2456-8686, viii(i), 2024:001-014
https://doi.org/10.26524/cm181

ORIGINAL RESEARCH Open Access

Hyers - Gavruta Type Stability Of Generalized Composite Functional Equations In Generalized 2 - Banach Spaces

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Received: 19 Feb 2024/ Accepted: 01 March 2024/ Published online: 19 June 2024

Sacred Heart Research Publications 2017

Abstract

In this paper, we explore the Hyers - Gavruta Type stability of generalized composite functional equations in Generalized 2- Banach Spaces.

Key words: Additive functional equations, Composite Functional Equations, Generalized 2-Banach Spaces, Hyers Direct Method, Gavruta stability.

AMS classification: 39B52, 32B72, 32B82

1 Introduction

The stability of functional equations is a hot topic that was delt in the last eight decades. In 1940, S.M. Ulam [29], gave a widespread of talk before a Mathematical Colloquium at the University of Wisconsin in which he introduced the number of important unsolved problems. One of them is the first point of a new line of investigation, the Stability Problem.

The first result regarding the stability of functional equations was presented in 1941 by D.H. Hyers [16]. He has completely answered the question of Ulam for the case the groups are Banach spaces. He proved the following theorem.

Theorem 1.1 [16] Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping

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2456-8686, viii(i), 2024:001-014
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satisfying

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon \tag{1}$$

for all $x, y \in X$. Then the limit

$$a(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \tag{2}$$

exists for all $x \in X$ and $a: X \to Y$ is the unique additive mapping satisfying

$$\|f(x)-a(x)\| \leq \epsilon \tag{3}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function a is linear.

This pioneer result can be expressed as " for any pair of Banach spaces Cauchy functional equation is stable". The method which was determined by Hyers and the additive function which he produces will be called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. This stability result is called Hyers-Ulam stability of functional equations.
In 1951, T. Aoki [4] generalized the Hyers theorem in Banach spaces for approximately linear transformation, by lowering the condition for the Cauchy difference for sum of powers of norms. Then a similar case was investigated by Th.M. Rassias [24] in 1978 (see: L. Maligranda [20]). Both of them proved the following Hyers-Ulam-Aoki-Rassias theorem for the sum.

Theorem 1.2 [4, 24] Let X and Y be two Banach spaces. Let $\theta \in [0,\infty)$ and $p \in [0,1)$. If a function $f: X \to Y$ satisfies the inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\left(||x||^p+||y||^p\right) \tag{4}$$

for all $x, y \in X$, then there exists a unique additive mapping $T: X \to Y$ such that

$$\|f(x)-T(x)\| \leq \frac{2\theta}{2-2^p}||x||^p \tag{5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function T is linear.

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In 1982, J.M. Rassias [21] replaced the sum by the product of powers of norms and proved the following Ulam - Gavruta - Rassias theorem.

Theorem 1.3 [21] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^p \tag{6}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{7}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{8}$$

for all $x \in E$. If $p < 0$, then the inequality (6) holds for $x, y \neq 0$ and (8) for $x \neq 0$. If $p > \frac{1}{2}$ the inequality (6) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{9}$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p} \tag{10}$$

for all $x \in E$. If, in addition $f : E \to E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}-$ linear mapping.

A generalization of all of the above stability results was obtained in 1994 by P.Gavruta [15] and prove the following theorem.

Theorem 1.4 [15] Let E be a abelian group, F be a Banach space and let $\phi : E \times E \to [0, \infty)$ be a function satisfying

$$\Phi(x, y) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \phi\left(2^k x, 2^k y\right) < +\infty \tag{11}$$

for all $x, y \in E$. If a function $f : E \to F$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \phi(x,y) \tag{12}$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to F$ which satisfies

$$\|f(x) - T(x)\| \leq \Phi(x,y) \tag{13}$$

for all $x \in E$. If moreover $f(tx)$ is continuous in t for fixed $x \in E$, then T is linear.

This stability theorem is called Gavruta stabiliy via Hyers method of functional equations.
A special case of Gavruta's theorem was seized by K. Ravi et.al. [26] in 2008, by considering the summation both sum and product of two p- norms introduced by J. M. Rassias. Hence it is called J. M. Rassias stability of functional equations.

