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Higher Order Laplace Transform

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Abstract

In this paper we extend the Laplace transform to higher order Laplace transform. Several new identities are derived using gamma function and maclaurine series. Also we discuss some applications with diagrams.

Keywords: Laplace Transform, Fourier transforms, Complex value and Fourier Series.

AMS Classification: 44A10, 65T50 , 28A10, 42A16 .

1 Introduction

The Laplace Transform is a powerful mathematical tool used in engineering and applied mathematics to analyze linear time-invariant (LTI) systems. It provides a convenient way to analyze and solve linear differential equations, making it an invaluable tool in various fields such as control theory, signal processing, and circuit analysis. [11]While the basic Laplace Transform is widely used, the concept of Higher Order Laplace Transforms extends its applicability to a broader range of problems.The integral transforms like mellin, Laplace and Fourier were applied to obtain the solution of differential equations.

The standard Laplace Transform is defined for a function $f(t)$ in equation (1). Here, $'s'$ is the complex frequency parameter. For higher order derivatives, the Higher Order Laplace Transform introduces additional parameters to account for the various derivatives. This extension facilitates the transformation of higher order differential equations into algebraic equations in the Laplace domain, making their analysis and solution more tractable.

Authors in [9] continued the work derived in [3] by defining fractional frequency Laplace transform with fractional factor $e^{-s\frac{1}{v}t}$. Also they presented the convolution product and several properties of the fractional transforms for functions like

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2456-8686, vii(ii), 2023: 107-126
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polynomial factorial and trigonometric functions.

The Higher Order Laplace Transform is an extension of the standard Laplace Transform, allowing for the transformation of higher-order derivatives of a function. In the context of differential equations, this enables the analysis of systems with multiple levels of complexity, where traditional Laplace Transforms fall short.

The Higher Order Laplace Transform finds applications in diverse scientific and engineering disciplines. It is particularly useful when dealing with complex dynamic systems, where multiple derivatives are involved. [4]The Higher Order Laplace Transform is employed to analyze and design these circuits. Mechanical Systems: The dynamics of mechanical systems, especially those with multiple degrees of freedom, can be modeled using high-order differential equations. The Higher Order Laplace Transform aids in transforming these equations into the Laplace domain, facilitating analysis and design.

With the factor e^{-st}, the Laplace transform (LT) and discrete Laplace transform (DLT) efficiently convert a signal (function) from the time domain to the frequency domain. Many authors have covered a number of LT and DLT applications [5][10]. Applications of the n-dimensional Laplace transform can be found in fluid dynamics modeling, heat equations, and wave equations [2] [8]. Authors have recently discovered fractional difference equation solutions [1]. In progress to it, authors in [6] derived the frequency Laplace transforms of the products of two and three functions with tuning factors and proposed the Laplace transform for certain types of multiseries of circular functions as well. Further in [7] defined $n-$dimensional fractional frequency Laplace transform with shift values.

2 Preliminaries

Definition 2.1 If a function $f(t)$ is defined for values of t, and then if $\int\limits_0^\infty e^{-st^n} f(t)dt$ exists for some values of n, then the n^{th} Order or higher-order Laplace transform is defined as:

$$\mathbf{L}_n[f(t)] = \int\limits_0^\infty e^{-st^n} f(t)dt = \mathbf{F}_n(s) \tag{1}$$

If $n = 1$, then this becomes the ordinary or first-order Laplace transform.

Property 2.1 Properties of Higher Order Laplace Transform

(i) $\mathbf{L}_n[cf(t)] = c\mathbf{L}_n[f(t)]$, where c is a constant.

(ii) $\mathbf{L}_n[f(t)+g(t)] = \mathbf{L}_n[f(t)] + \mathbf{L}_n[g(t)]$
(iii) $\mathbf{L}_n[f'(t)] = ns\mathbf{L}_n[f(t)t^{n-1}] - f(0)$

Proof: Let's see the proof of (iii):

$$\mathbf{L}_n[f'(t)] = \int_0^\infty e^{-st^n} f'(t)dt$$

$$u = e^{-st^n} \quad \Rightarrow \quad du = -nse^{-st^n}t^{n-1}dt$$

$$dv = f'(t)dt \quad \Rightarrow \quad v = f(t)$$

$$= [e^{-st^n} f(t)]_0^\infty + ns\int_0^\infty e^{-st^n} f(t)t^{n-1}dt$$

$$= -f(0) + ns\mathbf{L}_n[f(t)t^{n-1}]$$

So, thus we get (iii).

3 Transforms on polynomials and circular functions

Theorem 3.1 If n is a positive integer and $\Gamma(\nu)$ is gamma function, then

$$\mathbf{L}_n[1] = \frac{\Gamma\left(\frac{1}{n}\right)}{n\cdot\sqrt[n]{s}},\ s>0. \tag{2}$$

Proof: From the definition of higher order Laplace transform,

$$\mathbf{L}_n[f(t)] = \int_0^\infty e^{-st^n} f(t)dt,\ \text{if } f(t)=1=t^0$$

$$\Rightarrow \mathbf{L}_n[1] = \int_0^\infty e^{-st^n}dt;\ \text{let } u=t^n \Rightarrow t=\sqrt[n]{u}$$

$$\Rightarrow t^{n-1} = u^{1-\frac{1}{n}}; n\cdot t^{n-1} = \frac{du}{dt} \Rightarrow dt = \frac{du}{n\cdot u^{1-\frac{1}{n}}}$$

If $t=0 \Rightarrow u=0$; and if $t=\infty \Rightarrow u=\infty$

$$\mathbf{L}_n[1] = \frac{1}{n}\int_0^\infty e^{-su}u^{\frac{1}{n}-1}du;\ \text{we know that from Laplace transform } \mathbf{L}[t^r] = \frac{r!}{s^{r+1}}$$

$$\Rightarrow \mathbf{L}_n[1] = \frac{\left(\frac{1}{n}-1\right)!}{n\cdot s^{\frac{1}{n}}} = \frac{\Gamma\left(\frac{1}{n}\right)}{n\cdot\sqrt[n]{s}}, \text{ where } \Gamma\left(\frac{1}{n}\right) \text{ is the gamma function.}$$

Thus we get (2).

