 : $k \equiv 3(mod 4)$

$$h(u_i) = \begin{cases} 2 & i \equiv 0\,(mod 4) \\ 0 & i \equiv 1\,(mod 4) \quad 1 \leq i \leq n \\ 2 & i \equiv 2\,(mod 4) \\ 0 & i \equiv 3\,(mod 4) \end{cases}$$

$$h(v_i) = \begin{cases} 3 & i \equiv 0\,(mod 4) \\ 1 & i \equiv 1\,(mod 4) \quad 1 \leq i \leq n \\ 3 & i \equiv 2\,(mod 4) \\ 1 & i \equiv 3\,(mod 4) \end{cases}$$

2456-8686, vii(i), 2023 : 073-099
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If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

2456-8686, vii(i), 2023 : 073-099
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$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod 4) \\ 1 & j \equiv 1\,(mod 4) \quad 1 \le j \le k \\ 0 & j \equiv 2\,(mod 4) \\ 3 & j \equiv 3\,(mod 4) \end{cases}$$

The induced edge labeling is,

$$h^*(uv) = \begin{cases} 1 & \text{if } \; 2|(h(u)+h(v)) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } uv \in E([P_n : S_k])$$

Examination of vertex and edge demands are illustrated below.

$n \equiv 1(mod 4)$	Vertex values	Edge values
$k \equiv 0(mod 4)$	$v_h(0) = v_h(1) = \frac{(1+\frac{k}{2})(n-1)}{2} + (\frac{k}{4}+1)$ $v_h(2) = v_h(3) = \frac{(1+\frac{k}{2})(n-1)}{2} + \frac{k}{4}$	$e_h(0) = (1+\frac{k}{2})n - 1,$ $e_h(1) = (1+\frac{k}{2})n$
$k \equiv 1(mod 4)$	$v_h(0) = v_h(1) = v_h(3) = \frac{(k+2)(n-1)}{4} + (\frac{k-1}{4}+1)$ $v_h(2) = (\frac{(k+2)(n-1)}{4} + (\frac{k-1}{4})$	$e_h(0) = e_h(1) = (\frac{(k+2)(n-1)}{2} + (\frac{k-1}{2}+1)$
$k \equiv 2(mod 4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = (\frac{k-2}{4}+1)n$	$e_h(0) = (\frac{k-2}{2}+2)n,$ $e_h(1) = (\frac{k-2}{2}+2)n - 1$
$k \equiv 3(mod 4)$	$v_h(1) = \frac{(k+2)(n-1)}{4} + (\frac{k-3}{4}+2)$ $v_h(0) = v_h(2) = v_h(3) = \frac{(k+2)(n-1)}{4} + (\frac{k-3}{4}+1)$	$e_h(0) = e_h(1) = \frac{(k+2)(n-1)}{2} + (\frac{k-3}{2}+2)$

Table 2.7 : vertex and edge demands of path $[P_n : S_k]$, when $n \equiv 1(mod4)$

Consequently, the graph $[P_n : S_k]$, where $n \equiv 1(mod4)$, fulfills the demands $|e_h(0) - e_h(1)| \leq 1$ and $|v_h(\alpha) - v_h(\beta)| \leq 1$ for all $\alpha, \beta \in \{0,1,2,3\}$.

Accordingly, $[P_n : S_k]$, where $n \equiv 1(mod4)$ is a perfect mean cordial graph.

Example 2.4 Illustration of the perfect mean cordial graph $[P_5 : S_6]$ is shown in the Figure 2.2.

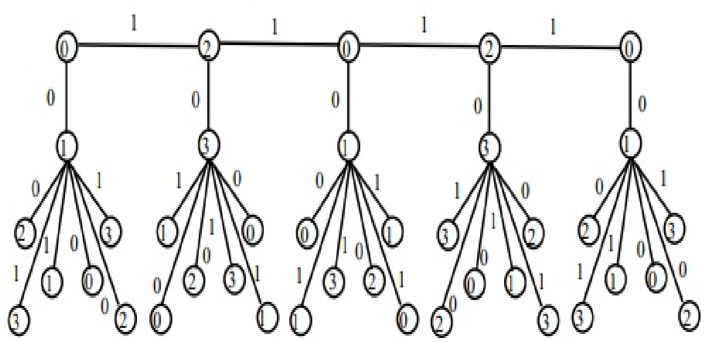

Figure .2.2. Perfect mean cordial labeling of $[P_5 : S_6]$

Theorem 2.5 $[P_n : S_k]$ is perfect mean cordial graph, when $n \equiv 2(mod4)$.