Theorem 1.5 [26] Let $(E, \perp)$ denote an orthogonality normed space with norm $\| \, . \, \|_E$ and $(F, \| \, . \, \|_E)$ is a Banach space and $f : E \to F$ be a mapping which satisfying the inequality

$$\begin{aligned} &\left\| f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y) \right\|_F \\ &\leq \epsilon \left\{ \|x\|_E^p \, \|y\|_E^p + \left(\|x\|_E^{2p} + \|y\|_E^{2p} \right) \right\} \end{aligned} \tag{14}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either $m > 1; p < 1$ or $m < 1; p > 1$ with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$.
Then the limit

$$Q(x) = \lim_{n \to \infty} \frac{f(m^n x)}{m^{2n}} \tag{15}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2|m^2 - m^{2p}|} \, \|x\|_E^{2p} \tag{16}$$

for all $x \in E$.

The famous Cauchy additive functional equation is

$$\mathcal{A}\ (x_1 + x_2) = \mathcal{A}\ (x_1) + \mathcal{A}\ (x_2) \tag{17}$$

The Hyers type stability of (17) and other types of additive functional equations in various normed spaces were investigated in [2, 3, 6, 7, 9, 13, 14, 17, 18, 19, 27, 22, 25, 28] and references cited there in.

In 2017, J.M. Rassias et.al., [23] introduced the following generalized composite functional equation

$$\mathcal{A}\left(k\ \mathcal{A}(x_1) - \sum_{i=2}^{k+1}\ \mathcal{A}(x_i)\right) + k\ \mathcal{A}(x_1) + \sum_{i=2}^{k+1}\ \mathcal{A}(x_i) = \sum_{i=2}^{k+1}\ \mathcal{A}(x_1 + x_i) + \sum_{i=2}^{k+1}\ \mathcal{A}(x_1 - x_i) \tag{18}$$

for any real $k \in \mathbb{R}^+ - \{0\}$ and prove its fundamental stabilities in non-Archimedean normed spaces.

In this paper, we explore the Hyers - Gavruta Type stability of generalized composite functional equations (18) in Generalized 2 - Banach Spaces.

2. Basics of Generalized 2 - Banach Spaces

In this section, we give the basic definitions about Generalized 2- Banach Spaces given in [1, 8, 10, 11, 12].

Definition 2.1 [1] Let X be linear space. A function $N(.,.) : X \times X \to [0, \infty)$ is called a generalized 2-normed space if it satisfies to following :

(i) $N(x, y) = 0$ if and only if x and y are linearly independent vectors;
(ii) $N(x, y) = N(y, x)$ for all $x, y \in X$;
(iii) $N(\lambda x, y) = |\lambda| N(x, y)$ for all $x, y \in X$;
(iv) $N(x + y, z) \leq N(x, z) + N(y, z)$ for all $x, y, z \in X$.

The generalized 2-normed space is denoted by $(X, N(.,.))$.

Definition 2.2 [1] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called convergent if there exist $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y) = 0$ then

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https://doi.org/10.26524/cm181

$\lim_{n\to\infty} N(x_n, y) = N(x, y)$ for all $y \in X$.

Definition 2.3 [1] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called Cauchy sequence if there exist two lineary independent elements y and z in X such that $\{N(x_n, y)\}$ and $\{N(x_n, z)\}$ are real Cauchy sequences.

Definition 2.4 [1] A generalized 2-normed space $(X, N(.,.))$ is called generalized 2-Banach space if every Cauchy sequence is convergent.

3 Hyers - Gavruta Type Stability Results

To prove stability theorems, throughout this paper consider Z_1 be a generalized 2-normed space and Z_2 be generalized 2-Banach space, respectively.

