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Stability of Mixed AQ_2CQ_4 - Functional Equation In Fuzzy Banach Spaces

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Abstract

In this research article, we discuss the stabilities of mixed AQ_2CQ_4 functional equation in Fuzzy Banach spaces with the help of the Hyers method.

Key words: Additive functional equation, quadratic functional equation, cubic functional equation, quartic functional equations, mixed type AQCQ functional equation, Generalized Hyers-Ulam stability, Fuzzy Banach space.

AMS classification: 39B52, 32B72, 32B82

1. Introduction

In 1940, the Stability of Functional Equation has been studied with the help of mapping idealogy by Ulam who used the famous Ulam Stability problem [41], Then further it was developed by Hyers [19] in 1941 for additive mappings. Further T. Aoki [3], Th.M. Rassias [36], P. Gavruta [17] and Rassias [38] generalized Hyers result in various settings.

Several types of mixed type functional equations in various normed spaces are introduced and investigated in [6, 7, 10, 13, 14, 30, 32, 34, 35, 40]. Side by side other types of AQCQ functional equations were introduced and its stability were discussed in [8, 9, 18, 23, 28, 29, 33, 39].

In this research article, the authors prove the stabilities of mixed AQ_2CQ_4

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functional equation

$$\sum_{k=1}^{2}\Big(f(tv_k+w_k)+f(w_k-tv_k)\Big)$$
$$=t^2\sum_{k=1}^{2}\Big(f(v_k+w_k)+f(w_k-v_k)\Big)+2\left(1-t^2\right)\sum_{k=1}^{2}\Big(f(w_k)\Big)$$
$$+\frac{\left(t^4-t^2\right)}{12}\sum_{k=1}^{2}\Big(f(2v_k)+f(-2v_k)-4f(v_k)-4f(-v_k)\Big) \quad (1)$$

where t is a real number with $t\neq 0,\pm1$ in Fuzzy Banach spaces with the help of Hyers methods. The above functional equation is different from the previous investigations and we trigger the stability by different substitutions.

Now, we recall the definitions and notations on fuzzy normed spaces given in [11] and [24, 25, 26, 27].

Definition 1.1 [24, 25, 26, 27] Let X be a real linear space. A function $N : X\times\mathbb{R}\longrightarrow [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,
$(FNS1)$ $N(x,c)=0$ for $c\geq 0$;
$(FNS2)$ $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
$(FNS3)$ $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$;
$(FNS4)$ $N(x+y,s+t)\geq min\{N(x,s),N(y,t)\}$;
$(FNS5)$ $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty}N(x,t)=1$;
$(FNS6)$ for $x\neq 0, N(x,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t.

Example 1.2 [24, 25, 26, 27] The fuzzy norm

$$N(x,t)=\begin{cases}\dfrac{t}{t+\|x\|}, & t>0,\ x\in X,\\ 0, & t\geq 0,\ x\in X\end{cases}$$

fuzzy normed linear space.

Definition 1.3 [24, 25, 26, 27] Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and one can denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 1.4 [24, 25, 26, 27] A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, one can have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

Definition 1.5 [24, 25, 26, 27] Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

The stability of various functional equations in fuzzy normed spaces were given in [4, 5, 7, 8, 9, 10, 14, 35, 40, 24, 25, 26, 27] and references cited there in.

For proving stability results, let us take $H_3, (H_1,N)$ and (H_2,N') are linear space, fuzzy normed space and fuzzy Banach space. Define a mapping $f : H_1 \longrightarrow H_2$ by

$$\begin{aligned} F^{1234}\Big(v_1,w_1,v_2,w_2\Big) = &\sum_{k=1}^{2}\Big(f\left(tv_k+w_k\right)+f\left(w_k-tv_k\right)\Big) \\ &-t^2\sum_{k=1}^{2}\Big(f\left(v_k+w_k\right)+f\left(w_k-v_k\right)\Big)-2\left(1-t^2\right)\sum_{k=1}^{2}\Big(f\left(w_k\right)\Big) \\ &-\frac{\left(t^4-t^2\right)}{12}\sum_{k=1}^{2}\Big(f\left(2v_k\right)+f\left(-2v_k\right)-4f\left(v_k\right)-4f\left(-v_k\right)\Big) \end{aligned}$$

for all $v_1,w_1,v_2,w_2 \in H_1$.

2. Fuzzy Stability Theorem: Odd Case

Theorem 2.1 Assume $p = \pm 1$ and let $f : H_1 \longrightarrow H_2$ be an odd mapping satisfying the functional inequality

$$\mathscr{F}\ \Big(F^{1234}\Big(v_1,\ \ w_1,\ \ v_2,\ \ w_2\Big),\mathscr{K}\Big) \geq \mathscr{F}'\ \Big(\mathscr{Z}_O\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\Big) \tag{2}$$

where $\mathscr{Z}_O, \mathscr{Z}_O^{ODD} : H_1^4 \longrightarrow H_3$ are functions with the conditions

$$\lim_{q\to\infty} \mathscr{F}' \left(\mathscr{Z}_O\left(2^{pq}v_1, 2^{pq}w_1, 2^{pq}v_2, 2^{pq}w_2\right), 2^{pq}\,\mathscr{K}\right) = 1 \tag{3}$$

$$\mathscr{F}' \left(\mathscr{Z}_O^{ODD}\left(2^p v, 2^p v, 2^p v, 2^p v\right), \mathscr{K}\right) \geq \mathscr{F}' \left(r^p\,\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right) \tag{4}$$

for all $v_1, w_1, v_2, w_2, v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{2}\right)^p < 1$. Then there exists a unique additive mapping $\mathscr{A} : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and

$$\mathscr{F}\left(f_1(v) - \mathscr{A}(v), T_O\,\mathscr{K}\right) = \mathscr{F}\left(f(2v) - 8f(v) - \mathscr{A}(v), T_O\,\mathscr{K}\right) \geq \mathscr{F}' \left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right) \tag{5}$$

where $\mathscr{Z}_O^{ODD}\left(v,v,v,v\right)$, $\mathscr{A}(v)$ and T_O are defined by

$$\begin{aligned}
&\mathscr{F}' \left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right)\\
&= \min\Big\{\mathscr{F}' \left(\mathscr{Z}_O\left(v,v,v,v\right), 2(1-t^2)\mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(v,2v,v,2v\right), t^2\mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(2v,v,2v,v\right), \mathscr{K}\right),\\
&\qquad \mathscr{F}' \left(\mathscr{Z}_O\left(v,(1+t)v,v,(1+t)v\right), \mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(v,(1-t)v,v,(1-t)v\right), \mathscr{K}\right),\\
&\qquad \mathscr{F}' \left(\mathscr{Z}_O\left(2v,2v,2v,2v\right), t^2\mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(2v,v,2v,v\right), 2(1-t^2)\mathscr{K}\right),\\
&\qquad \mathscr{F}' \left(\mathscr{Z}_O\left(v,v,v,v\right), \mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(3v,v,3v,v\right), \mathscr{K}\right),\\
&\qquad \mathscr{F}' \left(\mathscr{Z}_O\left(v,(1+2t)v,v,(1+2t)v\right), \mathscr{K}\right), \mathscr{F}' \left(\mathscr{Z}_O\left(v,(1-2t)v,v,(1-2t)v\right), \mathscr{K}\right)\Big\}
\end{aligned} \tag{6}$$

$$\lim_{q\to\infty} \mathscr{F}\left(\mathscr{A}(v) - \frac{f_1(2^{pq}v)}{2^{pq}}, \mathscr{K}\right) = 1; \quad T_O = \left(\frac{(16-3t^2)}{2(t^4-t^2)}\right) \tag{7}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

Proof: For oddness of f in (2), one can get

$$\begin{aligned}
&\mathscr{F}\Bigg(\sum_{k=1}^{2}\left(f(tv_k+w_k)+f(w_k-tv_k)\right) - t^2\sum_{k=1}^{2}\left(f(v_k+w_k)+f(w_k-v_k)\right)\\
&\qquad -2\left(1-t^2\right)\sum_{k=1}^{2}\left(f(w_k)\right), \mathscr{K}\Bigg) \geq \mathscr{F}' \left(\mathscr{Z}_O\left(v_1,w_1,v_2,w_2\right), \mathscr{K}\right)
\end{aligned} \tag{8}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$. Replacing (v_1, w_1, v_2, w_2) by (v, v, v, v) in (8), one can reach

$$\mathscr{F}\ \left(2f((1+t)v) + 2f((1-t)v) - 2t^2 f(2v) - 4\left(1-t^2\right) f(v), \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v, v, v, v\Big), \mathscr{K}\right) \tag{9}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (9), one can obtain

$$\begin{aligned}\mathscr{F}\ \big(4(1-t^2)f((1+t)v) + 4(1-t^2)f((1-t)v) - 4(1-t^2)t^2 f(2v) \\ -8(1-t^2)^2 f(v), 2(1-t^2)\mathscr{K}\big) \geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v, v, v, v\Big), \mathscr{K}\right)\end{aligned} \tag{10}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Replacing (v_1, w_1, v_2, w_2) by $(v, 2v, v, 2v)$ in (8), one can have