Theorem 3.2 If $\left(-\frac{1}{n}-m\right)!$ is not a negative integer then,

$$\mathbf{L}_n[t^{mn}] = \frac{(-1)^m\Gamma\left(\frac{1}{n}\right)\left(-\frac{1}{n}\right)!s^{\left(-\frac{1}{n}-m\right)}}{n\left(-\frac{1}{n}-m\right)!}, \tag{3}$$

Proof: From higher order Laplace transform,

$$\mathbf{L}_n[t^{mn}] = \int_0^\infty e^{-st^n}t^{mn}dt$$

$$\frac{d}{ds}e^{-st^n} = -t^n e^{-st^n}, \frac{d^2}{ds^2}e^{-st^n} = t^{2n}e^{-st^n},$$

$$\Rightarrow \frac{d^m}{ds^m} = (-1)^m e^{-st^n}t^{mn} \Rightarrow e^{-st^n}t^{mn} = (-1)^m\frac{d^m}{ds^m}e^{-st^n}$$

$$\Rightarrow \int_0^\infty e^{-st\cdot}t^{mn}dt = (-1)^m\frac{d^m}{ds^m}\int_0^\infty e^{-st^n}dt = (-1)^m\frac{d^m}{ds^m}\frac{\Gamma\left(\frac{1}{n}\right)s^{\left(-\frac{1}{n}\right)}}{n} \quad \text{[from above theorem]}$$

$$= (-1)^m\frac{\Gamma\left(\frac{1}{n}\right)}{n}\cdot\left(-\frac{1}{n}\right)!\cdot\frac{s^{\left(-\frac{1}{n}-m\right)}}{\left(-\frac{1}{n}-m\right)!} \quad \left[\text{because, } \frac{d^\alpha}{dx^\alpha}x^n = \frac{n!\cdot x^{(n-\alpha)}}{(n-\alpha)!}\right]$$

$$= \frac{(-1)^m\Gamma\left(\frac{1}{n}\right)\left(-\frac{1}{n}\right)!s^{\left(-\frac{1}{n}-m\right)}}{n\cdot\left(-\frac{1}{n}-m\right)!}$$

Hence Theorem 3.2 has been proved by using the above theorem.
Now let's see some wired infinite series which we got from finding the value of the second-order Laplace Transform of $\cos(\alpha t)$.

Theorem 3.3 $\mathbf{L}_2[\cos(\alpha t)] = \frac{\sqrt{\pi}\cdot e^{-\frac{\alpha^2}{4s}}}{2\cdot\sqrt{s}}$

Proof: Here we are going to give the Feynman approach.

$$\mathbf{L}_2[\cos(\alpha t)] = \int_0^\infty e^{-st^2}\cos(\alpha t)dt = f(\alpha) \Rightarrow f'(\alpha) = \int_0^\infty -e^{-st^2}t\sin(\alpha t)dt$$

2456-8686, vii(ii), 2023: 107-126
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$$\Rightarrow \int_0^\infty -e^{-st^2} t\sin(\alpha t)dt = \left[\frac{\sin(\alpha t)e^{-st^2}}{2s}\right]_0^\infty - \frac{\alpha}{2s}\int_0^\infty e^{-st^2}\cos(\alpha t)dt \quad \text{[by integration by parts]}$$

$$\Rightarrow \lim_{t\to\infty}\frac{\sin(\alpha t)e^{-st^2}}{2s} = 0; \lim_{t\to 0}\frac{\sin(\alpha t)e^{-st^2}}{2s} = 0$$

$$\Rightarrow \int_0^\infty -e^{-st^2} t\sin(\alpha t)dt = -\frac{\alpha}{2s}\int_0^\infty e^{-st^2}\cos(\alpha t)dt \Rightarrow f'(\alpha) = -\frac{\alpha}{2s}f(\alpha)$$

$$\Rightarrow \frac{d}{d\alpha}f(\alpha) = -\frac{\alpha}{2s}f(\alpha) \Rightarrow \frac{df(\alpha)}{f(\alpha)} = -\frac{\alpha}{2s}d\alpha \Rightarrow \int \frac{1}{f(\alpha)}df(\alpha) = -\frac{\alpha^2}{4s} + c$$

$\ln(f(\alpha)) = -\dfrac{\alpha^2}{4s} + c \Rightarrow f(\alpha) = e^{-\frac{\alpha^2}{4s}+c}$. To find the value of the constant, let $\alpha = 0$

$$\Rightarrow f(0) = \int_0^\infty e^{-st^2}dt = e^{-\frac{0}{4s}}\cdot e^c \Rightarrow e^c = \int_0^\infty e^{-st^2}dt = \frac{\sqrt{\pi}}{2\sqrt{s}}$$

$$\therefore f(\alpha) = \mathbf{L}_2[cos(\alpha t)] = \int_0^\infty e^{-st^2}\cos(\alpha)dt = \frac{\sqrt{\pi}e^{-\frac{\alpha^2}{4s}}}{2\sqrt{s}}.$$

The above theorems of $\mathbf{L}_n[f(t)] = \int_0^\infty e^{-st^n} f(t)dt = \mathbf{F}_n(s)$, appears in various areas of mathematics, physics, and engineering. In quantum mechanics, the integral appears when calculating transition probabilities or studying wave functions. In mechanical engineering, this integral can be used to analyze the response of a mechanical system to harmonic forces or vibrations. It's particularly relevant when studying systems with damping. In statistics, this integral can appear when dealing with probability density functions, especially in cases involving normally distributed random variables. It can be used to compute probabilities and moments. The integral can also appear in problems related to heat conduction, where it is used to describe how heat propagates through materials when subjected to a varying temperature profile.

3.1 Infinite series and its Identities

Theorem 3.4

(i) $\frac{1}{(\frac{-1}{2})!} - \frac{1}{2!(\frac{-3}{2})!} + \frac{1}{4!(\frac{-5}{2})!} - \frac{1}{6!(\frac{-7}{2})!} + \frac{1}{8!(\frac{-9}{2})!} - \ldots = \sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!\cdot(\frac{-1}{2}-r)!} = \frac{e^{\frac{1}{4}}}{\sqrt{\pi}}$

(ii) $\frac{1}{(\frac{-1}{2})!} + \frac{1}{2!(\frac{-3}{2})!} + \frac{1}{4!(\frac{-5}{2})!} + \frac{1}{6!(\frac{-7}{2})!} + \frac{1}{8!(\frac{-9}{2})!} + \ldots = \sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!\cdot(\frac{-1}{2}-r)!} = \frac{e^{\frac{-1}{4}}}{\sqrt{\pi}}$

2456-8686, vii(ii), 2023: 107-126
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(iii) $1 - \frac{1}{2!\cdot 8} + \frac{3}{4!\cdot 2!\cdot 8^2} - \frac{3\cdot 5}{6!\cdot 3!\cdot 8^3} + \frac{3\cdot 5\cdot 7}{8!\cdot 4!\cdot 8^4} - \ldots = \sum_{r=0}^{\infty} \frac{(-1)^r (2r-1)_{(2)}^{(r)}}{(2r)!\cdot 8^r \cdot r!} = 1$