Theorem 3.1 Assume $\mathcal{A} : Z_1 \to Z_2$ be a mapping for which there exist a function $\Psi : Z_1^{k+1} \to [0, \infty)$ and all $\Delta \in Z_1$ with the condition

$$\lim_{\delta\to\infty} \frac{1}{2^{\delta\alpha}} \Psi\left(\left(2^{\delta\alpha}x_1, 2^{\delta\alpha}x_2, \cdots, 2^{\delta\alpha}x_k, 2^{\delta\alpha}x_{k+1}\right), \Delta\right) = 0 \tag{19}$$

such that the functional inequality

$$\mathbf{N}\Bigg(\Bigg(\mathcal{A}\Big(k\ \mathcal{A}(x_1) - \sum_{i=2}^{k+1} \mathcal{A}(x_i)\Big) + k\ \mathcal{A}(x_1) + \sum_{i=2}^{k+1} \mathcal{A}(x_i) - \sum_{i=2}^{k+1} \mathcal{A}(x_1 + x_i) - \sum_{i=2}^{k+1} \mathcal{A}(x_1 - x_i)\Bigg),\ \Delta\Bigg) \leq \Psi\Big((x_1, x_2, x_3, \cdots, x_k, x_{k+1}),\ \Delta\Big) \tag{20}$$

for all $x_1, \cdots, x_k, x_{k+1} \in Z_1$. Then there exists a unique additive mapping $B : Z_1 \to Z_2$ satisfying the functional equation (18) and

$$\mathbf{N}\left(\left(\mathcal{A}(x) - B(x)\right),\ \Delta\right) \leq \frac{1}{2k} \sum_{\beta=\frac{1-\alpha}{2}}^{\infty} \frac{1}{2^{\beta\alpha}}\ \Psi\Big((2^{\beta\alpha}x, 2^{\beta\alpha}x, 2^{\beta\alpha}x, \cdots, 2^{\beta\alpha}x, 2^{\beta\alpha}x),\ \Delta\Big) \tag{21}$$

for all $x \in Z_1$. The mapping $B(x)$ is defined by

$$\lim_{\delta\to\infty} \mathbf{N}\left(\frac{\mathcal{A}(2^{\delta\alpha}x)}{2^{\delta\alpha}}, \Delta\right) = \mathbf{N}\left(B(x), \Delta\right) \tag{22}$$

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for all $x \in Z_1$ and $\alpha = \pm 1$.
Proof. Substituting $x = x_1 = x_2 = \cdots = x_k = x_{k+1}$ in (20) , we get

$$\mathbf{N}\left((2k\ \mathcal{A}(x) - k\ \mathcal{A}(2x)),\ \Delta\right) \leq \Psi\Big((x, x, \cdots, x, x),\ \Delta\Big) \tag{23}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Using (G2N3) in (23), we land

$$\mathbf{N}\left(\left(\mathcal{A}(x) - \frac{1}{2}\ \mathcal{A}(2x)\right),\ \Delta\right) \leq \frac{1}{2\ k}\ \Psi\Big((x, x, \cdots, x, x),\ \Delta\Big) \tag{24}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Again substituting x by $2x$ and dividing by 2 in (24), we reach

$$\mathbf{N}\left(\left(\frac{1}{2}\ \mathcal{A}(2x) - \frac{1}{2^2}\ \mathcal{A}\left(2^2 x\right)\right),\ \Delta\right) \leq \frac{1}{2^2\ k}\ \Psi\Big((2x, 2x, \cdots, 2x, 2x),\ \Delta\Big) \tag{25}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Using (G2N4), from (24) and (25), we have

$$\begin{aligned}\mathbf{N}&\left(\left(\mathcal{A}(x) - \frac{1}{2^2}\ \mathcal{A}\left(2^2 x\right)\right),\ \Delta\right) \\ &\leq \frac{1}{2\ k}\left\{\Psi\Big((x, x, \cdots, x, x),\ \Delta\Big) + \frac{1}{2}\ \Psi\Big((2x, 2x, \cdots, 2x, 2x),\ \Delta\Big)\right\}\end{aligned} \tag{26}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Using induction on a positive integer δ , we arrive

$$\mathbf{N}\left(\left(\mathcal{A}(x) - \frac{1}{2^\delta}\ \mathcal{A}\left(2^\delta x\right)\right),\ \Delta\right) \leq \frac{1}{2\ k}\sum_{\beta=0}^{\delta-1}\frac{1}{2^\beta}\ \Psi\Big((2^\beta x, 2^\beta x, \cdots, 2^\beta x, 2^\beta x),\ \Delta\Big) \tag{27}$$

for all $x \in Z_1$and all $\Delta \in Z_1$. By substituting x by 2^γ and dividing by 2^γ in (27),