$$\begin{aligned}\mathscr{F}\ \left(2f((2-t)v) + 2f((2+t)v) - 2t^2 f(3v) - 2t^2 f(v) - 4\left(1-t^2\right) f(2v), \mathscr{K}\right) \\ \geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v, 2v, v, 2v\Big), \mathscr{K}\right)\end{aligned} \tag{11}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (11), one can get

$$\begin{aligned}\mathscr{F}\ \left(2t^2 f((2-t)v) + 2t^2 f((2+t)v) - 2t^4 f(3v) - 2t^4 f(v) - 4t^2\left(1-t^2\right) f(2v), t^2\mathscr{K}\right) \\ \geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v, 2v, v, 2v\Big), \mathscr{K}\right)\end{aligned} \tag{12}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Interchanging

$$\Big(v_1, w_1, v_2, w_2\Big) \ \text{by} \ \Big(2v, v, 2v, v\Big); \Big(v, (1+t)v, v, (1+t)v\Big); \Big(v, (1-t)v, v, (1-t)v\Big)$$

in (8), one can observe the inequalities

$$\begin{aligned}\mathscr{F}\ \left(2f((1+2t)v) + 2f((1-2t)v) - 2t^2 f(3v) + 2t^2 f(v) - 4\left(1-t^2\right) f(v), \mathscr{K}\right) \\ \geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(2v, v, 2v, v\Big), \mathscr{K}\right)\end{aligned} \tag{13}$$

$$\mathscr{F}\ \left(2f\left((1+2t)v\right)+2f\left(v\right)-2t^2f\left((2+t)v\right)-2t^2f\left(tv\right)-4\left(1-t^2\right)f\left((1+t)v\right),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1+t)v,v,(1+t)v\Big),\mathscr{K}\right) \qquad (14)$$

$$\mathscr{F}\ \left(2f\left((1-2t)v\right)+2f\left(v\right)-2t^2f\left((2-t)v\right)+2t^2f\left(tv\right)-4\left(1-t^2\right)f\left((1-t)v\right),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1-t)v,v,(1-t)v\Big),\mathscr{K}\right) \qquad (15)$$

for all $v\in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS4)$ and (10), (12), (13), (14), (15), one can find that

$$\mathscr{F}\ \left(5f\left(v\right)+2(t^4-t^2)\Big(f\left(3v\right)-4f\left(2v\right)\Big),\Big(2(1-t^2)+t^2+3\Big)\mathscr{K}\right)$$
$$\geq \min\Big\{\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,v,v,v\Big),2(1-t^2)\mathscr{K}\right),\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,2v,v,2v\Big),t^2\mathscr{K}\right),$$
$$\mathscr{F}'\ \left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),\mathscr{K}\right),\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1+t)v,v,(1+t)v\Big),\mathscr{K}\right),$$
$$\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1-t)v,v,(1-t)v\Big),\mathscr{K}\right)\Big\} \qquad (16)$$

for all $v\in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (16), one can arrive

$$\mathscr{F}\ \left(4(t^4-t^2)\Big(f\left(3v\right)-4f\left(2v\right)+5f\left(v\right)\Big),\Big(4(1-t^2)+2t^2+6\Big)\mathscr{K}\right)$$
$$\geq \min\Big\{\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,v,v,v\Big),2(1-t^2)\mathscr{K}\right),\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,2v,v,2v\Big),t^2\mathscr{K}\right),$$
$$\mathscr{F}'\ \left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),\mathscr{K}\right),\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1+t)v,v,(1+t)v\Big),\mathscr{K}\right),$$
$$\mathscr{F}'\ \left(\mathscr{Z}_O\Big(v,(1-t)v,v,(1-t)v\Big),\mathscr{K}\right)\Big\} \qquad (17)$$

for all $v\in H_1$ and all $\mathscr{K}>0$. Replacing v by $2v$ in (9), one can reach

$$\mathscr{F}\ \left(2f\left(2(1+t)v\right)+2f\left(2(1-t)v\right)-2t^2f\left(4v\right)-4\left(1-t^2\right)f\left(2v\right),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(2v,2v,2v,2v\Big),\mathscr{K}\right) \qquad (18)$$

for all $v\in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (18), one can obtain

$$\mathscr{F}\ \left(2t^2f\left(2(1-t)v\right)+2t^2f\left(2(1+t)v\right)-2t^4f\left(4v\right)-4\left(t^2-t^4\right)f\left(2v\right),t^2\mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_O\Big(2v,2v,2v,2v\Big),\mathscr{K}\right) \qquad (19)$$

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for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (13), one can get

$$\mathscr{F}\left(4(1-t^2)f((1+2t)v)+4(1-t^2)f((1-2t)v)-4(t^2-t^4)f(3v)\right.$$
$$\left.+4(t^2-t^4)f(v)-8(1-t^2)^2f(v),2(1-t^2)\mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_O\left(2v,v,2v,v\right),\mathscr{K}\right) \quad (20)$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Replacing

$$\left(v_1,w_1,v_2,w_2\right) \text{ by } \left(3v,v,3v,v\right);\left(v,(1+2t)v,v,(1+2t)v\right);\left(v,(1-2t)v,v,(1-2t)v\right)$$

in (8), one can observe the inequalities

$$\mathscr{F}\left(2f((1+3t)v)+2f((1-3\,t)v)-2t^2f(4v)+2t^2f(2v)-4\left(1-t^2\right)f(v),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\left(\mathscr{Z}_O\left(3v,v,3v,v\right),\mathscr{K}\right) \quad (21)$$

$$\mathscr{F}\left(2f((1+3t)\,v)+2f((1+t)\,v)-2t^2f(2(1+t)v)-2t^2f(2tv)-4\left(1-t^2\right)f((1+2t)v),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\left(\mathscr{Z}_O\left(v,(1+2t)v,v,(1+2t)v\right),\mathscr{K}\right) \quad (22)$$

$$\mathscr{F}\left(2f((1-3t)v)+2f((1-t)v)-2t^2f(2(1-t)v)+2t^2f(2tv)-4\left(1-t^2\right)f((1-2t)v),\mathscr{K}\right)$$
$$\geq \mathscr{F}'\left(\mathscr{Z}_O\left(v,(1-2t)v,v,(1-2t)v\right),\mathscr{K}\right) \quad (23)$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS4)$ and (9), (19), (20), (21), (22), (23), one can arrive

$$\mathscr{F}\left(2(t^4-t^2)\left(f(4v)-2f(3v)-2f(2v)+6f(v)\right),\left(t^2+2(1-t^2)+4\right)\mathscr{K}\right)$$
$$\geq \min\Big\{\mathscr{F}'\left(\mathscr{Z}_O\left(2v,2v,2v,2v\right),t^2\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\left(2v,v,2v,v\right),2(1-t^2)\mathscr{K}\right),$$
$$\mathscr{F}'\left(\mathscr{Z}_O\left(v,v,v,v\right),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\left(3v,v,3v,v\right),\mathscr{K}\right),$$
$$\mathscr{F}'\left(\mathscr{Z}_O\left(v,(1+2t)v,v,(1+2t)v\right),\mathscr{K}\right),$$
$$\mathscr{F}'\left(\mathscr{Z}_O\left(v,(1-2t)v,v,(1-2t)v\right),\mathscr{K}\right)\Big\} \quad (24)$$

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for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS4)$ and (17), (24), one can have

$$\begin{aligned}
&\mathscr{F}\left(2(t^4-t^2)\Big(f(4v)-10f(2v)+16f(v)\Big),\Big(16-3t^2\Big)\mathscr{K}\right)\\
&\geq \min\Big\{\mathscr{F}'\left(\mathscr{Z}_O\Big(v,v,v,v\Big),2(1-t^2)\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,2v,v,2v\Big),t^2\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1+t)v,v,(1+t)v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1-t)v,v,(1-t)v\Big),\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(2v,2v,2v,2v\Big),t^2\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),2(1-t^2)\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,v,v,v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(3v,v,3v,v\Big),\mathscr{K}\right),\\
&\quad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1+2t)v,v,(1+2t)v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1-2t)v,v,(1-2t)v\Big),\mathscr{K}\right)\Big\}
\end{aligned} \tag{25}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS4)$ and (25), one can find

$$\begin{aligned}
&\mathscr{F}\left(f(4v)-10f(2v)+16f(v),\left(\frac{(16-3t^2)}{2(t^4-t^2)}\right)\mathscr{K}\right)\\
&\geq \min\Big\{\mathscr{F}'\left(\mathscr{Z}_O\Big(v,v,v,v\Big),2(1-t^2)\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,2v,v,2v\Big),t^2\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1+t)v,v,(1+t)v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1-t)v,v,(1-t)v\Big),\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(2v,2v,2v,2v\Big),t^2\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(2v,v,2v,v\Big),2(1-t^2)\mathscr{K}\right),\\
&\qquad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,v,v,v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(3v,v,3v,v\Big),\mathscr{K}\right),\\
&\quad \mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1+2t)v,v,(1+2t)v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_O\Big(v,(1-2t)v,v,(1-2t)v\Big),\mathscr{K}\right)\Big\}\\
&= \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\Big(v,v,v,v\Big),\mathscr{K}\right)
\end{aligned} \tag{26}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Set $T_O = \left(\frac{(16-3t^2)}{2(t^4-t^2)}\right)$ in (26), one can see