Proof: Let $V([P_n : S_k]) = \{u_i, v_i, v_{ij} : 1 \leq i \leq n, , 1 \leq j \leq k\}$

Let $E([P_n : S_k]) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup$
$\{v_i v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$

We construct vertex labeling $h : V([P_n : S_k]) \to \{0,1,2,3\}$ as follows:

$$h(u_i) = \begin{cases} 3 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 1 & i \equiv 2\,(mod4) \\ 2 & i \equiv 3\,(mod4) \end{cases}$$

2456-8686, vii(i), 2023 : 073-099
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$$h(v_i) = \begin{cases} 1 & i \equiv 0\,(mod4) \\ 2 & i \equiv 1\,(mod4) \quad 1 \leq i \leq n \\ 3 & i \equiv 2\,(mod4) \\ 0 & i \equiv 3\,(mod4) \end{cases}$$

Case 1 $: k \equiv 0(mod4)$

$$\text{For all } i,\ h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 1 & j \equiv 3\,(mod4) \end{cases}$$

Case 2 $: k \equiv 1(mod4)$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 1 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 1 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 0 & j \equiv 3\,(mod4) \end{cases}$$

Case 3 $: k \equiv 2(mod4)$

If $i \equiv 0\,(mod4)$,

2456-8686, vii(i), 2023 : 073-099
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$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 1 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod4) \\ 0 & j \equiv 3\,(mod4) \end{cases}$$

2456-8686, vii(i), 2023 : 073-099
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Case 4 : $k \equiv 3(mod4)$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 0 & j \equiv 3\,(mod4) \end{cases}$$

If, $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

2456-8686, vii(i), 2023 : 073-099
https://doi.org/10.26524/cm164

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod 4) \\ 0 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

The induced edge labeling is,

$$h^*(uv) = \begin{cases} 1 & \text{if } \; 2|(h(u) + h(v)) \\ & \qquad\qquad \text{for all } uv \in E([P_n : S_k]) \\ 0 & \qquad \text{otherwise} \end{cases}$$

Examination of vertex and edge demands are illustrated below.

$n \equiv 2(mod4)$	Vertex values	Edge values
$k \equiv 0(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = \frac{(1+\frac{k}{2})(n-2)}{2} + (\frac{k}{2}+1)$	$e_h(0) = (1+\frac{k}{2})n-1$, $e_h(1) = (1+\frac{k}{2})n$
$k \equiv 1(mod4)$	$v_h(0) = v_h(3) = \frac{(k+2)(n-2)}{4} + (\frac{k+3}{2})$ $v_h(1) = v_h(2) = (\frac{(k+2)(n-2)}{4} + (\frac{k+3}{2} - 1)$	$e_h(0) = \frac{(k+2)(n-2)}{2} + (k+2)$ $e_h(1) = \frac{(k+2)(n-2)}{2} + (k+1)$
$k \equiv 2(mod4)$	$v_h(0) = v_h(1) = v_h(2) = v_h(3) = (\frac{k-2}{4}+1)n$	$e_h(0) = (\frac{k-2}{2} + 2)n-1$, $e_h(1) = (\frac{k-2}{2} + 2)n$
$k \equiv 3(mod4)$	$v_h(0) = v_h(2) = \frac{(k+2)(n-2)}{4} + (\frac{k-3}{2}+2)$ $v_h(1) = v_h(3) = (\frac{(k+2)(n-2)}{4} + (\frac{k-3}{2}+3)$	$e_h(0) = \frac{(k+2)(n-2)}{2} + (k+1)$ $e_h(1) = \frac{(k+2)(n-2)}{2} + (k+2)$

Table 2.8 : vertex and edge demands of path $[P_n : S_k]$, when $n \equiv 2(mod4)$

Consequently, the graph $[P_n : S_k]$ fulfills the demands $|e_h(0) - e_h(1)| \leq 1$ and $|v_h(\alpha) - v_h(\beta)| \leq 1$ for all $\alpha, \beta \in \{0,1,2,3\}$.
Accordingly, $[P_n : S_k]$ is a perfect mean cordial graph.

Example 2.6 Illustration of the perfect mean cordial graph $[P_6 : S_7]$ is shown in the Figure 2.3.