(iv) $1 - \frac{1}{2!} + \frac{3}{4!\cdot 2!} - \frac{3\cdot 5}{6!\cdot 3!} + \frac{3\cdot 5\cdot 7}{8!\cdot 4} - \ldots = \sum_{r=0}^{\infty} \frac{(-1)^r (2r-1)_{(2)}^{(r)}}{(2r)!\cdot r!} = 1$

Proof: By Machnaularin series, $\cos(\alpha t) = \sum_{r=0}^{\infty} \frac{(-1)^r \alpha^{2r} t^{2r}}{(2r)!}$

Taking the second order Laplace Transform on both side gives

$$\mathbf{L}_2[cos(\alpha t)] = \sum_{r=0}^{\infty} \frac{(-1)^r \alpha^{2r}}{(2r)!} \int_0^{\infty} e^{-st^2} t^{2r} dt = \frac{\sqrt{\pi} e^{\frac{-\alpha^2}{4s}}}{2\sqrt{s}}$$

$$\Rightarrow \frac{\sqrt{\pi} e^{\frac{-\alpha^2}{4s}}}{2\sqrt{s}} = \sum_{r=0}^{\infty} \frac{(-1)^r \alpha^{2r}}{(2r)!} \cdot \frac{(-1)^r \Gamma\left(\frac{1}{2}\right) \left(\frac{-1}{2}\right)! s^{\left(\frac{-1}{2}-r\right)}}{2\cdot \left(\frac{-1}{2}-r\right)!} [\text{from theorem 1}]$$

$$\Rightarrow \frac{e^{\frac{-\alpha^2}{4s}}}{\sqrt{s}} = \sum_{r=0}^{\infty} \frac{\alpha^{2r}\sqrt{\pi} s^{\left(\frac{-1}{2}-r\right)}}{(2r)!\cdot 2\cdot \left(\frac{-1}{2}-r\right)!} \Rightarrow e^{\frac{-\alpha^2}{4s}} = \sum_{r=0}^{\infty} \frac{\alpha^{2r}\sqrt{\pi s}\cdot s^{\frac{-1}{2}}}{s^r (2r)!\cdot 2\cdot \left(\frac{-1}{2}-r\right)!}, \quad \text{let } k = -\alpha^2$$

$$\Rightarrow e^{\frac{k}{4s}} = \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^r \frac{(-1)^r \sqrt{\pi}}{(2r)!\left(\frac{-1}{2}-r\right)!}, \Rightarrow \frac{e^{\frac{k}{4s}}}{\sqrt{\pi}} = \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^r \frac{(-1)^r}{(2r)!\left(\frac{-1}{2}-r\right)!}$$

If $k = s$ we will get the first infinite series, if $k = -1$ and $s = 1$ we will get the second infinite series.

$$\frac{e^{\frac{k}{4s}}}{\sqrt{\pi}} = \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^r \frac{(-1)^r}{(2r)!\left(\frac{-1}{2}-r\right)!}, \left(\frac{-1}{2}-r\right)! = \frac{(-2)^r\sqrt{\pi}}{(2r-1)_{(2)}^{(r)}} \qquad [\text{Let's sub the value}]$$

$$\Rightarrow e^{\frac{k}{4s}} = \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^r \frac{(2r-1)_{(2)}^{(r)}}{(2r)!2^r}, \text{but} \quad e^{\frac{-k}{4s}} = \sum_{r=0}^{\infty} \left(\frac{-k}{4s}\right) \frac{1}{r!}.$$

$$\Rightarrow \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^{2r} \frac{(-1)^r (2r-1)_(2)^(r)}{(2r)! r! 8^r} = 1$$

if $k = s$ we get the (iii) infinite series. If $k = 2\sqrt{2}, s = 1$ we get the (iv)infinite series. Like the four infinite series we can write many series by substituting the corresponding values of k and s. In more generally,

$$\sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^{2r} \frac{(-1)^r (2r-1)_{(2)}^{(r)}}{(2r)!\cdot r!\cdot 8^r} = 1, \quad \sum_{r=0}^{\infty} \left(\frac{k}{s}\right)^r \frac{(-1)^r}{(2r)!\cdot \left(\frac{-1}{2}-r\right)!} = \frac{e^{\frac{k}{4s}}}{\sqrt{\pi}}$$

4 Higher order Laplace Transform by taking complex values of s

In the previous sections we have see about the Higher order Laplace Transform by taking the values of s which $s \in R$. But in this section we are dealing with the values of s which $s \in Z$.

$$\mathbf{F}_n(s) = \mathbf{L}_n[f(t)] = \int_0^\infty e^{-st^n} f(t)dt$$

It is an Higher order Laplace Transform which we have defined before.But in this section we are going to deal with the $\mathbf{F}_n(is) = \mathbf{L}_n[f(t)]_{s\to is} = \int_0^\infty e^{-ist^n} f(t)dt$ You may have an doubt that while applying the limit we have face a problem that $e^{-i\infty}$.We have an solution for that.In this section we should take $-i\infty = -\infty$.If we take $-i\infty = -\infty$ we are getting the solutions of some problems in short steps.There is an another reason why we are saying that $-i\infty = -\infty$, that is

$$\lim_{x\to 0} -i \cdot \frac{1}{x} = \lim_{x\to 0} -\frac{1}{-ix} \quad \text{but} \lim_{x\to 0} \pm ix = 0 \Rightarrow \lim_{x\to 0} -\frac{1}{-ix} = -\infty.$$

Theorem 4.1 By assuming $-i\infty = -\infty$, we have

$$\mathbf{L}_n[1]_{s\to is} = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{is}}, \ s \notin Z.$$

Proof: Grom the higher order Laplace and limit property,

$$\mathbf{L}_n[1]_{s\to is} = \int_0^\infty e^{-ist^n} dt = \mathbf{F}_n(is)$$

$$\text{let}, ist^n = u \Rightarrow t = \sqrt[n]{\frac{u}{is}}, \quad t^{n-1} = \frac{u^{(1-\frac{1}{n})}}{(is)^{(1-\frac{1}{n})}} \Rightarrow n \cdot is \cdot t^{(n-1)} = \frac{du}{dt}$$

$$\Rightarrow dt = \frac{du \cdot (is)^{(1-\frac{1}{n})}}{n \cdot u^{(1-\frac{1}{n})}} \quad \text{if} t = 0 \Rightarrow u = 0; \text{if} t = \infty \Rightarrow u = \infty$$

$$\Rightarrow \frac{1}{n \cdot \sqrt[n]{is}} \int_0^\infty e^{-u} u^{(\frac{1}{n}-1)} du = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{is}} \quad \text{[by Gamma function]}$$

$$\therefore \mathbf{L}_n[1]_{s\to is} = \int_0^\infty e^{-ist^n}dt = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{is}}$$