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for any $\gamma, \delta > 0$, we deduce

$$\begin{aligned}
&\mathbf{N}\left(\left(\frac{1}{2^{\gamma}}\ \mathcal{A}\left(2^{\gamma}x\right) - \frac{1}{2^{\delta+\gamma}}\ \mathcal{A}\left(2^{\delta+\gamma}x\right)\right),\ \Delta\right)\\
&\qquad = \frac{1}{2^{\gamma}}\ \mathbf{N}\left(\left(\mathcal{A}\left(2^{\gamma}x\right) - \frac{1}{2^{\delta}}\ \mathcal{A}\left(2^{\delta+\gamma}x\right)\right),\ \Delta\right)\\
&\qquad \leq \frac{1}{2\ k}\sum_{\beta=0}^{\delta-1}\ \frac{1}{2^{\beta+\gamma}}\ \Psi\Big((2^{\beta+\gamma}x, 2^{\beta+\gamma}x, 2^{\beta+\gamma}x, \cdots, 2^{\beta+\gamma}x, 2^{\beta+\gamma}x),\ \Delta\Big)\\
&\qquad \to 0 \quad as\ \ \gamma \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{N}\left(\left(\frac{1}{2^{\gamma}}\ \mathcal{A}\left(2^{\gamma}x\right) - \frac{1}{2^{\delta+\gamma}}\ \mathcal{A}\left(2^{\delta+\gamma}x\right)\right),\ \Delta_1\right)\\
&\qquad = \frac{1}{2^{\gamma}}\ \mathbf{N}\left(\left(\mathcal{A}\left(2^{\gamma}x\right) - \frac{1}{2^{\delta}}\ \mathcal{A}\left(2^{\delta+\gamma}x\right)\right),\ \Delta_1\right)\\
&\qquad \leq \frac{1}{2\ k}\sum_{\beta=0}^{\delta-1}\ \frac{1}{2^{\beta+\gamma}}\ \Psi\Big((2^{\beta+\gamma}x, 2^{\beta+\gamma}x, 2^{\beta+\gamma}x, \cdots, 2^{\beta+\gamma}x, 2^{\beta+\gamma}x),\ \Delta_1\Big)\\
&\qquad \to 0 \quad as\ \ \gamma \to \infty
\end{aligned}$$

for all $x \in Z_1$ and all $\Delta, \Delta_1 \in Z_1$.

Hence there exists two linearly independent elements Δ and Δ_1 in Z_1 such that

$$\left\{\mathbf{N}\left(\frac{1}{2^{\delta}}\ \mathcal{A}(2^{\delta}x), \Delta\right)\right\} and \left\{\mathbf{N}\left(\frac{1}{2^{\delta}}\ \mathcal{A}(2^{\delta}x), \Delta_1\right)\right\}$$

are real Cauchy sequences. Hence the sequence

$$\left\{\frac{1}{2^{\delta}}\ \mathcal{A}(2^{\delta}x)\right\},$$

is a Cauchy sequence. Since Z_2 is complete, there exists a mapping $B : Z_1 \to Z_2$ such that

$$\lim_{\delta\to\infty} \mathbf{N}\left(\frac{1}{2^{\delta}}\ \mathcal{A}(2^{\delta}x), \Delta\right) = N\Big(B(x), \Delta\Big)$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. In (27) approaching $\delta \to \infty$, we observe that (21) holds for all $x \in Z_1$ and all $\Delta \in Z_1$ for $\alpha = 1$.

To prove that $B(x)$ satisfies (18), substituting

$$(x_1, x_2, x_3, \cdots, x_k, x_{k+1}) \quad by \quad (2^\delta x_1, 2^\delta x_2, 2^\delta x_3, \cdots, 2^\delta x_k, 2^\delta x_{k+1})$$