Theorem 2.7 $[P_n : S_k]$ is perfect mean cordial graph, when $n \equiv 3(mod4)$.
Proof:

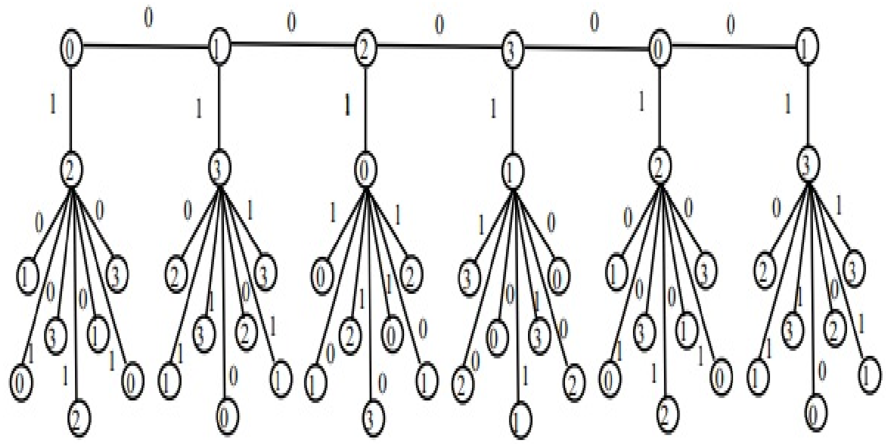

Figure .2.3. Perfect mean cordial labeling of $[P_6 : S_7]$

Let $V([P_n : S_k]) = \{u_i, v_i, v_{ij} : 1 \le i \le n,, 1 \le j \le k\}$
Let $E([P_n : S_k]) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\} \cup \{v_i v_{ij} : 1 \le i \le n, 1 \le j \le k\}$
We construct vertex labeling $h : V([P_n : S_k]) \to \{0,1,2,3\}$ as follows:

Case 1 : $k \equiv 0(mod4)$

$$h(u_i) = \begin{cases} 3 & i \equiv 0\,(mod4) \\ 0 & i \equiv 1\,(mod4) \quad 1 \le i \le n \\ 1 & i \equiv 2\,(mod4) \\ 2 & i \equiv 3\,(mod4) \end{cases}$$

$$h(v_i) = \begin{cases} 1 & i \equiv 0\,(mod 4) \\ 2 & i \equiv 1\,(mod 4) \quad 1 \le i \le n \\ 3 & i \equiv 2\,(mod 4) \\ 0 & i \equiv 3\,(mod 4) \end{cases}$$

$$\text{For all } i,\ h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod 4) \\ 0 & j \equiv 1\,(mod 4) \quad 1 \le j \le k \\ 1 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

Case 2 : $k \not\equiv 0(mod 4)$

$$h(u_i) = \begin{cases} 3 & i \equiv 0\,(mod 4) \\ 0 & i \equiv 1\,(mod 4) \quad 1 \le i \le n \\ 2 & i \equiv 2\,(mod 4) \\ 1 & i \equiv 3\,(mod 4) \end{cases}$$

$$h(v_i) = \begin{cases} 0 & i \equiv 0\,(mod 4) \\ 2 & i \equiv 1\,(mod 4) \quad 1 \le i \le n \\ 3 & i \equiv 2\,(mod 4) \\ 1 & i \equiv 3\,(mod 4) \end{cases}$$

Subcase 1 : $k \equiv 1\,(mod 4)$
If $i \equiv 0\,(mod 4)$,

2456-8686, vii(i), 2023 : 073-099
https://doi.org/10.26524/cm164

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod 4) \\ 2 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod 4) \\ 0 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 1\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod 4) \\ 3 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 2\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod 4) \\ 1 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod 4) \\ 3 & j \equiv 3\,(mod 4) \end{cases}$$

If $i \equiv 3\,(mod 4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod 4) \\ 0 & j \equiv 1\,(mod 4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod 4) \\ 2 & j \equiv 3\,(mod 4) \end{cases}$$

Subcase 2 $: k \equiv 2(mod4)$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 0 & j \equiv 3\,(mod4) \end{cases}$$

If $i \not\equiv 0\,(mod4)$, we define $h(v_{ij})$ as in Subcase 1.

Subcase 4 $: k \equiv 3(mod4)$

If $i \equiv 0\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 3 & j \equiv 0\,(mod4) \\ 2 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 1 & j \equiv 2\,(mod4) \\ 0 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 1\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 2 & j \equiv 0\,(mod4) \\ 3 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 0 & j \equiv 2\,(mod4) \\ 1 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 2\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 1 & j \equiv 0\,(mod4) \\ \\ 0 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 3 & j \equiv 2\,(mod4) \\ 2 & j \equiv 3\,(mod4) \end{cases}$$

If $i \equiv 3\,(mod4)$,

$$h(v_{ij}) = \begin{cases} 0 & j \equiv 0\,(mod4) \\ \\ 1 & j \equiv 1\,(mod4) \quad 1 \leq j \leq k \\ 2 & j \equiv 2\,(mod4) \\ 3 & j \equiv 3\,(mod4) \end{cases}$$

In all cases, the induced edge labeling is,

$$h^*(uv) = \begin{cases} 1 & \text{if } \ 2|(h(u)+h(v)) \\ & \qquad\qquad\qquad \text{for all } uv \in E([P_n : S_k]) \\ 0 & \qquad \text{otherwise} \end{cases}$$

Examination of vertex and edge demands are illustrated below.