By taking the $-i\infty = -\infty$ we have been proved the above theorem.In upcoming sections we are going give reason what will be the use for taking $-i\infty = -\infty$. **Theorem 4.2** If $(s, a) \in R$ and s is not a a negative integer or fraction then

$$\mathbf{L}_n[e^{st^n} cos(at^n)] = \mathbf{F}_n(s) = \frac{\Gamma(\frac{1}{n})}{n\cdot\sqrt[n]{a}}\cdot cos\left(\frac{\pi}{2n}\right)$$

Proof: Taking higher order laplace on $\cos at^n$,

$$\mathbf{L}_n[e^{st^n} cos(at^n)] = \int_0^\infty e^{-st^n}\cdot e^{st^n} cos(at^n)dt \quad \text{By eulers formula } cos(at^n) = RPe^{-iat^n}$$

$$\Rightarrow \int_0^\infty cos(at^n)dt = RP\int_0^\infty e^{-iat^n}dt \quad \text{From above theorem}$$

$$\int_0^\infty cos(at^n)dt = RP\left[\frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\cdot i^{\left(\frac{-1}{n}\right)}\right]; \quad i^{\left(\frac{-1}{n}\right)} = e^{\left(\frac{-i\pi}{2n}\right)} = cos\left(\frac{\pi}{2n}\right) + i\cdot sin\left(\frac{\pi}{2n}\right)$$

$$\Rightarrow RP\left[\frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\cdot i^{\left(\frac{-1}{n}\right)}\right] = RP\left[cos\left(\frac{\pi}{2n}\right) - i\cdot sin\left(\frac{\pi}{2n}\right)\right]\left[\frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\right] = \frac{\Gamma(\frac{1}{n})}{n\cdot\sqrt[n]{a}}\cdot cos\left(\frac{\pi}{2n}\right)$$

$$\therefore \mathbf{L}_n[e^{st^n} cos(at^n)] = \mathbf{F}_n(s) = \int_0^\infty cos(at^n)dt = \frac{\Gamma(\frac{1}{n})}{n\cdot\sqrt[n]{a}}\cdot cos\left(\frac{\pi}{2n}\right)$$

Theorem 4.3 The n^{th} order Laplace transform of $\sin at^n$,

$$\mathbf{L}_n[e^{st^n} sin(at^n)] = \mathbf{F}_n(s) = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\cdot sin\left(\frac{\pi}{2n}\right)$$

Proof: Taking n^{th} Laplace on $\sin at^n$,

$$\mathbf{L}_n[e^{st^n} sin(at^n)] = \int_0^\infty sin(at^n)dt, \quad \text{By eulers formula } sin(at^n) = IP - e^{-iat^n}$$

$$\Rightarrow \int_0^\infty sin(at^n)dt = IP - \int_0^\infty e^{-iat^n}dt \quad \text{[From the theorem 3.1]}$$

$$\int_0^\infty sin(at^n)dt = IP\frac{-\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot sin\left(\frac{\pi}{2n}\right), \quad (i)^{\frac{-1}{n}} = e^{\frac{-i\pi}{2n}} = cos\left(\frac{\pi}{2n}\right) - i \cdot sin\left(\frac{\pi}{2n}\right)$$

$$\int_0^\infty sin(at^n)dt = IP\left[\left(i \cdot sin\left(\frac{\pi}{2n}\right) - cos\left(\frac{\pi}{2n}\right)\right) \cdot \left(\frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\right)\right] = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot sin\left(\frac{\pi}{2n}\right)$$

$$\therefore \mathbf{L}_n[e^{st^n} sin(at^n)] = \mathbf{F}_n(s) = \int_0^\infty sin(at^n)dt = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot sin\left(\frac{\pi}{2n}\right)$$

By taking $-i\infty = -\infty$ we can able to proof the above theorem. There is another method to prove this theorem,but that will take more steps than this. So by taking $-i\infty = -\infty$ we are getting solutions for problems much easier and in few steps, lets verify the above theorems.

$$\int_0^\infty e^{-iat^n}dt = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{ia}}, \quad \text{[from eulers formula } e^{-iat^n} = cos(at^n) - i \cdot sin(at^n)\text{]}$$

$$\Rightarrow \int_0^\infty e^{-iat^n}dt = \int_0^\infty cos(at^n)dt - i \cdot \int_0^\infty sin(at^n)dt \quad \text{[From the the theorem 4.1, 4.2 and 4.3]}$$

$$\frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{ia}} = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot cos\left(\frac{\pi}{2n}\right) - i \cdot \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot sin\left(\frac{\pi}{2n}\right) \Rightarrow \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{ia}} = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\left[cos\left(\frac{\pi}{2n}\right) - i \cdot sin\left(\frac{\pi}{2n}\right)\right]$$

$$cos(\frac{\pi}{2n}) - i \cdot sin(\frac{\pi}{2n}) = e^{\frac{i\pi}{2n}} = i^{\frac{-1}{n}}; \quad \text{so,} \quad \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}}\left[cos\left(\frac{\pi}{2n}\right) - i \cdot sin\left(\frac{\pi}{2n}\right)\right] = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{a}} \cdot i^{(\frac{-1}{n})}$$

$$\therefore \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{ia}} = \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{ia}}$$

Theorem 4.4 If $s \notin Z$ then we have

$$\mathbf{L}_n[t^{mn}]_{s\to is} = \mathbf{F}_n(is) = \frac{(i)^{(m-\frac{1}{n})} \cdot \Gamma(\frac{1}{n})\left(\frac{-1}{n}\right)! \cdot s^{\left(\frac{-1}{n}-m\right)}}{n \cdot \left(-\frac{1}{n} - m\right)!}$$

