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Stability of a Generalised Difference Functional Equation

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Abstract

In this paper, we discuss the generalised Ulam Hyers stability of a generalized difference functional equation

$$A(g + \gamma h) - A(g + (\gamma - 1)h) = 1!A(g)$$

where $\gamma \neq 0, \pm 1$ in Fuzzy Banach space using direct and fixed point methods.

Key words: Difference Functional Equations, Fuzzy Banach Spaces, Hyers Direct Method,

AMS classification: 39B52, 32B72, 32B82

1. Introduction

The stability of functional equations is a hot topic that was delt in the last eight decades. In 1940, S.M. Ulam , gave a widespread of talk before a Mathematical Colloquium at the University of Wisconsin in which he introduced the number of important unsolved problems. One of them is the first point of a new line of investigation, the Stability Problem.

The general solution and stability of the following additive functional equations

$$A(x + y) = A(x) + A(y) \tag{1}$$

$$f(2x - y) + f(x - 2y) = 3f(x) - 3f(y) \tag{2}$$

$$f(x + y - 2z) + f(2x + 2y - z) = 3f(x) + 3f(y) - 3f(z) \tag{3}$$

$$M(p) + M(q + r) = M(p + q) + M(r) \tag{4}$$

$$p(2a \pm b \pm c) + p(a \pm b) + p(a \pm c) \tag{5}$$

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were investigated by J. Aczel and J. Dhombres [2], D. O. Lee [25], K. Ravi, M. Arunkumar [35] and M.Arunkumar [5, 6].

In this paper, the authors present and establish the generalized Ulam-Hyers stability of a generalized difference functional equation

$$A(g+\gamma h) - A(g+(\gamma-1)h) = 1!A(h) \tag{6}$$

where $\gamma \neq 0, \pm 1$ in Fuzzy Banach spaces, via two alternate methods.

2.General Solution

Now, we provide the general solution of the functional equation (6).

Theorem 2.1 Let V_1 and V_2 are real Vector Spaces. The mapping $A : V_1 \to V_2$ satisfying the functional equation (1) if and only if $A : V_1 \to V_2$ be a function satisfying the functional equation (6) for all $g, h \in V_1$.

Proof. Given $A : V_1 \to V_2$ is a functional satisying functional eqaution (1).
Put $x = 0$ and $y = 0$ in (1) , we get

$$A(0) = 0 \tag{7}$$

Put $x = h$ and $y = -h$ in (1), we obtain

$$-A(h) = A(-h) \tag{8}$$

for all $h \in V_1$.Put $x = h$ and $y = h$ in (1), we receive

$$A(2h) = 2A(h) \tag{9}$$

for all $h \in V_1$.Put $x = h$ and $y = 2h$ in (1) and using (9), we have

$$A(3h) = 3A(h) \tag{10}$$

for all $h \in V_1$.From (9) and (10), in common for any positive integer K, we have

$$A(Kh) = KA(h) \tag{11}$$

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for all $h \in V_1$.Put $x = h$ and $y = g + (\gamma - 1)h$ in (1), we arrive

$$A(g + \gamma h) - A(g + (\gamma - 1)h) = 1!A(h) \quad (12)$$

for all $g, h \in V_1$.
Conversly.$A : V_1 \to V_2$ is a functional satisying functional eqaution (6).
Put $g = 0$ and $h = 0$ in (6) , we get

$$A(0) = 0 \quad (13)$$

Put $g = -\gamma h$ and $h = h$ in (6), we obtain

$$-A(h) = A(-h) \quad (14)$$

for all $h \in V_1$.Put $g = -h - \gamma h$ and $h = h$ in (6) and using (14), we receive

$$A(2h) = 2A(h) \quad (15)$$

for all $h \in V_1$.Put $g = -h - \gamma 2h$ and $h = 2h$ in (6) and using (15), we have

$$A(3h) = 3A(h) \quad (16)$$

for all $h \in V_1$.From (15) and (16), in common for any positive integer k, we have

$$A(kh) = kA(h) \quad (17)$$

for all $h \in V_1$.Put $g = g - \gamma h$ and $h = h$ in (6), we arrive

$$A(g - h) = A(g) - A(h) \quad (18)$$

for all $g, h \in V_1$.Put $g = x$ and $h = -y$ in (6), we arrive

$$A(x + y) = A(x) + A(y) \quad (19)$$

for all $x, y \in V_1$. Hence the proof of theorem complete.

Definitions about Fuzzy Normed Space

Now we use the basic definition of fuzzy normed spaces discussed in [15] and [28, 29].