$n\equiv 3(mod4)$	Vertex values	Edge values
$k\equiv 0(mod4)$	$v_h(0)=v_h(2)=\frac{(1+\frac{k}{2})(n-3)}{2}+(\frac{k-4}{4}+\frac{k+4}{2}+1)$ $v_h(1)=v_h(3)=\frac{(1+\frac{k}{2})(n-3)}{2}+(\frac{k-4}{4}+\frac{k+4}{2})$	$e_h(0)=(1+\frac{k}{2})n-1$, $e_h(1)=(1+\frac{k}{2})n$
$k\equiv 1(mod4)$	$v_h(0)=v_h(2)=v_h(3)=\frac{(k+2)(n-3)}{4}+(\frac{k+1}{2}+\frac{k-1}{4}+1)$ $v_h(1)=\frac{(k+2)(n-3)}{4}+(\frac{k+1}{2}+\frac{k-1}{4}+2)$	$e_h(0)=e_h(1)=3+(\frac{(k-1)(n-3)}{2})+(k+3+\frac{k-1}{2})$
$k\equiv 2(mod4)$	$v_h(0)=v_h(1)=v_h(2)=v_h(3)=(\frac{k-2}{4}+1)n$	$e_h(0)=(\frac{k-2}{2}+2)n-1$, $e_h(1)=(\frac{k-2}{2}+2)n$
$k\equiv 3(mod4)$	$v_h(0)=\frac{(k+2)(n-3)}{4}+(\frac{k-3}{2}+\frac{k-3}{4}+3)$ $v_h(1)=v_h(2)=v_h(3)=\frac{(k+2)(n-3)}{4}+(\frac{k-3}{2}+\frac{k-3}{4}+4)$	$e_h(0)=e_h(1)=\frac{(k+2)(n-3)}{2}+(\frac{k-3}{2}+k+4)$

Table 2.9 : vertex and edge demands of path $[P_n : S_k]$, when $n\equiv 3(mod4)$

Consequently, the graph $[P_n : S_k]$ where $n\equiv 3(mod4)$ fulfills the demands $|e_h(0)-e_h(1)|\leq 1$ and $|v_h(\alpha)-v_h(\beta)|\leq 1$ for all $\alpha,\beta\in\{0,1,2,3\}$.

Accordingly, $[P_n : S_k]$ where $n\equiv 3(mod4)$ is a perfect mean cordial graph.

Example 2.8 Illustration of the perfect mean cordial graph $[P_7 : S_4]$ is shown in the Figure 2.4.

2456-8686, vii(i), 2023 : 073-099
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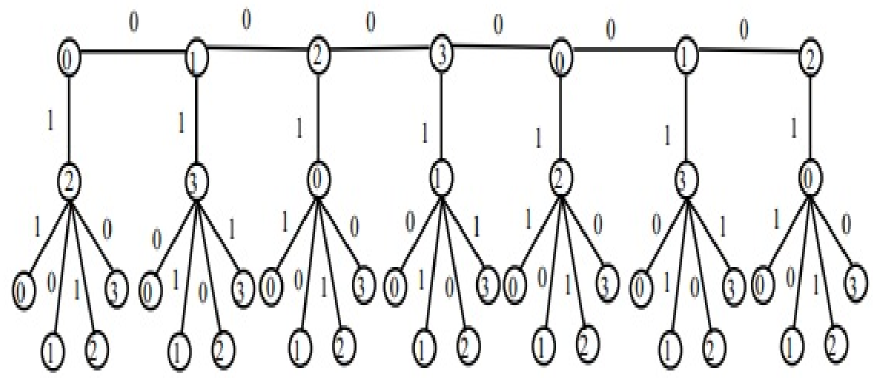

Figure 2.4.Perfect mean cordial labeling of $[P_7 : S_4]$

3 Conclusion

In this paper we derived a number of fundamental theorems for q and $q(\alpha)$ difference operators. Suitable examples are provided for verification.

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