and dividing by 2^δ in (20), we reach

$$\mathbf{N}\left(\frac{1}{2^\delta}\left(\mathcal{A}\Big(k\ \mathcal{A}\left(2^\delta\ x_1\right) - \sum_{i=2}^{k+1} \mathcal{A}\left(2^\delta\ x_i\right)\Big) + k\ \mathcal{A}\left(2^\delta\ x_1\right) + \sum_{i=2}^{k+1} \mathcal{A}\left(2^\delta\ x_i\right) - \sum_{i=2}^{k+1} \mathcal{A}\left(2^\delta\ (x_1 + x_i)\right) - \sum_{i=2}^{k+1} \mathcal{A}\left(2^\delta\ (x_1 - x_i)\right)\right),\ \Delta\right) \leq \frac{1}{2^\delta}\ \Psi\Big((2^\delta x_1, 2^\delta x_2, 2^\delta x_3, \cdots, 2^\delta x_k, 2^\delta x_{k+1}),\ \Delta\Big) \tag{28}$$

for all $x_1, x_2, x_3, \cdots, x_k, x_{k+1} \in Z_1$ and all $\Delta \in Z_1$. Approaching $\delta \to \infty$ in (28) using (19), definition of $B(x)$ and (G2N1), we see that

$$B\left(k\ \mathcal{A}\left(x_1\right) - \sum_{i=2}^{k+1} B\left(x_i\right)\right) + k\ B\left(x_1\right) + \sum_{i=2}^{k+1} B\left(x_i\right) = \sum_{i=2}^{k+1} B\left(x_1 + x_i\right) + \sum_{i=2}^{k+1} B\left(x_1 - x_i\right).$$

So, $B(x)$ satisfying the functional equation (18) for all $x_1, x_2, x_3, \cdots, x_k, x_{k+1} \in Z_1$ and all $\Delta \in Z_1$.

Finally, to show $B(x)$ is unique, let $B'(x)$ be another additive mapping satisfying (18) and (21), then

$$\begin{aligned}
\mathbf{N}\left(B(x) - B'(x), \Delta\right) &= \frac{1}{2^\gamma}\mathbf{N}\left(B(2^\gamma x) - B'(2^\gamma x), \Delta\right) \\
&\leq \frac{1}{2^\gamma}\left\{\mathbf{N}\left(B(2^\gamma x) - \mathcal{A}(2^\gamma x), \Delta\right) + \mathbf{N}\left(\mathcal{A}(2^\gamma x) - B'(2^\gamma x), \Delta\right)\right\} \\
&\leq \frac{2}{2\ k}\sum_{\beta=0}^{\infty} \frac{1}{2^{\beta+\gamma}}\ \Psi\Big((2^{\beta+\gamma}x, 2^{\beta+\gamma}x, \cdots, 2^{\beta+\gamma}x, 2^{\beta+\gamma}x),\ \Delta\Big) \\
&\to 0 \quad as\ \ \gamma \to \infty
\end{aligned}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Thus $B'(x)$ is unique.

Therefore, the theorem is true for $\alpha = 1$.

Now, substituting $x = \frac{x}{2}$ in (23), we observe

$$\mathbf{N}\left(\left(2k\ \mathcal{A}\left(\frac{x}{2}\right) - k\ \mathcal{A}\left(x\right)\right),\ \Delta\right) \leq \Psi\Big(\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}, \frac{x}{2}\right),\ \Delta\Big) \tag{29}$$

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for all $x \in Z_1$ and all $\Delta \in Z_1$. Using (G2N3) in (29), we land

$$\mathbf{N}\left(\left(2\mathcal{A}\left(\frac{x}{2}\right) - \mathcal{A}\left(x\right)\right),\ \Delta\right) \leq \frac{1}{k}\ \Psi\Big(\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}, \frac{x}{2}\right),\ \Delta\Big) \tag{30}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Again substituting x by $\frac{x}{2}$ and multiply by 2 in (30), we reach

$$\mathbf{N}\left(\left(2^2\mathcal{A}\left(\frac{x}{2^2}\right) - 2\mathcal{A}\left(\frac{x}{2}\right)\right),\ \Delta\right) \leq \frac{2}{k}\ \Psi\Big(\left(\frac{x}{2^2}, \frac{x}{2^2}, , \cdots, \frac{x}{2^2}, \frac{x}{2^2}\right),\ \Delta\Big) \tag{31}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Using (G2N4) it follows from (30) and (31), we have