$$\mathscr{F}\left(f(4v)-10f(2v)+16f(v),T_O\,\mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\Big(v,v,v,v\Big),\mathscr{K}\right) \tag{27}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Define a function

$$f_1(v) = f(2v) - 8f(v) \tag{28}$$

for all $v \in H_1$. Using (28) in (27), one can achieve

$$\mathscr{F}\left(f_1(2v)-2f_1(v), T_O\,\mathscr{K}\right)=\mathscr{F}\left(\left(f(4v)-8f(2v)\right)-2\left(f(2v)-8f(v)\right), T_O\,\mathscr{K}\right)$$
$$\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,\ v,\ v,\ v\right),\mathscr{K}\right) \tag{29}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (29), one can have

$$\mathscr{F}\left(\frac{f_1(2v)}{2}-f_1(v),\frac{T_O\,\mathscr{K}}{2}\right)\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right),\mathscr{K}\right) \tag{30}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. Setting v by $2^q v$ in (30), one can obtain

$$\mathscr{F}\left(\frac{f_1(2^{q+1}v)}{2}-f_1(2^q v),\frac{T_O\,\mathscr{K}}{2}\right)\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(2^q v,2^q v,2^q v,2^q v\right),\mathscr{K}\right) \tag{31}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. Using (4), $(FNS3)$ in (31), one can arrive

$$\mathscr{F}\left(\frac{f_1(2^{q+1}v)}{2}-f_1(2^q v),\frac{T_O\,\mathscr{K}}{2}\right)\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right),\frac{\mathscr{K}}{r^q}\right) \tag{32}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (32), one can see

$$\mathscr{F}\left(\frac{f_1(2^{q+1}v)}{2^{(q+1)}}-\frac{f_1(2^q v)}{2^q},\frac{T_O\,\mathscr{K}}{2\cdot 2^q}\right)\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right),\frac{\mathscr{K}}{r^q}\right) \tag{33}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. Putting $\mathscr{K}$ by $\mathscr{K}\ r^q$ in (33), one can get

$$\mathscr{F}\left(\frac{f_1(2^{q+1}v)}{2^{(q+1)}}-\frac{f_1(2^q v)}{2^q},\frac{T_O\,\mathscr{K}}{2}\cdot\left(\frac{r}{2}\right)^q\right)\geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right),\mathscr{K}\right) \tag{34}$$

for all $v \in H_1$ and all $\mathscr{K}>0$. One can verify that

$$\frac{f_1(2^q v)}{2^q}-f_1(v)=\sum_{s=0}^{q-1}\left[\frac{f_1(2^{s+1}v)}{2^{(s+1)}}-\frac{f_1(2^s v)}{2^s}\right] \tag{35}$$

for all $v \in H_1$. From (34) and (35), one can have

$$\begin{aligned}
&\mathscr{F}\left(\frac{f_1(2^q v)}{2^q} - f_1(v), \frac{T_O\ \mathscr{K}}{2} \cdot \sum_{s=0}^{q-1}\left(\frac{r}{2}\right)^s\right) \\
&\quad \geq \min\left\{\bigcup_{s=0}^{q-1} \mathscr{F}\left(\frac{f_1(2^{s+1}v)}{2^{(s+1)}} - \frac{f_1(2^s v)}{2^s}, \frac{T_O\ \mathscr{K}}{2} \cdot \left(\frac{r}{2}\right)^d\right)\right\} \\
&\quad \geq \min\bigcup_{s=0}^{q-1}\left\{\mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right)\right\} = \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right)
\end{aligned} \tag{36}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Substituting v by $2^{q'}v$ in (36) and using (4), $(FNS3)$ and changing $\mathscr{K}$ by $r^{q'}\mathscr{K}$, one can obtain

$$\mathscr{F}\left(\frac{f_1(2^{q+q'}v)}{2^{(q+q')}} - \frac{f_1(2^{q'}v)}{2^{q'}}, \frac{T_O\ \mathscr{K}}{2} \cdot \sum_{s=q'}^{q+q'-1}\left(\frac{r}{2}\right)^s\right) \geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right) \tag{37}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$ and all $q' > q \geq 0$. It follows from (37), one can get

$$\mathscr{F}\left(\frac{f_1(2^{q+q'}v)}{2^{(q+q')}} - \frac{f_1(2^{q'}v)}{2^{q'}}, T_O\ \mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \frac{\mathscr{K}}{\frac{1}{2} \cdot \sum_{s=q'}^{q+q'-1}\left(\frac{r}{2}\right)^s}\right) \tag{38}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. By data

$$0 < r < 2 \quad and \quad \sum_{s=0}^{q}\left(\frac{r}{2}\right)^s < \infty,$$

and $(FNS5)$ implies that

$$\left\{\frac{f_1(2^q v)}{2^q}\right\} \ \textit{is Cauchy sequence in} \ (H_2, N').$$

Since (H_2, N') is complete and this sequence converges to some point $\mathscr{A} \in H_2$. So one can define a mapping $\mathscr{A} : H_1 \longrightarrow H_2$ by

$$\lim_{q\to\infty} \mathscr{F}\left(\mathscr{A}(v) - \frac{f_1(2^q v)}{2^q}, \mathscr{K}\right) = 1 \tag{39}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Taking $q' = 0$ and $q \to \infty$ in (38), one can get

$$\mathscr{F}\left(\mathscr{A}(v) - f_1(v), T_O\,\mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}(2-r)\right)$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. To prove $\mathscr{A}$ satisfies the (1), substituting (v_1, w_1, v_2, w_2) by $(2^q v_1, 2^q w_1, 2^q v_2, 2^q w_2)$ in (4), one can obtain

$$\begin{aligned}\mathscr{F}\left(\mathscr{A}^{1234}\left(v_1,w_1,v_2,w_2\right),\mathscr{K}\right) &= \mathscr{F}\left(\frac{1}{2^q}F^{1234}\left(2^q v_1, 2^q w_1, 2^q v_2, 2^q w_2\right),\mathscr{K}\right)\\ &\geq \mathscr{F}'\left(\mathscr{Z}_O\left(2^q v_1, 2^q w_1, 2^q v_2, 2^q w_2\right), 2^q\,\mathscr{K}\right) \end{aligned} \quad (40)$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$. Now,

$$\begin{aligned}&\mathscr{F}\Bigg(\sum_{k=1}^{2}\left(\mathscr{A}(tv_k+w_k)+\mathscr{A}(w_k-tv_k)\right)-t^2\sum_{k=1}^{2}\left(\mathscr{A}(v_k+w_k)+\mathscr{A}(w_k-v_k)\right)\\
&-2\left(1-t^2\right)\sum_{k=1}^{2}\left(\mathscr{A}(w_k)\right)-\frac{(t^4-t^2)}{12}\sum_{k=1}^{2}\left(\mathscr{A}(2v_k)+\mathscr{A}(-2v_k)-4\mathscr{A}(v_k)-4\mathscr{A}(-v_k)\right),\ 5\mathscr{K}\Bigg)\\
&\geq \min\Bigg\{\mathscr{F}\Bigg(\sum_{k=1}^{2}\left(\mathscr{A}(tv_k+w_k)+\mathscr{A}(w_k-tv_k)\right);\\
&\qquad -\frac{1}{2^q}\sum_{k=1}^{2}\left(f(t2^q v_k+2^q w_k)+f(2^q w_k-t2^q v_k)\right),\mathscr{K}\Bigg),\\
&\quad \mathscr{F}\Bigg(-t^2\sum_{k=1}^{2}\left(\mathscr{A}(v_k+w_k)+\mathscr{A}(w_k-v_k)\right);\\
&\qquad +\frac{t^2}{2^q}\sum_{k=1}^{2}\left(f(2^q v_k+2^q w_k)+f(2^q w_k-2^q v_k)\right),\mathscr{K}\Bigg),\\
&\quad \mathscr{F}\Bigg(-2\left(1-t^2\right)\sum_{k=1}^{2}\left(\mathscr{A}(w_k)\right)+\frac{2\left(1-t^2\right)}{2^q}\sum_{k=1}^{2}\left(f(2^q w_k)\right),\mathscr{K}\Bigg),\end{aligned}$$

$$\begin{aligned}\mathscr{F}\ &\Bigg(-\frac{(t^4-t^2)}{12}\sum_{k=1}^{2}\Big(\mathscr{A}(2v_k)+\mathscr{A}(-2v_k)-4\mathscr{A}(v_k)-4\mathscr{A}(-v_k)\Big)\\&+\frac{1}{2^q}\frac{(t^4-t^2)}{12}\sum_{k=1}^{2}\Big(f(2\cdot 2^q v_k)+f(-2\cdot 2^q v_k)-4f(2^q v_k)-4f(-2^q v_k)\Big),\mathscr{K}\Bigg),\\ \mathscr{F}\ &\Bigg(\frac{1}{2^q}\sum_{k=1}^{2}\Big(f(t2^q v_k+2^q w_k)+f(2^q w_k-t2^q v_k)\Big)\\&-\frac{t^2}{2^q}\sum_{k=1}^{2}\Big(f(2^q v_k+2^q w_k)+f(2^q w_k-2^q v_k)\Big)-\frac{2(1-t^2)}{2^q}\sum_{k=1}^{2}\Big(f(2^q w_k)\Big)\\&-\frac{1}{2^q}\frac{(t^4-t^2)}{12}\sum_{k=1}^{2}\Big(f(2\cdot 2^q v_k)+f(-2\cdot 2^q v_k)-4f(2^q v_k)-4f(-2^q v_k)\Big),\mathscr{K}\Bigg)\Bigg\}\end{aligned} \tag{41}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$. Using (39), (40), $(FNS5)$ in (41), one can reach