2456-8686, vii(ii), 2023: 107-126
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Proof: Applying Laplace on t^{mn},

$$\mathbf{L}_n[t^{mn}]_{s\to is} = \int_0^\infty e^{-ist^n} t^{mn} dt; \quad \text{[From Feymann integral technequie]}$$

$$\frac{d}{ds}e^{-ist^n} = -it^n e^{-ist^n}, \quad \frac{d^2}{ds^2}e^{-ist^n} = (-i)^2 t^{2n} e^{-ist^n}$$

$$\Rightarrow \frac{d^m}{ds^m}e^{-ist^n} = (-i)^m t^{mn} e^{-ist^n} \Rightarrow (i)^m \frac{d^m}{ds^m}e^{-ist^n} = t^{mn}e^{-ist^n}$$

$$\Rightarrow \int_0^\infty e^{-ist^n} t^{mn} dt = (i)^m \frac{d^m}{ds^m} \int_0^\infty e^{-ist^n} dt \quad \text{[From theorem 4.1]}$$

$$(i)^m \frac{d^m}{ds^m} \int_0^\infty e^{-ist^n} dt = (i)^m \frac{d^m}{ds^m} \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{s}} \cdot (i)^{\left(\frac{-1}{n}\right)} = (i)^{\left(m-\frac{1}{n}\right)} \frac{d^m}{ds^m} \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{s}}$$

$$\Rightarrow (i)^{\left(m-\frac{1}{n}\right)} \frac{d^m}{ds^m} \frac{\Gamma(\frac{1}{n})}{n\sqrt[n]{s}} = \frac{(i)^{\left(m-\frac{1}{n}\right)} \cdot \Gamma(\frac{1}{n}) \left(\frac{-1}{n}\right)! \cdot s^{\left(\frac{-1}{n}-m\right)}}{n \cdot \left(-\frac{1}{n}-m\right)!} \quad \left[\text{because, } \frac{d^\alpha}{dx^\alpha}x^n = \frac{n! \cdot x^{(n-\alpha)}}{(n-\alpha)!}\right]$$

$$\therefore \mathbf{L}_n[t^{mn}]_{s\to is} = \mathbf{F}_n(is) = \frac{(i)^{\left(m-\frac{1}{n}\right)} \cdot \Gamma(\frac{1}{n}) \left(\frac{-1}{n}\right)! \cdot s^{\left(\frac{-1}{n}-m\right)}}{n \cdot \left(-\frac{1}{n}-m\right)!}$$

While approaching $\mathbf{L}_n[t^{mn}]_{s\to is}$ we will get in a complex valued answer, in this theorem also we got complex valued RHS.

5 Reduction of higher order Laplace transform to first order Laplace Transform

In the previous sections we have seen to find the higher order laplace transform of some existing function. In this section we are going to see how to reduce the higher order to first order Laplace transform. We can able to reduce the higher order to first order for some functions and some specific input for the functions.

Theorem 5.1 The relation between higher order and first order Laplace transform is given by

$$\mathbf{L}_n[f(t^n)t^{n-1}] = \frac{1}{n}\mathbf{L}[f(t)]$$

Proof: From the definition of Laplace Transform,

$$\mathbf{L}_n[f(t^n)t^{n-1}] = \int_0^\infty e^{-st^n} f(t^n)t^{n-1}dt \quad \text{let, } u = t^n \Rightarrow n \cdot t^{n-1} = \frac{du}{dt}$$

$$\Rightarrow \frac{du}{n} = t^{n-1}dt \quad \text{if } t = 0 \Rightarrow u = o; \quad t = \infty \Rightarrow u = \infty$$

$$\Rightarrow \int_0^\infty e^{-st^n} f(t^n)t^{n-1}dt = \frac{1}{n}\int_0^\infty e^{-su} f(u)du = \frac{1}{n}\mathbf{L}_n[f(t)]$$

$$\therefore \mathbf{L}_n[f(t^n)t^{n-1}] = \frac{1}{n}\mathbf{L}[f(t)]$$

In the above theorem we have seen that the reduction of higher order to first order for the specific input for the function. Now we are going to see some another type of function which we have already used in property (iii), but here we are using different input for the function.

Theorem 5.2 Let $s > 0$. The higher order Laplace Transform of differentiable function is given by

$$\mathbf{L}_n[f'(t^n)] = \mathbf{L}[f'(t)] = s \cdot \mathbf{L}[f(t)] - f(0)$$

Proof: By the third property of Higher order Laplace transform i.e $\mathbf{L}_n[f'(t)] = ns \cdot \mathbf{L}_n[f(t)t^{n-1}] - f(0)$, we can say by using the theorem 5.1 that

$$\mathbf{L}_n[f'(t^n)] = ns \cdot \mathbf{L}_n[f(t^n)t^{n-1}] - f(0)$$
$$\mathbf{L}_n[f'(t^n)] = ns \cdot \mathbf{L}_n[f(t^n)t^{n-1}] - f(0) = s \cdot \mathrm{L}[f(t)] - f(0)$$
$$\therefore \mathbf{L}_n[f'(t^n)] = \mathbf{L}[f'(t)] = s \cdot \mathbf{L}[f(t)] - f(0)$$

Here also we can able to reduce the higher order into first order by using the specific input for the function. In properties of Higher Order Laplace Transform, we have used the $f'(t)$ but for reduction into first order we have used the function $f'(t^n)$. So for the reduction of higher into first order we need to use the specific inputs for the function.

Theorem 5.3 If $f^{(n)}(t^n) = \frac{d^n}{dt^n} f(t^n)$ then

$$\mathbf{L}_n[f^{(n)}(t^n)] = \mathbf{L}[f^{(n)}(t)] - s \cdot f^{(n-1)}(0)$$

Proof: From the definition of Laplace Transform,

$$\mathbf{L}_n[f^{(n)}(t^n)] = ns \cdot \mathbf{L}_n[f^{(n-1)}(t^n) \cdot t^{n-1}] - f^{(n-1)}(0) \quad \text{[By the (iii) property.]}$$
$$ns \cdot \mathbf{L}_n[f^{(n-1)}(t^n) \cdot t^{n-1}] - f^{(n-1)}(0) = \frac{ns}{n}\mathbf{L}[f^{(n-1)}(t)] - f^{(n-1)}(0) \quad \text{[By the theorem 5.2].}$$
$$s \cdot \left[s^{(n-1)}\mathbf{L}[f(t)] - s^{(n-2)}f(0) - s^{(n-3)}f'(0) - \cdots - f^{(n-1)}(0)\right] - f^{(n-1)}(0)$$
$$\Rightarrow s^{(n)}\mathbf{L}[f(t)] - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0) - sf^{(n-1)}(0)$$
$$\therefore \mathbf{L}_n[f^{(n)}(t^n)] = \mathbf{L}[f^{(n)}(t)] - s \cdot f^{(n-1)}(0)$$

So this would be the extended version of the previous theorem. Like this there are many functions we can use these above theorems to reduce its higher order to first order Laplace transform. **Property 5.4** Shifting Property:

(i) $\mathbf{L}_n[e^{-at^n}f(t)] = \mathbf{L}_n[f(t)]_{s\to s+a}$
(ii) $\mathbf{L}_n[e^{at^n}f(t)] = \mathbf{L}_n[f(t)]_{s\to s-a}$