$$\begin{aligned}&\mathbf{N}\left(\left(2^2\mathcal{A}\left(\frac{x}{2^2}\right) - \mathcal{A}\left(x\right)\right),\ \Delta\right)\\ &\leq \frac{1}{k}\Big\{\ \Psi\Big(\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}, \frac{x}{2}\right),\ \Delta\Big) + 2\ \Psi\Big(\left(\frac{x}{2^2}, \frac{x}{2^2}, , \cdots, \frac{x}{2^2}, \frac{x}{2^2}\right),\ \Delta\Big)\Big\}\end{aligned} \tag{32}$$

for all $x \in Z_1$ and all $\Delta \in Z_1$. Using induction on a positive integer δ , we arrive

$$\begin{aligned}\mathbf{N}\left(\left(2^\delta\mathcal{A}\left(\frac{x}{2^\delta}\right) - \mathcal{A}\left(x\right)\right),\ \Delta\right) &\leq \frac{1}{k}\sum_{\beta=1}^{\delta}\ 2^{\beta-1}\ \ \Psi\Big(\left(\frac{x}{2^\beta}, \frac{x}{2^\beta}, \frac{x}{2^\beta}, \cdots, \frac{x}{2^\beta}, \frac{x}{2^\beta}\right),\ \Delta\Big)\\ &= \frac{1}{2\ k}\sum_{\beta=1}^{\delta}\ 2^{\beta}\ \ \Psi\Big(\left(\frac{x}{2^\beta}, \frac{x}{2^\beta}, \cdots, \frac{x}{2^\beta}, \frac{x}{2^\beta}\right),\ \Delta\Big)\end{aligned} \tag{33}$$

for all $x \in Z_1$and all $\Delta \in Z_1$. The rest of the proof is similar to that of $\alpha = 1$. Thus the theorem holds for $\alpha = -1$. This completes the proof of the theorem.

Corollary 3.2 Assume $\mathcal{A} : Z_1 \to Z_2$ be a mapping for which there exists constants

a, b and all $\Delta \in Z_1$ such that the functional inequality

$$\mathbf{N}\left(\left(\mathcal{A}\Big(k\ \mathcal{A}(x_1) - \sum_{i=2}^{k+1} \mathcal{A}(x_i)\Big) + k\ \mathcal{A}(x_1) + \sum_{i=2}^{k+1} \mathcal{A}(x_i) - \sum_{i=2}^{k+1} \mathcal{A}(x_1 + x_i) + \sum_{i=2}^{k+1} \mathcal{A}(x_1 - x_i)\right),\ \Delta\right) \leq \begin{cases} a, \\ a\ \sum_{i=1}^{k+1} ||x_i,\ \Delta||^b, & b \neq 1; \\ a\ \sum_{i=1}^{k+1} ||x_i,\ \Delta||^{b_i}, & b_i \neq 1; \\ a\ \prod_{i=1}^{k+1} ||x_i,\ \Delta||^b, & (k+1)b \neq 1; \\ a\ \prod_{i=1}^{k+1} ||x_i,\ \Delta||^{b_i}, & \sum_{i=1}^{k+1} b_i \neq 1; \end{cases} \tag{34}$$

for all $x_1, x_2, x_3, \cdots, x_k, x_{k+1} \in Z_1$. Then there exists a unique additive mapping $B : Z_1 \to Z_2$ satisfying the functional equation (18) and

$$\mathbf{N}\left((\mathcal{A}(x) - B(x)),\ \Delta\right) \leq \begin{cases} \dfrac{a}{|k|}, \\ \dfrac{a\ (k+1)||x,\ \Delta||^b}{k\ |2 - 2^b|}, \\ \displaystyle\sum_{i=1}^{k+1} \dfrac{a\ ||x,\ \Delta||^{b_i}}{k\ |2 - 2^{b_i}|}, \\ \dfrac{a\ ||x,\ \Delta||^{(k+1)b}}{k\ |2 - 2^{(k+1)b}|}, \\ \dfrac{a\ ||x,\ \Delta||^{\sum_{i=1}^{k+1} b_i}}{k\ |2 - 2^{\sum_{i=1}^{k+1} b_i}|}, \end{cases} \tag{35}$$

for all $x \in Z_1$.

4 Conclusion

The authors analyze the Gavruta stability of generalized composite functional equations in Generalized 2- Banach Spaces via Hyers method is analyzed.

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