$$\begin{aligned}&\mathscr{F}\ \Bigg(\sum_{k=1}^{2}\Big(\mathscr{A}(tv_k+w_k)+\mathscr{A}(w_k-tv_k)\Big)-t^2\sum_{k=1}^{2}\Big(\mathscr{A}(v_k+w_k)+\mathscr{A}(w_k-v_k)\Big)\\&-2(1-t^2)\sum_{k=1}^{2}\Big(\mathscr{A}(w_k)\Big)-\frac{(t^4-t^2)}{12}\sum_{k=1}^{2}\Big(\mathscr{A}(2v_k)+\mathscr{A}(-2v_k)-4\mathscr{A}(v_k)-4\mathscr{A}(-v_k)\Big),\ 5\mathscr{K}\Bigg)\\&\geq \min\Big\{1,1,1,1,\mathscr{F}'\ \Big(\mathscr{Z}_O\Big(2^q v_1, 2^q w_1, 2^q v_2, 2^q w_2\Big), 2^q\ \mathscr{K}\Big)\Big\}\end{aligned} \tag{42}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$. Applying (3), $(FNS2)$ and approaching q tends to infinity in (42) one can see that $\mathscr{A}$ satisfies the functional equation (1). To prove $\mathscr{A}(v)$ is unique, let $\mathscr{A}'(v)$ be another additive functional equation satisfying (1) and (7). So,

$$\begin{aligned}\mathscr{F}\ (\mathscr{A}(v)-\mathscr{A}'(v),\mathscr{K}) &= \mathscr{F}\ \left(\frac{\mathscr{A}(2^q v)}{2^q}-\frac{\mathscr{A}'(2^q v)}{2^q},\mathscr{K}\right)\\&\geq \min\left\{\mathscr{F}\ \left(\frac{\mathscr{A}(2^q v)}{2^q}-\frac{f_1(2^q v)}{2^q},\frac{\mathscr{K}}{2}\right),\mathscr{F}\ \left(\frac{\mathscr{A}'(2^q v)}{2^q}-\frac{f_1(2^q v)}{2^q},\frac{\mathscr{K}}{2}\right)\right\}\\&\geq \mathscr{F}\ '\left(\mathscr{Z}_O^{ODD}(2^q v,2^q v,2^q v,2^q v),\frac{\mathscr{K}(2-r)2^q}{2}\right)\\&= \mathscr{F}\ '\left(\mathscr{Z}_O^{ODD}(v,v,v,v),\frac{\mathscr{K}(2-r)2^q}{2\cdot r^q}\right)\end{aligned} \tag{43}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Since

$$\lim_{q\to\infty} \frac{(2-r)2^q}{2\cdot r^q} = \infty, \quad and \quad \lim_{q\to\infty} \mathscr{F}'\left(\mathscr{Z}_O^{ODD}\left(v,\ v,\ v,\ v\right), \frac{\mathscr{K}(2-r)2^q}{2\cdot r^q}\right) = 1$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. From (43), one can have

$$N(\mathscr{A}\ (\ v\ ) - \mathscr{A}'\ (\ v\ ), \mathscr{K}) = 1$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. So, $\mathscr{A}(v) = \mathscr{A}'(v)$. Hence $\mathscr{A}(v)$ is unique. Therefore for $p = 1$ the theorem holds.

Replacing v by $\frac{v}{2}$ in (29), one can arrive

$$\mathscr{F}\ \left(f_1(v) - 2f_1\left(\frac{y}{2}\right), T_O\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_O^{ODD}\left(\frac{v}{2}, \frac{v}{2}, \frac{v}{2}, \frac{v}{2}\right), \mathscr{K}\right) \tag{44}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. The rest of the proof is similar to that of case $p = 1$. Hence the theorem holds for the case $p = -1$. This completes the proof of the theorem.

Corollary 2.2 An odd mapping $f : H_1 \longrightarrow H_2$ is gratifying the functional inequality

$$\mathscr{F}\ \left(F^{1234}\left(v_1, w_1, v_2, w_2\right), \mathscr{K}\right) \geq \begin{cases} \mathscr{F}'\ (\ i, \mathscr{K}\ ) \\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j + ||w_k||^j\right), \mathscr{K}\ \right), \end{cases} \tag{45}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$ with i being positive constant. Then there exists a unique additive mapping $\mathscr{A} : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\begin{aligned} &\mathscr{F}\ (f_1(v) - \mathscr{A}(v), T_O\ \mathscr{K}\ ) = \mathscr{F}\ (f(2v) - 8f(v) - \mathscr{A}(v), T_O\ \mathscr{K}\ ) \\ &\geq \begin{cases} \mathscr{F}'\ \left(\ 11\ i, \left[4 + 2t^2 + 7\right]\ \left|2 - 1\right| \mathscr{K}\right), \\ \mathscr{F}'\ \left(i\left[24 + 5\cdot 2^{j+1} + 2(1+t)^j + 2(1-t)^j + 2.3^j + 2(1+2t)^j + 2(1-2t)^j\right] ||v||^j, \right. \\ \qquad\qquad \left. \left[4 + 2t^2 + 7\right]\left|2 - 2^j\right|\ \mathscr{K}\right), j \neq 1; \end{cases} \end{aligned} \tag{46}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

Theorem 2.3 Assume $p=\pm 1$ and let $f: H_1 \longrightarrow H_2$ be an odd mapping satisfying the functional inequality (2) where $\mathscr{Z}_O, \mathscr{Z}_O^{ODD}: H_1^4 \longrightarrow H_3$ are functions with conditions

$$\lim_{q\to\infty} \mathscr{F}' \left(\mathscr{Z}_O\left(2^{pq}v_1, 2^{pq}w_1, 2^{pq}v_2, 2^{pq}w_2\right), 8^{pq}\ \mathscr{K}\right) = 1 \tag{47}$$

and (4) for all $v_1, w_1, v_2, w_2, v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{8}\right)^p < 1$. Then there exists a unique cubic mapping $\mathscr{C}: H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ (f_3(v) - \mathscr{C}(v), T_O\ \mathscr{K}) = \mathscr{F}\ (f(2v) - 2f(v) - \mathscr{C}(v), T_O\ \mathscr{K}) \geq \mathscr{F}'\ \left(\mathscr{Z}_O^{ODD}\left(v\ , v,\ v,\ v\right), \mathscr{K}\right) \tag{48}$$

where $\mathscr{Z}_O^{ODD}\left(v,v,v,v\right)$ and T_O are defined in (6), (7) and $\mathscr{C}(v)$ is given by

$$\lim_{q\to\infty} \mathscr{F}\ \left(\mathscr{C}(v) - \frac{f_3(2^{pq}v)}{8^{pq}}, \mathscr{K}\right) = 1; \tag{49}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.
Proof: Define a function

$$f_3(v) = f(2v) - 2f(v) \tag{50}$$

for all $v \in H_1$. Using (50) in (27), one can achieve

$$\begin{aligned}\mathscr{F}\ (f_3(2v) - 8f_3(v), T_O\ \mathscr{K}) &= \mathscr{F}\ \left(\left(f(4v) - 2f(2v)\right) - 8\left(f(2v) - 2f(v)\right), T_O\ \mathscr{K}\right) \\ &\geq \mathscr{F}'\ \left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right)\end{aligned} \tag{51}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. The rest of the proof is similar to that of Theorem 2.1.

Corollary 2.4 Suppose $f: H_1 \longrightarrow H_2$ be an odd mapping satisfying the functional inequality

$$\mathscr{F}\ \left(F^{1234}\left(v_1, w_1, v_2, w_2\right), \mathscr{K}\right) \geq \begin{cases} \mathscr{F}'\ (\ i, \mathscr{K}) \\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j + ||w_k||^j\right), \mathscr{K}\right), \end{cases} \tag{52}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$ with i being positive constant. Then there exists a unique cubic mapping $\mathscr{C} : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ (f_3(v) - \mathscr{C}(v), T_O\ \mathscr{K}) = \mathscr{F}\ (f(2v) - 2f(v) - \mathscr{C}(v), T_O\ \mathscr{K})$$
$$\geq \begin{cases} \mathscr{F}'\ \left(\ 11\ i,\ \left[4+2t^2+7\right]\ \left|8-1\right|\mathscr{K}\right), \\ \mathscr{F}'\ \left(i\left[24+5\cdot 2^{j+1}+2(1+t)^j+2(1-t)^j+2.3^j+2(1+2t)^j+2(1-2t)^j\right]||v||^j, \right. \\ \qquad\qquad\qquad\qquad \left. \left[4+2t^2+7\right]\left|8-2^j\right|\ \mathscr{K}\right), j \neq 3; \end{cases} \tag{53}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