Proof: The proof for the shifting property

$$\mathbf{L}_n[e^{-at^n}f(t)] = \int_0^\infty e^{-st^n}e^{-at^n}f(t)dt$$
$$= \int_0^\infty e^{-t^n(s+a)}f(t)dt$$
$$\therefore \mathbf{L}_n[e^{-at^n}f(t)] = \mathbf{L}_n[f(t)]_{s\to s+a}$$
$$\mathbf{L}_n[e^{at^n}f(t)] = \mathbf{L}_n[f(t)]_{s\to s-a}$$

Replace $-a$ as a in above property.

$$\mathbf{L}_n[e^{at^n}f(t)] = \int_0^\infty e^{-st^n}e^{at^n}f(t)dt = \int_0^\infty e^{-t^n(s-a)}f(t)dt$$
$$\therefore \mathbf{L}_n[e^{at^n}f(t)] = \mathbf{L}_n[f(t)]_{s\to s-a}$$

6 Applications and the Examples

In before sections we have approaches some formulas for the Higher Order Laplace Transform, with the help of the above theorems we are going to approaches many

2456-8686, vii(ii), 2023: 107-126
https://doi.org/10.26524/cm179

examples and applications.

6.1 Applications of function when $f(t) = t^8$

$$\mathbf{L}_2[t^8] = \int_0^\infty e^{-st^n} t^{(2\cdot 4)} = dt = 5.815 \quad s^{(-\frac{1}{2}-4)}$$

$$\mathbf{L}_n[t^{mn}] = \frac{(-1)^m \Gamma\left(\frac{1}{n}\right)\left(-\frac{1}{n}\right)! s^{\left(-\frac{1}{n}-m\right)}}{n\left(-\frac{1}{n}-m\right)!}$$

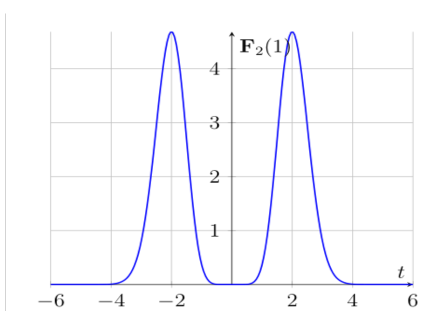
$$\Rightarrow \mathbf{L}_2[t^8] = \mathbf{L}_2[t^{2\cdot 4}] = \int_0^\infty e^{-st^2} t^{2\cdot 4} dt \quad [\text{here, } n = 2 \text{ and } m = 4]$$

$$\Rightarrow \mathbf{L}_2[t^8] = \frac{\Gamma(\frac{1}{2})(-\frac{1}{2})! \cdot s^{(-\frac{1}{2}-4)}}{2\cdot(-\frac{1}{2}-4)!} = 5.815 \quad s^{(-\frac{1}{2}-4)}$$

$$\therefore \mathbf{L}_2[t^8] = \int_0^\infty e^{-st^n} t^{(2\cdot 4)} = dt = 5.815 \cdot s^{(-\frac{1}{2}-4)}$$

In quantum mechanics they deal with the integral $\int_{-\infty}^{\infty} e^{-st^2} t^8 dt$, if we multiply by 2 in $\mathbf{L}_2[t^8]$ we get the value of $\int_{-\infty}^{\infty} e^{-st^2} t^8 dt$,which is equal to $11.63 \cdot s^{(-\frac{1}{2}-4)}$. The above integral took placce in the areas of mathematics such as; Probability theory and statistics, Quantum mechanics, signal processing, Control theory, Information theory. In quantum mechanics, the integral $\int_{-\infty}^{\infty} e^{-st^2} t^8 dt$ appear in the context of calculating higher order moments and correlation functions of quantum mechanics observables.

6.1.1 Applications to quantum mechanics

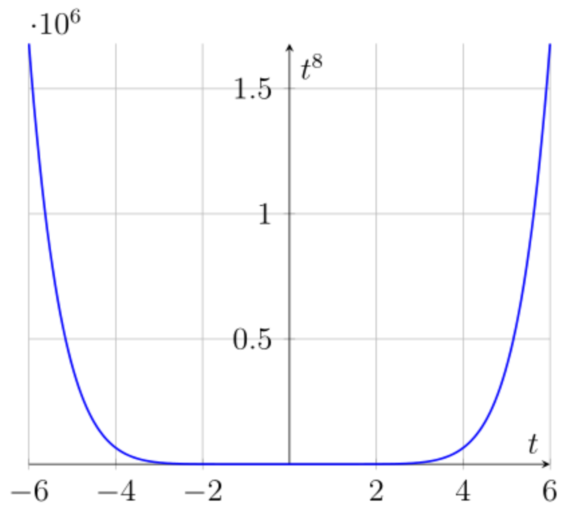


This is the graph of $e^{-t^2} t^8$, where $s = 1$. In quantum mechanics and many fields of

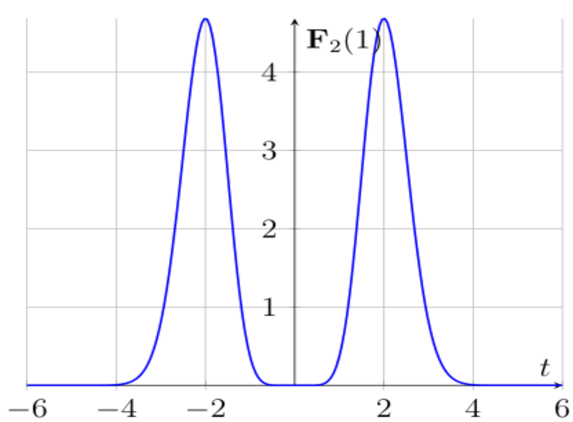
2456-8686, vii(ii), 2023: 107-126
https://doi.org/10.26524/cm179

mathematics we have to find the area from $-\infty$ to ∞, which is $2\mathbf{L}_2[t^8]$. The area for the function is 11.631728.