Theorem 2.5 Assume $p = \pm 1$ and let $f : H_1 \longrightarrow H_2$ be an odd mapping gratifying the functional inequality (2) where $\mathscr{Z}_O, \mathscr{Z}_O^{ODD} : H_1^4 \longrightarrow H_3$ are functions with the conditions (3), (4), (47) for all $v_1, w_1, v_2, w_2, v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{8}\right)^p < \left(\frac{r}{2}\right)^p < 1$. Then there exists a unique additive mapping $\mathscr{A} : H_1 \longrightarrow H_2$ and a unique cubic mapping $\mathscr{C} : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ \left(f(v) - \mathscr{A}(v) - \mathscr{C}(v), \frac{2}{6}T_O\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_{1A}^{3C}\left(v,v,v,v\right), \mathscr{K}\right) \tag{54}$$

where $T_O\ \mathscr{A}(v)\ , \mathscr{C}(v)$ are given in (7), (7), (49) for all $v \in H_1$ and all $\mathscr{K} > 0$. Proof: By Theorem 2.1, there exists a unique additive mapping $\mathscr{A}^O : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ \left(f(2v) - 8f(v) - \mathscr{A}^O(v), T_O\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right) \tag{55}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. By Theorem 2.3, there exists a unique cubic mapping $\mathscr{C}^O : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ \left(f(2v) - 2f(v) - \mathscr{C}^O(v), T_O\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_O^{ODD}\left(v,v,v,v\right), \mathscr{K}\right) \tag{56}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Now,

$$\begin{aligned}
&\mathscr{F}\ \left(-6f(v)-\mathscr{A}^{O}(v)+\mathscr{C}^{O}(v),2T_{O}\ \mathscr{K}\right)\\
&=\mathscr{F}\ \left(f\left(2v\right)-8f\left(v\right)-\mathscr{A}^{O}(v)-f\left(2v\right)+2f\left(v\right)+\mathscr{C}^{O}(v),2T_{O}\ \mathscr{K}\right)\Big\}\\
&\geq\min\left\{\mathscr{F}\ \left(f\left(2v\right)-8f\left(v\right)-\mathscr{A}^{O}(v),T_{O}\ \mathscr{K}\right),\mathscr{F}\ \left(f\left(2v\right)-2f\left(v\right)-\mathscr{C}^{O}(v),T_{O}\ \mathscr{K}\right)\right\}\\
&\geq\min\left\{\mathscr{F}'\ \left(\mathscr{Z}_{O}^{ODD}\left(v,v,v,v\right),\mathscr{K}\right),\mathscr{F}'\ \left(\mathscr{Z}_{O}^{ODD}\left(v,v,v,v\right),\mathscr{K}\right)\right\}\\
&=\mathscr{F}'\ \left(\mathscr{Z}_{1A}^{3C}\left(v,v,v,v\right),\mathscr{K}\right) \qquad (57)
\end{aligned}$$

for all $v\in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (56), one can get

$$\mathscr{F}\ \left(f(v)+\frac{1}{6}\mathscr{A}^{O}(v)-\frac{1}{6}\mathscr{C}^{O}(v),\frac{2}{6}T_{O}\ \mathscr{K}\right)\geq\mathscr{F}'\ \left(\mathscr{Z}_{1A}^{3C}\left(v,v,v,v\right),\mathscr{K}\right) \qquad (58)$$

for all $v\in H_1$ and all $\mathscr{K}>0$. Thus in (58), if we take

$$\mathscr{A}\left(v\right)=-\frac{1}{6}\mathscr{A}^{O}(v);\quad \mathscr{C}\left(v\right)=\frac{1}{6}\mathscr{C}^{O}(v)$$

we arrive (54) as desired.

Corollary 2.6 Suppose $f:H_1\longrightarrow H_2$ be an odd mapping satisfying the functional inequality

$$\mathscr{F}\ \left(F^{1234}\left(v_1,w_1,v_2,w_2\right),\mathscr{K}\right)\geq\begin{cases}\mathscr{F}'\ \left(\ i,\mathscr{K}\right)\\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j+||w_k||^j\right),\mathscr{K}\right),\end{cases} \qquad (59)$$

for all $v_1,w_1,v_2,w_2\in H_1$ and all $\mathscr{K}>0$ with i being positive constant. Then there exists a unique additive mapping $\mathscr{A}:H_1\longrightarrow H_2$ and a unique cubic mapping $\mathscr{C}:$

$H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f(v)-\mathscr{A}(v)-\mathscr{C}(v), \frac{2}{6}T_O\,\mathscr{K}\right)$$
$$\geq \begin{cases} \mathscr{F}'\left(22\,i, \left[4+2t^2+7\right]\left\{\left|2-1\right|+\left|8-1\right|\right\}\mathscr{K}\right), \\ \mathscr{F}'\left(2i\left[24+5\cdot 2^{j+1}+2(1+t)^j+2(1-t)^j+2.3^j+2(1+2t)^j+2(1-2t)^j\right]||v||^j,\right. \\ \qquad\left.\left[4+2t^2+7\right]\left\{\left|2-2^j\right|+\left|8-2^j\right|\right\}\mathscr{K}\right), j\neq 1,3; \end{cases} \tag{60}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

3. Fuzzy Stability Theorem: Even Case

Theorem 3.1 Assume $p = \pm 1$ and let $f : H_1 \longrightarrow H_2$ be an even mapping satisfying the functional inequality

$$\mathscr{F}\left(F^{1234}\left(v_1,w_1,v_2,w_2\right),\mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_E\left(v_1,w_1,v_2,w_2\right),\mathscr{K}\right) \tag{61}$$

where $\mathscr{Z}_E, \mathscr{Z}_E^{EVEN} : H_1^4 \longrightarrow H_3$ are functions with the conditions

$$\lim_{q\to\infty}\mathscr{F}'\left(\mathscr{Z}_E\left(2^{pq}v_1,2^{pq}w_1,2^{pq}v_2,2^{pq}w_2\right),4^{pq}\mathscr{K}\right)=1 \tag{62}$$

$$\mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\left(2^pv,2^pv,2^pv,2^pv\right),\mathscr{K}\right) \geq \mathscr{F}'\left(r^p\,\mathscr{Z}_E^{EVEN}\left(v,v,v,v\right),\mathscr{K}\right) \tag{63}$$

for all $v_1,w_1,v_2,w_2,v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{4}\right)^p < 1$. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f_2(v)-\mathscr{Q}_2(v),T_E\,\mathscr{K}\right)=\mathscr{F}\left(f(2v)-16f(v)-\mathscr{Q}_2(v),T_E\,\mathscr{K}\right)\geq\mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\left(v,v,v,v\right),\mathscr{K}\right) \tag{64}$$

where $\mathscr{Z}_E^{EVEN}\left(v,v,v,v\right)$, $\mathscr{Q}_2(v)$ and T_E are defined by

$$\mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\left(v,v,v,v\right),\mathscr{K}\right)$$
$$=\min\left\{\mathscr{F}'\left(\mathscr{Z}_E\left(v,0,v,0\right),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_E\left(2v,0,2v,0\right),\mathscr{K}\right),\right.$$
$$\left.\mathscr{F}'\left(\mathscr{Z}_E\left(v,v,v,v\right),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_E\left(v,tv,v,tv\right),\mathscr{K}\right)\right\} \tag{65}$$

$$\lim_{q\to\infty}\mathscr{F}\left(\mathscr{Q}_2(v)-\frac{f_2(2^{pq}v)}{4^{pq}},\mathscr{K}\right)=1;\quad T_E=\left(\frac{48}{2(t^4-t^2)}\right) \tag{66}$$

for all $v\in H_1$ and all $\mathscr{K}>0$.
Proof: For evenness of f in (61), one can arrive

$$\mathscr{F}\left(f(tv_1+w_1)+f(w_1-tv_1)+f(tv_2+w_2)+f(w_2-tv_2)\right.$$
$$-t^2\left[f(v_1+w_1)+f(w_1-v_1)+f(v_2+w_2)+f(w_2-v_2)\right]-2\left(1-t^2\right)\left[f(w_1)+f(w_2)\right]$$
$$\left.-\frac{2\left(t^4-t^2\right)}{12}\left[f(2v_1)-4f(v_1)+f(2v_2)-4f(v_2)\right],\mathscr{K}\right)\geq\mathscr{F}'\left(\mathscr{Z}_E\left(v_1,w_1,v_2,w_2\right),\mathscr{K}\right) \tag{67}$$

for all $v_1,w_1,v_2,w_2\in H_1$ and all $\mathscr{K}>0$. Replacing (v_1,w_1,v_2,w_2) by $(v,0,v,0)$ in (67), one can reach

$$\mathscr{F}\left(4f(tv)-4t^2f(v)-\frac{2\left(t^4-t^2\right)}{12}\left[f(2v)-4f(v)\right],\mathscr{K}\right)\geq\mathscr{F}'\left(\mathscr{Z}_E\left(v,0,v,0\right),\mathscr{K}\right) \tag{68}$$

for all $v\in H_1$ and all $\mathscr{K}>0$. With the help of $(FNS3)$ in (68), one can obtain

$$\mathscr{F}\left(48(1-t^2)f(tv)-48(1-t^2)t^2f(v)-4\left(t^4-t^2\right)(1-t^2)\left[f(2v)-4f(v)\right],12(1-t^2)\mathscr{K}\right)$$
$$\geq\mathscr{F}'\left(\mathscr{Z}_E\left(v,0,v,0\right),\mathscr{K}\right) \tag{69}$$

for all $v\in H_1$ and all $\mathscr{K}>0$. Replacing v by $2v$ in (68), one can have

$$\mathscr{F}\left(4f(2tv)-4t^2f(2v)-\frac{2\left(t^4-t^2\right)}{12}\left[f(4v)-4f(2v)\right],\mathscr{K}\right)\geq\mathscr{F}'\left(\mathscr{Z}_E\left(2v,0,2v,0\right),\mathscr{K}\right) \tag{70}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (70), one can get

$$\mathscr{F}\ \left(24f(2tv) - 24t^2 f(2v) - 2\left(t^4 - t^2\right)[f(4v) - 4f(2v)], 6\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_E\left(2v, 0, 2v, 0\right), \mathscr{K}\right) \tag{71}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Interchanging (v_1, w_1, v_2, w_2) by (v, v, v, v) in (67), one can find

$$\mathscr{F}\ \left(2f((1+t)v) + 2f((1-t)v) - 2t^2 f(2v) - 4\left(1-t^2\right)f(v) - \frac{2\left(t^4 - t^2\right)}{12}[f(2v) - 4f(v)], \mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_E\left(v, v, v, v\right), \mathscr{K}\right) \tag{72}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (72), one can reach

$$\mathscr{F}\ \left(24t^2 f((1+t)v) + 24t^2 f((1-t)v) - 24t^4 f(2v) - 48t^2\left(1-t^2\right)f(v)\right.$$
$$\left. -4\left(t^4 - t^2\right)t^2[f(2v) - 4f(v)], 12t^2\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_E\left(v, v, v, v\right), \mathscr{K}\right) \tag{73}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Interchanging (v_1, w_1, v_2, w_2) by (v, tv, v, tv) in (67), one can find

$$\mathscr{F}\ \left(2f(2tv) - 2t^2[f((1+t)v) + f((1-t)v)] - 4\left(1-t^2\right)f(tv) - \frac{2\left(t^4 - t^2\right)}{12}[f(2v) - 4f(v)], \mathscr{K}\right)$$
$$\geq \mathscr{F}'\ \left(\mathscr{Z}_E\left(v, tv, v, tv\right), \mathscr{K}\right) \tag{74}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (74), one can get

$$\mathscr{F}\ \left(24f(2tv) - 24t^2[f((1+t)v) + f((1-t)v)] - 48\left(1-t^2\right)f(tv)\right.$$
$$\left. -4\left(t^4 - t^2\right)[f(2v) - 4f(v)], 12\ \mathscr{K}\right) \geq \mathscr{F}'\ \left(\mathscr{Z}_E\left(v, tv, v, tv\right), \mathscr{K}\right) \tag{75}$$

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for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS4)$ and (69), (71), (73), (75), one can find that

$$\begin{aligned}
&\mathscr{F}\ \Big(2(t^4-t^2)\Big(f(4v)-20f(2v)+64f(v)\Big),48\ \mathscr{K}\Big)\\
&\geq \min\Big\{\mathscr{F}\ \big(48(1-t^2)f(tv)-48(1-t^2)t^2f(v)-4\left(t^4-t^2\right)(1-t^2)[f(2v)\\
&\qquad\qquad -4f(v)],12(1-t^2)\mathscr{K}\big),\\
&\qquad \mathscr{F}\ \big(24f(2tv)-24t^2f(2v)-2\left(t^4-t^2\right)[f(4v)-4f(2v)],6\mathscr{K}\big),\\
&\qquad \mathscr{F}\ \big(24t^2f((1+t)v)+24t^2f((1-t)v)-24t^4f(2v)-48t^2\left(1-t^2\right)f(v)\\
&\qquad\qquad -4\left(t^4-t^2\right)t^2[f(2v)-4f(v)],12t^2\mathscr{K}\big),\\
&\qquad \mathscr{F}\ \big(24f(2tv)-24t^2[f((1+t)v)+f((1-t)v)]-48\left(1-t^2\right)f(tv)\\
&\qquad\qquad -4\left(t^4-t^2\right)[f(2v)-4f(v)],12\mathscr{K}\big)\Big\}\\
&\geq \min\Big\{\mathscr{F}'\ \Big(\mathscr{Z}_E\Big(v,0,v,0\Big),\mathscr{K}\Big),\mathscr{F}'\ \Big(\mathscr{Z}_E\Big(2v,0,2v,0\Big),\mathscr{K}\Big),\\
&\qquad \mathscr{F}'\ \Big(\mathscr{Z}_E\Big(v,v,v,v\Big),\mathscr{K}\Big),\mathscr{F}'\ \Big(\mathscr{Z}_E\Big(v,tv,v,tv\Big),\mathscr{K}\Big)\Big\}\\
&= \mathscr{F}'\ \Big(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big),\mathscr{K}\Big) \qquad (76)
\end{aligned}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (76), one can find

$$\mathscr{F}\ \left(f(4v)-20f(2v)+64f(v),\frac{48}{2(t^4-t^2)}\ \mathscr{K}\right) \geq \mathscr{F}'\ \Big(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big),\mathscr{K}\Big) \quad (77)$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Set $T_E = \left(\frac{48}{2(t^4-t^2)}\right)$ in (77), one can see

$$\mathscr{F}\ \left(f(4v)-20f(2v)+64f(v),T_E\ \mathscr{K}\right) \geq \mathscr{F}'\ \Big(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big),\mathscr{K}\Big) \quad (78)$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Define a function

$$f_2(v) = f(2v) - 16f(v) \quad (79)$$

for all $v \in H_1$. Using (79) in (78), one can achieve

$$\begin{aligned}
\mathscr{F}\ \left(f_2(2v)-4f_2(v),T_E\ \mathscr{K}\right) &= \mathscr{F}\ \Big(\Big(f(4v)-16f(2v)\Big)-4\Big(f(2v)-16f(v)\Big),T_E\ \mathscr{K}\Big)\\
&\geq \mathscr{F}'\ \Big(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big),\mathscr{K}\Big) \qquad (80)
\end{aligned}$$

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for all $v \in H_1$ and all $\mathscr{K} > 0$. The rest of the proof is similar to that of Theorem 2.1.

Corollary 3.2 Suppose $f : H_1 \longrightarrow H_2$ be an even mapping gratifying functional inequality

$$\mathscr{F}\ \left(F^{1234}\left(v_1, w_1, v_2, w_2\right), \mathscr{K}\right) \geq \begin{cases} \mathscr{F}'\ (\ i, \mathscr{K}) \\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j + ||w_k||^j\right), \mathscr{K}\right), \end{cases} \tag{81}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$ with i being positive constant. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\begin{aligned}\mathscr{F}\ (f_2(v) - \mathscr{Q}_2(v), T_O\ \mathscr{K}) &= \mathscr{F}\ (f(2v) - 16f(v) - \mathscr{Q}_2(v), T_O\ \mathscr{K}) \\ &\geq \begin{cases} \mathscr{F}'\ \left(i, \left|4-1\right|\ \mathscr{K}\right), \\ \mathscr{F}'\ \left(i\left[8 + 2^{j+1} + 2\cdot t^j\right]||v||^j, 4\ \left|4-2^j\right|\ \mathscr{K}\right), j \neq 2; \end{cases}\end{aligned} \tag{82}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

Theorem 3.3 Assume $p = \pm 1$ and let $f : H_1 \longrightarrow H_2$ be an even mapping satisfying the functional inequality (61) where $\mathscr{Z}_E, \mathscr{Z}_E^{EVEN} : H_1^4 \longrightarrow H_3$ are functions with conditions

$$\lim_{q\to\infty} \mathscr{F}'\ \left(\mathscr{Z}_E\left(2^{pq}v_1, 2^{pq}w_1, 2^{pq}v_2, 2^{pq}w_2\right), 16^{pq}\mathscr{K}\right) = 1 \tag{83}$$

and (63) for all $v_1, w_1, v_2, w_2, v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{16}\right)^p < 1$. Then there exists a unique quartic mapping $\mathscr{Q}_4 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\ (f_4(v) - \mathscr{Q}_4(v), T_E\ \mathscr{K}) = \mathscr{F}\ (f(2v) - 4f(v) - \mathscr{Q}_4(v), T_E\ \mathscr{K}) \geq \mathscr{F}'\ \left(\mathscr{Z}_E^{EVEN}\left(v, v, v, v\right), \mathscr{K}\right) \tag{84}$$

where $\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big)$, and T_E (65), (66) and $\mathscr{Q}_4(v)$ is given by

$$\lim_{q\to\infty}\mathscr{F}\left(\mathscr{Q}_4(v)-\frac{f_4(2^{pq}v)}{4^{pq}},\mathscr{K}\right)=1 \tag{85}$$

for all $v\in H_1$ and all $\mathscr{K}>0$.
Proof: Define a function

$$f_4(v)=f(2v)-4f(v) \tag{86}$$

for all $v\in H_1$. Using (86) in (78), one can achieve

$$\begin{aligned}\mathscr{F}\ (f_4(2v)-16f_4(v),T_E\ \mathscr{K})&=\mathscr{F}\ \Big(\Big(f(4v)-4f(2v)\Big)-16\Big(f(2v)-4f(v)\Big),T_E\ \mathscr{K}\Big)\\&\geq\mathscr{F}'\ \Big(\mathscr{Z}_E^{even}\Big(v,v,v,v\Big),\mathscr{K}\Big)\end{aligned} \tag{87}$$

for all $v\in H_1$ and all $\mathscr{K}>0$. The rest of the proof is similar to that of Theorem 2.1.