6.1.2 Application to the signal related



When the time domain of $f(t) = t^8$, and the graph of second order Laplace transform of t^8 would be,

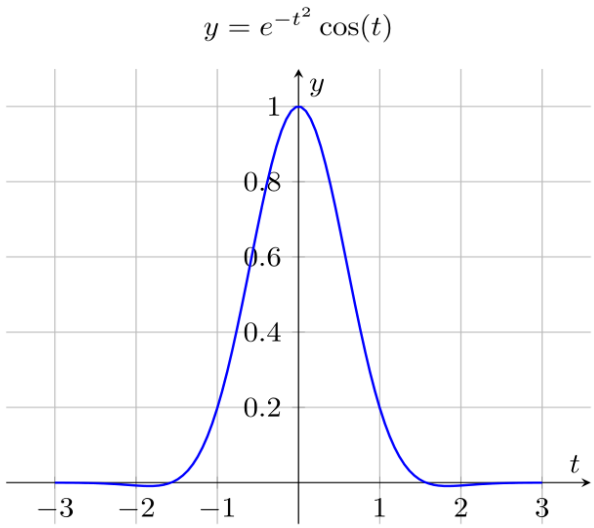


If we integrate in the limit of 0 to ∞ we get the value as 5.815. By taking the different values of s we get the different graphs of the function $f(t) = t^8$, by taking the corresponding values of s and if we substitue in the function $\mathbf{F}_2(s) = 5.815 \cdot s^{(-\frac{1}{2}-4)}$, we get the value of 2nd order laplace transform of t^8.

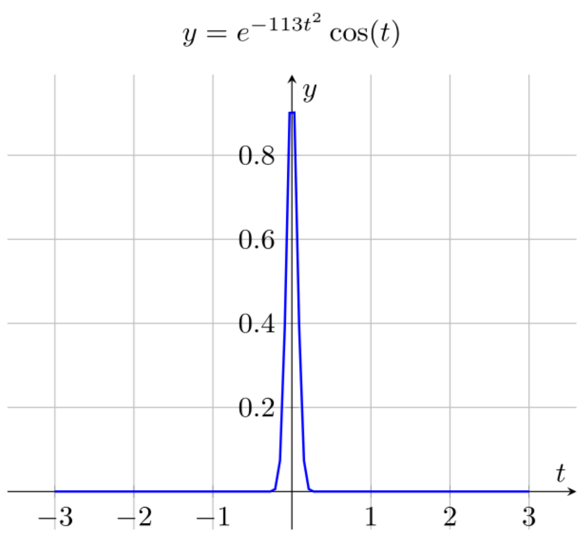
2456-8686, vii(ii), 2023: 107-126
https://doi.org/10.26524/cm179

6.2 Applications of function when $f(t) = cos(\alpha t)$

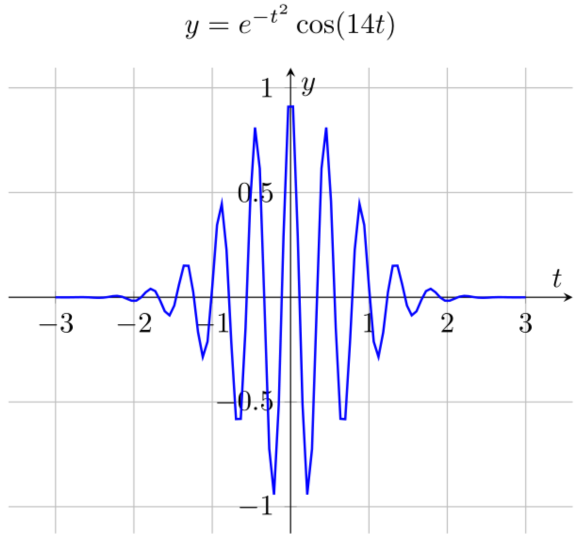
This function is very interesting function. To understand this lets see the graph of the function without the integral. The function is $e^{-t^2} cos(t)$.



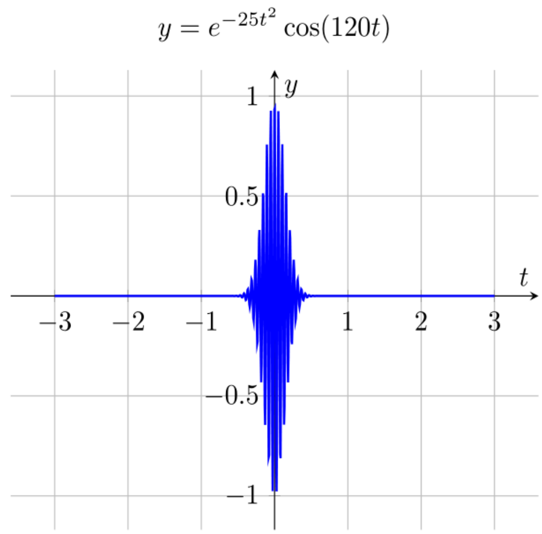
When the value of s is increasing, then the curve's width will decrease. To understand this let's take $s = 113$.



Since the doamin of the both graphs is $[-3, 3]$, we can see the curve's width have been decreased. When α value is greater than 1, then the curve will oscillate from -1 to 1 in the y-axis. To understand better let's take $\alpha = 14$.



Therefore, $\lim_{s\to\infty} e^{-st^2}cos(t)$ then the width of the curve of the graph would be decreasing. Similarily, $\lim_{\alpha\to\infty} e^{-t^2}cos(\alpha t)$ the number of oscillation of the curve of the graph will increase. If both the varibles is increasing the the both of the properties will the curve behave. To understand let's take $s = 25$ and $\alpha = 120$.



2456-8686, vii(ii), 2023: 107-126
https://doi.org/10.26524/cm179

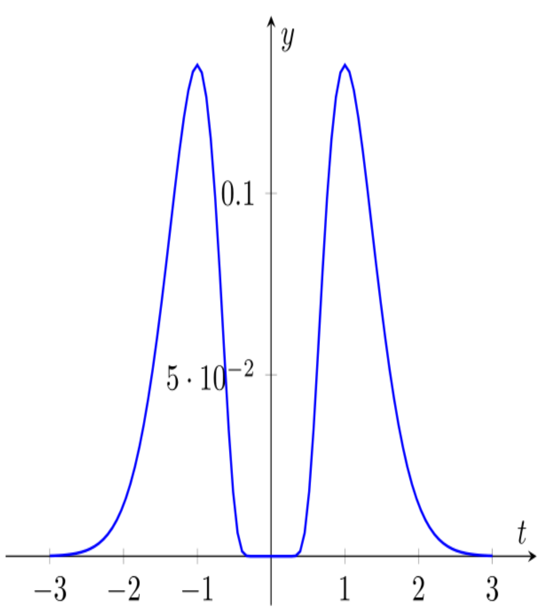
6.2.1 Applications to different fields

1. **Signal Processing and Fourier Transforms:**$\mathbf{L}_2[cos(\alpha t)]$ is a common form encountered in the context of Fourier Transforms, especially in the field of signal processing. Fourier Transforms are used to analyze functions in the frequency domain with real valued signals.
2. **Probability Theory and Statistics:** $\mathbf{L}_2[cos(\alpha t)]$ can be found in probability theory, particularly in the study of probability distributions. It might appear when dealing with the characteristic function of a probability distribution.
3. **Vibrations and Oscillations:** Systems exhibiting harmonic motion or oscillations can lead to integrals similar to the $\mathbf{L}_2[cos(\alpha t)]$. Mechanical and electrical systems, as well as problems in acoustics, can involve equations with similar forms.