Corollary 3.4 Suppose $f:H_1\longrightarrow H_2$ be an even mapping satisfying the functional inequality

$$\mathscr{F}\ \Big(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\Big)\geq\begin{cases}\mathscr{F}'\ (\ i,\mathscr{K})\\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j+||w_k||^j\right),\mathscr{K}\right),\end{cases} \tag{88}$$

for all $v_1,w_1,v_2,w_2\in H_1$ and all $\mathscr{K}>0$ with i being positive constant. Then there exists a unique quartic mapping $\mathscr{Q}_4:H_1\longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\begin{aligned}\mathscr{F}\ (f_4(v)-\mathscr{Q}_4(v),T_E\ \mathscr{K})&=\mathscr{F}\ (f(2v)-4f(v)-\mathscr{Q}_4(v),T_E\ \mathscr{K})\\&\geq\begin{cases}\mathscr{F}'\ \Big(i,\Big|16-1\Big|\ \mathscr{K}\Big),\\ \mathscr{F}'\ \Big(i\Big[8+2^{j+1}+2\cdot t^j\Big]||v||^j,4\ \Big|16-2^j\Big|\ \mathscr{K}\Big),j\neq 4;\end{cases}\end{aligned} \tag{89}$$

for all $v\in H_1$ and all $\mathscr{K}>0$.

Theorem 3.5 Assume $p=\pm 1$ and let $f:H_1\longrightarrow H_2$ be an even mapping satisfying the functional inequality (61) where $\mathscr{Z}_E,\mathscr{Z}_E^{EVEN}:H_1^4\longrightarrow H_3$ are function with conditions (62), (63), (83) for all $v_1, w_1, v_2, w_2, v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{16}\right)^p < \left(\frac{r}{4}\right)^p < 1$. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$ and a unique quartic mapping $\mathscr{Q}_4 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f(v) - \mathscr{Q}_2(v) - \mathscr{Q}_4(v), \frac{2}{12} T_E\, \mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big), \mathscr{K}\right) \tag{90}$$

where T_E, $\mathscr{Q}_2(v)$, $\mathscr{Q}_4(v)$ are given in (66) , (85) for all $v \in H_1$ and all $\mathscr{K} > 0$.
Proof: By Theorem 3.1, there exists a unique quadratic mapping $\mathscr{Q}_2^O : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f(2v) - 16f(v) - \mathscr{Q}_2(v), T_E\, \mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big), \mathscr{K}\right) \tag{91}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. By Theorem 3.3, there exists a unique quartic mapping $\mathscr{Q}_4^O : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f(2v) - 4f(v) - \mathscr{Q}_4(v), T_E\, \mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big), \mathscr{K}\right) \tag{92}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Now,

$$\begin{aligned}
&\mathscr{F}\left(-12f(v) - \mathscr{Q}_2^E(v) + \mathscr{Q}_4^E(v), 2T_E\, \mathscr{K}\right)\\
&= \mathscr{F}\left(f(2v) - 16f(v) - \mathscr{Q}_2^E(v) - f(2v) + 4f(v) + \mathscr{Q}_4^E v), 2T_E\, \mathscr{K}\right)\Big\}\\
&\geq \min\Big\{\mathscr{F}\left(f(2v) - 16f(v) - \mathscr{Q}_2^E(v), T_E\, \mathscr{K}\right), \mathscr{F}\left(f(2v) - 4f(v) - \mathscr{Q}_4^E(v), T_E\, \mathscr{K}\right)\Big\}\\
&\geq \min\Big\{\mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big), \mathscr{K}\right), \mathscr{F}'\left(\mathscr{Z}_E^{EVEN}\Big(v,v,v,v\Big), \mathscr{K}\right)\Big\}\\
&= \mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big), \mathscr{K}\right)
\end{aligned} \tag{93}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. With the help of $(FNS3)$ in (93), one can get

$$\mathscr{F}\left(f(v) + \frac{1}{12}\mathscr{Q}_2^E(v) - \frac{1}{12}\mathscr{Q}_4^E(v), \frac{2}{12} T_E\, \mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big), \mathscr{K}\right) \tag{94}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Thus in (94), if we take

$$\mathscr{Q}_2(v) = -\frac{1}{12}\mathscr{Q}_2^E(v); \quad \mathscr{Q}_4(v) = \frac{1}{12}\mathscr{Q}_4^E(v)$$

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we arrive (90) as desired.

Corollary 3.6 Suppose $f : H_1 \longrightarrow H_2$ be a mapping satisfying the functional inequality

$$\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right) \geq \begin{cases} \min\Big\{\mathscr{F}'\ \big(\ i,\mathscr{K}\big)\Big\}, \\ \min\Big\{\mathscr{F}'\ \big(\ i\sum_{k=1}^{2}\ \big(||v_k||^j+||w_k||^j\big),\mathscr{K}\big)\Big\}, \end{cases} \tag{95}$$

for all $v_1,w_1,v_2,w_2 \in H_1$ and all $\mathscr{K} > 0$ with i being positive constant. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$ and a unique quartic mapping $\mathscr{Q}_4 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\begin{aligned}&\mathscr{F}\left(f(v)-\mathscr{Q}_2(v)-\mathscr{Q}_4(v),\frac{2}{12}T_E\ \mathscr{K}\right)\\ &\quad\geq \begin{cases}\mathscr{F}'\ \Big(2i,\Big\{\Big|4-1\Big|+\Big|16-1\Big|\Big\}\ \mathscr{K}\Big),\\ \mathscr{F}'\ \Big(2i\Big[8+2^{j+1}+2\cdot t^j\Big]||v||^j,4\ \Big\{\Big|4-2^j\Big|+\Big|16-2^j\Big|\Big\}\ \mathscr{K}\Big), j\neq 2,4;\end{cases}\end{aligned} \tag{96}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

4. Fuzzy Stability Theorem: Odd - Even Case

Theorem 4.1 Assume $p = \pm 1$ and let $f : H_1 \longrightarrow H_2$ be a mapping satisfying the functional inequality

$$\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right) \geq \mathscr{F}'\left(\mathscr{Z}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right) \tag{97}$$

where $\mathscr{Z} : H_1^4 \longrightarrow H_3$ are functions with conditions (3), (4), (47), (62), (63), (83), for all $v_1,w_1,v_2,w_2,v \in H_1$ and all $\mathscr{K} > 0$, for some $r > 0$ with $0 < \left(\frac{r}{16}\right)^p < \left(\frac{r}{8}\right)^p < \left(\frac{r}{4}\right)^p < \left(\frac{r}{2}\right)^p < 1$. Then there exists a unique additive mapping $\mathscr{A} : H_1 \longrightarrow H_2$, a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$, a unique cubic mapping $\mathscr{C} : H_1 \longrightarrow H_2$ and a unique quartic mapping $\mathscr{Q}_4 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1)

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and the inequality

$$\mathscr{F}\left(f(v)-\mathscr{A}(v)-\mathscr{Q}_2(v)-\mathscr{C}(v)-\mathscr{Q}_4(v),\left(\frac{2}{6}T_O+\frac{2}{12}T_E\right)\mathscr{K}\right)$$
$$\geq \min\Big\{\mathscr{F}'\left(\mathscr{Z}_{1A}^{3C}\Big(v,v,v,v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_{1A}^{3C}\Big(-v,-v,-v,-v\Big),\mathscr{K}\right),$$
$$\mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big),\mathscr{K}\right),\mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(-v,-v,-v,-v\Big),\mathscr{K}\right)\Big\} \tag{98}$$

where ($\mathscr{Z}_{1A}^{3C}\Big(v,v,v,v\Big)$, $\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big)$; are defined in (56, (93), and $\mathscr{A}(v)$, $\mathscr{C}(v)$, $\mathscr{Q}_2(v)$, $\mathscr{Q}_4(v)$,T_O, T_E is given in (7) , (49) (66) , (85) for all $v\in H_1$ and all $\mathscr{K}>0$. Proof: Assume a function

$$f_{AC}(v)=\frac{f(v)-f(-v)}{2} \tag{99}$$

for all $v\in H_1$. It is easy to see from (99) that

$$f_{AC}(v)=0 \;\; and \;\; f_{AC}(-v)=-f_{AC}(v) \tag{100}$$

for all $v\in H_1$. So by $(FNS3)$ and $(FNS4)$ with the help of (99), one can see

$$\begin{aligned}
&\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)\\
&=\mathscr{F}\left(\frac{1}{2}\Big[F^{1234}\Big(v_1,w_1,v_2,w_2\Big)-F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big)\Big],\mathscr{K}\right)\\
&=\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big)-F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big),2\mathscr{K}\right)\\
&\geq\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)+\mathscr{F}\left(F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big),\mathscr{K}\right)\\
&\geq\mathscr{F}'\left(\mathscr{Z}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)+\mathscr{F}'\left(\mathscr{Z}\Big(-v_1,-w_1,-v_2,-w_2\Big),\mathscr{K}\right)
\end{aligned} \tag{101}$$

for all $v_1,w_1,v_2,w_2,v\in H_1$ and all $\mathscr{K}>0$.

By Theorem 2.5 and (101), there exists a unique additive mapping $\mathscr{A}:H_1\longrightarrow H_2$ and a unique cubic mapping $\mathscr{C}:H_1\longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f_{AC}(v)-\mathscr{A}(v)-\mathscr{C}(v),\frac{2}{6}T_O\,\mathscr{K}\right)\geq \mathscr{F}'\left(\mathscr{Z}_{1A}^{3C}\Big(v,v,v,v\Big),\mathscr{K}\right)+\mathscr{F}'\left(\mathscr{Z}_{1A}^{3C}\Big(-v,-v,-v,-v\Big),\mathscr{K}\right) \tag{102}$$

for all $v\in H_1$ and all $\mathscr{K}>0$.

Again assume a function

$$f_{QQ}(v)=\frac{f(v)+f(-v)}{2} \tag{103}$$

for all $v\in H_1$. It is easy to see from (103) that

$$f_{QQ}(v)=0 \quad and \quad f_{QQ}(-v)=f_{AC}(v) \tag{104}$$

for all $v\in H_1$. So by $(FNS3)$ and $(FNS4)$ with the help of (103), one can see

$$\begin{aligned}
&\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)\\
&=\mathscr{F}\left(\frac{1}{2}\Big[F^{1234}\Big(v_1,w_1,v_2,w_2\Big)+F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big)\Big],\mathscr{K}\right)\\
&=\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big)+F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big),2\mathscr{K}\right)\\
&\geq\mathscr{F}\left(F^{1234}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)+\mathscr{F}\left(F^{1234}\Big(-v_1,-w_1,-v_2,-w_2\Big),\mathscr{K}\right)\\
&\geq\mathscr{F}'\left(\mathscr{Z}\Big(v_1,w_1,v_2,w_2\Big),\mathscr{K}\right)+\mathscr{F}'\left(\mathscr{Z}\Big(-v_1,-w_1,-v_2,-w_2\Big),\mathscr{K}\right)
\end{aligned} \tag{105}$$

for all $v_1,w_1,v_2,w_2,v\in H_1$ and all $\mathscr{K}>0$.

By Theorem 3.5 and (105), there exists a unique quadratic mapping $\mathscr{Q}_2:H_1\longrightarrow H_2$ and a unique quartic mapping $\mathscr{Q}_4:H_1\longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f_{QQ}(v)-\mathscr{Q}_2(v)-\mathscr{Q}_4(v),\frac{2}{12}T_E\,\mathscr{K}\right)\geq \mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(v,v,v,v\Big),\mathscr{K}\right)+\mathscr{F}'\left(\mathscr{Z}_{2Q}^{4Q}\Big(-v,-v,-v,-v\Big),\mathscr{K}\right) \tag{106}$$

for all $v\in H_1$ and all $\mathscr{K}>0$.

Finally assume a function

$$f(v) = f_{AC}(v) + f_{QQ}(v) \tag{107}$$

for all $v \in H_1$. It follows from (107), (106) , (102) one can arrive

$$\begin{aligned}
&\mathscr{F}\ \left(f(v) - \mathscr{A}(v) - \mathscr{Q}_2(v) - \mathscr{C}(v) - \mathscr{Q}_4(v), \left(\frac{2}{6}T_O + \frac{2}{12}T_E\right)\ \mathscr{K}\right)\\
&= \mathscr{F}\ \left(f_{AC}(v) + f_{QQ}(v) - \mathscr{A}(v) - \mathscr{Q}_2(v) - \mathscr{C}(v) - \mathscr{Q}_4(v), \left(\frac{2}{6}T_O + \frac{2}{12}T_E\right)\ \mathscr{K}\right)\\
&= \mathscr{F}\ \left(\left(\left[f_{AC}(v) - \mathscr{A}(v) - \mathscr{C}(v)\right] + \left[f_{QQ}(v) - \mathscr{Q}_2(v) - \mathscr{Q}_4(v)\right]\right), \left(\frac{2}{6}T_O + \frac{2}{12}T_E\right)\ \mathscr{K}\right)\\
&\geq \min\left\{ \mathscr{F}\ \left(f_{AC}(v) - \mathscr{A}(v) - \mathscr{C}(v), \frac{2}{6}T_O\ \mathscr{K}\right), \mathscr{F}\ \left(f_{QQ}(v) - \mathscr{Q}_2(v) - \mathscr{Q}_4(v), \frac{2}{12}T_E\ \mathscr{K}\right)\right\}\\
&\geq \min\left\{ \mathscr{F}'\ \left(\mathscr{Z}_{1A}^{3C}\left(v,v,v,v\right), \mathscr{K}\right), \mathscr{F}'\ \left(\mathscr{Z}_{1A}^{3C}\left(-v,-v,-v,-v\right), \mathscr{K}\right),\right.\\
&\qquad\qquad \left. \mathscr{F}'\ \left(\mathscr{Z}_{2Q}^{4Q}\left(v,v,v,v\right), \mathscr{K}\right), \mathscr{F}'\ \left(\mathscr{Z}_{2Q}^{4Q}\left(-v,-v,-v,-v\right), \mathscr{K}\right)\right\}
\end{aligned}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$. Hence the proof is complete.

Corollary 4.2 Suppose $f : H_1 \longrightarrow H_2$ be an odd mapping satisfying the functional inequality

$$\mathscr{F}\ \left(F^{1234}\left(v_1, w_1, v_2, w_2\right), \mathscr{K}\right) \geq \begin{cases} \mathscr{F}'\ \left(\ i, \mathscr{K}\right) \\ \mathscr{F}'\ \left(\ i\sum_{k=1}^{2}\ \left(||v_k||^j + ||w_k||^j\right), \mathscr{K}\right), \end{cases} \tag{108}$$

for all $v_1, w_1, v_2, w_2 \in H_1$ and all $\mathscr{K} > 0$ with i being positive constant. Then there exists a unique additive mapping $\mathscr{A} : H_1 \longrightarrow H_2$, a unique quadratic mapping $\mathscr{Q}_2 : H_1 \longrightarrow H_2$, a unique cubic mapping $\mathscr{C} : H_1 \longrightarrow H_2$ and a unique quartic mapping

$\mathscr{Q}_4 : H_1 \longrightarrow H_2$ which satisfies the functional equation (1) and the inequality

$$\mathscr{F}\left(f(v)-\mathscr{A}v)-\mathscr{Q}_2(v)-\mathscr{C}(v)-\mathscr{Q}_4(v),\left(\frac{2}{6}T_O+\frac{2}{12}T_E\right)\mathscr{K}\right)$$
$$\geq\begin{cases}\mathscr{F}'\left(24i,\left(\left[4+2t^2+7\right]\left\{\left|2-1\right|+\left|8-1\right|\right\}+\left\{\left|4-1\right|+\left|16-1\right|\right\}\right)\mathscr{K}\right),\\ \mathscr{F}'\Big(2i\Big\{\left[24+5\cdot 2^{j+1}+2(1+t)^j+2(1-t)^j+2.3^j+2(1+2t)^j+2(1-2t)^j\right]\\ \qquad +\left[8+2^{j+1}+2\cdot t^j\right]\Big\}||v||^j,\\ \qquad \left[4+2t^2+7\right]\left\{\left|2-2^j\right|+\left|8-2^j\right|\right\}+4\left\{\left|4-2^j\right|+\left|16-2^j\right|\right\}\mathscr{K}\Big),\ j\neq 1,2,3,4;\end{cases} \tag{109}$$

for all $v \in H_1$ and all $\mathscr{K} > 0$.

5. Fuzzy Applications

In this section, we will discuss the fields where the concepts of fuzzy logic are extensively applied. To begin with in Aerospace Fuzzy logic is used in various areas such as Altitude control of Spacecraft, Satellite altitude control, flow and mixture regulation in aircraft deicing vehicles. In automotive, fuzzy logic is used in different sections such as trainable fuzzy systems for idle speed control, shift scheduling method for automatic transmission, Intelligent highway systems, Traffic Control and improving efficiency of automatic transmissions.

In addition, in business, this logic can play as a decision and personal evaluation in a large company. Likewise, in defense, this helps people the recognize the under water targets can spontaneously recognize the thermal infrared images, works as decision support aids, controller of a hypervelocity interceptor and this can be used in fuzzy set modelling of NATO decision making.

Further more, In Electronics, this Fuzzy logic puts its marks higher. Also in the manufacturing department it is used to optimize the cheese and milk production. In order to bring out the transportations to the next level the fuzzy logic used in different aspects with transports such as autopilot for ships, optimal route selection, control of autonomous under water vehicles, ship steering, automatic operation in train operation, schedule, acceleration breaking and stopping.

In Electronics while manufacturing of chips, the approximation between the chips we use functional equations in order to get the correct output. This logic is extended for medical field too. It is used as a medical diagnostic support system, control of a sterile pressure during anesthesia, Radiology diagnoses and diagnosis of diabetes and prostate cancer.

6. Conclusion

In this paper with the help of classical Hyers Method, we analyze the generalized Ulam - Hyers stability of a mixed AQ_2CQ_4 functional equation in Fuzzy Banach spaces. The stability results gives better possible upper bound when comparing to previous stability results.

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