These are some of the applications, by taking corresponding values of s and α we can able to find the solutions of the $\mathbf{L}_2[cos(\alpha t)]$.

6.3 Applications of function when $f(t) = e^{-\frac{1}{t^2}}$

When the $f(t) = e^{-\frac{1}{t^2}}$, $2[\mathbf{L}_2[e^{-\frac{1}{t^2}}]]_{s=1}$ gives more application. The graph of the function would be,



In many fields, they usually find the integral $\int\limits_{-\infty}^{\infty} e^{-t^2}e^{-\frac{1}{t^2}}dt$, which can be expressed as $2[\mathbf{L}_2[e^{-\frac{1}{t^2}}]]_{s=1}$.

$$2[\mathbf{L}_2[e^{-\frac{1}{t^2}}]]_{s=1} = \int\limits_{-\infty}^{\infty} e^{-t^2}e^{-\frac{1}{t^2}}dt = \frac{\sqrt{\pi}}{e^2}$$

To arrive the above result,

$$2\mathbf{L}_2[e^{-\frac{1}{t^2}}]_{s=1} = \int\limits_{-\infty}^{\infty} e^{-t^2}e^{-\frac{1}{t^2}}dt = 2\int\limits_{0}^{\infty} e^{-t^2}e^{-\frac{1}{t^2}}dt$$

$$2\mathbf{L}_2[e^{-\frac{1}{t^2}}]_{s=1} = \frac{2}{e^2}\int\limits_{0}^{\infty} e^{-\left(t-\frac{1}{t}\right)^2}dt \quad \left[\text{Because,} \quad e^{-t^2}e^{-\frac{1}{t^2}} = \frac{1}{e^2}e^{-\left(t-\frac{1}{t}\right)^2}\right]$$

Let, $\int\limits_{0}^{\infty} e^{-\left(t-\frac{1}{t}\right)^2}dt = I \quad \dots (1)$

Let, $\frac{1}{t} = u \Rightarrow t = \frac{1}{u} \Rightarrow \frac{du}{dt} = -\frac{1}{t^2} \Rightarrow dt = -\frac{1}{u^2}du,$ when, $t = 0 \Rightarrow u = \infty$
when, $t = \infty \Rightarrow u = 0$

$$\Rightarrow 2\mathbf{L}_2[e^{-\frac{1}{t^2}}]_{s=1} = -2\int\limits_{\infty}^{0} e^{-\left(u-\frac{1}{u}\right)^2}\frac{1}{u^2}du$$

$$\Rightarrow 2\mathbf{L}_2[e^{-\frac{1}{t^2}}]_{s=1} = 2\int\limits_{0}^{\infty} e^{-\left(u-\frac{1}{u}\right)^2}\frac{1}{u^2}du \quad \text{[Replace the } u \text{ as } t\text{, because } u \text{ is a dummy variable]}$$

$$\Rightarrow 2\mathbf{L}_2[e^{-\frac{1}{t^2}}]_{s=1} = 2\int\limits_{0}^{\infty} e^{\left(t-\frac{1}{t}\right)^2}\frac{1}{t^2}dt = I \dots (2) \quad \text{[Now let's add eqn (1) and (2)]}$$

$$\Rightarrow 2I = \frac{2}{e^2}\int\limits_{0}^{\infty} e^{(t-\frac{1}{t})^2}\left(1+\frac{1}{t^2}\right)dt \quad \text{Let, } \left(t-\frac{1}{t}\right) = \tau \Rightarrow d\tau = \left(1+\frac{1}{t}\right)dt$$

When $t = 0 \Rightarrow \tau = -\infty$, similarly, $t = \infty \Rightarrow \tau = \infty$

$$\Rightarrow I = \frac{1}{e^2}\int\limits_{-\infty}^{\infty} e^{-\tau^2}d\tau = \frac{\sqrt{\pi}}{e^2} \quad \text{[by Gaussian integral]}$$

The maximum value of the function when $s = 1$. That is $\frac{\sqrt{\pi}}{e^2}$.

6.4 Applications to the Fourier series

We have come across about the Fourier series, now we are going to deal the application of Higher order Laplace transform to the Fourier series.

Theorem 6.1 If $f(x)$ can be expressed as the Fourier series, and the $f(x)$ will converge when we compute the integral with the limit from zero to infinity then,

$$\int_{0}^{\infty} (f(x) - A_0)\, dx = \sum_{n=1}^{\infty} \frac{B_n L}{n\pi}$$

Proof: $f(x) = A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow (f(x) - A_0) = \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right)$$

Integrate on both sides with the limit from 0 to ∞

$$\Rightarrow \int_{0}^{\infty} (f(x) - A_0)\, dx = \sum_{n=1}^{\infty} A_n \int_{0}^{\infty} cos\left(\frac{n\pi x}{L}\right) dx + \sum_{n=1}^{\infty} B_n \int_{0}^{\infty} sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow \int_{0}^{\infty} (f(x) - A_0)\, dx = \sum_{n=1}^{\infty} \frac{B_n L}{n\pi} \qquad \text{[from Theorem 4.2 and 4.3]}$$

7. Conclusion

By taking the different values of n, the function w.r.t $'t'$ is trasformed into the function w.r.t $'s'$. The best property of the Higher order Laplace transform is fast converges comparing to the ordinary Laplace transform. The Higher order Laplace transform have many applications for different values of n, mainly it has more application in the filed of quantum mechanics. In quantum field the value of n will be in fraction, so n can be any number except complex numbers. In summary, the nth order Laplace transform of a function $f(t)$ is defined as $\mathbf{L}_n[f(t)] = \int_0^\infty e^{-st^n} f(t)\, dt$, denoted as $\mathbf{F}_n(s)$, where n represents the order of the transform. This generalized transform extends the concept of the ordinary or first-order Laplace transform ($\mathbf{L}_1[f(t)]$) for functions defined over certain values of t. If the integral $\int_0^\infty e^{-st^n} f(t)\, dt$ exists for specific values of n, the nth order Laplace transform provides a powerful tool in the study and analysis of various mathematical and engineering problems.

2456-8686, vii(ii), 2023: 107-126
https://doi.org/10.26524/cm179

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