Definition 2.2 [15] Let X be a real linear space. A function $N : X \times R \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$,

(F1) $N(x,t) = 0$ for $t \leq 0$;
(F2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(F3) For all $t > 0, N(cx,t) = N\Big(x, \frac{t}{|c|}\Big)$ if $c \neq 0$;
(F4) $N(x+y, s+t) \geq min\Big\{N(x,s), N(y,t)\Big\}$;
(F5) $N(x,.)$ is a non-decreasing function on R and $\lim_{t\to\infty} N(x,t) = 1$;
(F6) For $x \neq 0, N(x,.)$is (upper semi) continuous on R.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t.

Definition 2.3 Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$.In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.4 A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 2.5 Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

To prove the stability results, we assume that $V_1, (V_1, N)$ and (V_2, N') are linear space, fuzzy normed space and fuzzy banach space respectively.

3.Stability Results in Fuzzy Banach Space : Direct Method

In this part, the generalised Ulam-Hyers stability of a functional equation (6) is analyzed in Fuzzy Banach space applying the direct method.

Theorem 3.1 Let $P = \{1, -1\}$. Let $\phi : V_1^2 \to (0, \infty]$ be a mapping with $0 < (\frac{\tau}{2})^P < 1$.

$$N'\bigg(\phi\Big((-\gamma+2)2^{Pn}h, 2^{Pn}h\Big), r\bigg) \geq N'\bigg(\tau^{Pn}\phi\Big((-\gamma+2)h, h\Big), r\bigg) \tag{20}$$

for all $h \in V_1$ and all $\tau > 0$.

$$\lim_{n\to\infty} N^{'}\Big(\phi\big(2^{Pn}g, 2^{Pn}h\big), 2^{Pn}r\Big) = 1 \tag{21}$$

for all $g, h \in V_1$ and all $r > 0$. Suppose that a function $A : V_1 \to V_2$ satisfies the inequality

$$N\big(A(g+\gamma h) - A(g+(\gamma-1)h) - 1!A(h), r\big) \geq N^{'}(\phi(g,h), r) \tag{22}$$

for all $g, h \in V_1$ and all $r > 0$ and there exists a unique mapping $\mathcal{A} : V_1 \to V_2$ which satisfying (6) and

$$N\Big(\big(\mathcal{A}(h) - A(h)\big), r\Big) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r|2-\tau|\Big) \tag{23}$$

for all $h \in V_1$ and all $r > 0$. Then limit of $\mathcal{A}(h)$ is defined as

$$\lim_{n\to\infty} N\Big(\mathcal{A}(h) - \frac{A(2^{Pn}h)}{2^{Pn}}, r\Big) = 1 \tag{24}$$

Proof. First let us prove for $P = 1$. Replacing (g, h) as $((-\gamma+2)h, h)$ in (22), we obtain

$$N\big(A(2h) - 2A(h), r\big) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{25}$$

$$N\left(\Big(\frac{A(2h)}{2} - A(h)\Big), \frac{r}{2}\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{26}$$

for all $h \in V_1$ and all $r > 0$. Now replace in r as $2r$ in the above in equality , we get

$$N\Big(\Big(\frac{A(2h)}{2} - A(h)\Big), r\Big) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), 2r\Big) \tag{27}$$

for all $h \in V_1$ and all $r > 0$. Replace h as $2^n h$ in (27), we have

$$N\left(\Big(\frac{A(2^{n+1}h)}{2^{n+1}} - \frac{A(2^n h)}{2^n}\Big), \frac{r}{2^n}\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), \frac{2r}{\tau^n}\Big) \tag{28}$$

for all $h \in V_1$ and all $r > 0$. Replace r as $2^n r$ in (28), we have

$$N\left(\Big(\frac{A(2^{n+1}h)}{2^{n+1}} - \frac{A(2^n h)}{2^n}\Big), r\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), \frac{2^{n+1}r}{\tau^n}\Big) \tag{29}$$

for all $h \in V_1$ and all $r > 0$. Replace r as $\frac{r}{\frac{2^{n+1}}{\tau^n}}$ in (29) , we arrive

$$N\left(\Big(\frac{A(2^{n+1}h)}{2^{n+1}} - \frac{A(2^n h)}{2^n}\Big), \frac{r}{2}\left(\frac{\tau}{2}\right)^n\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{30}$$

for all $h \in V_1$ and all $r > 0$. It is easy to show that

$$\sum_{i=0}^{n-1}\Big(\frac{A(2^{i+1}h)}{2^{i+1}} - \frac{A(2^i h)}{2^i}\Big) = \Big(\frac{A(2^n h)}{2^n} - A(h)\Big) \tag{31}$$

for all $h \in V_1$. From (30) and (31) , we receive

$$N\left(\sum_{i=0}^{n-1}\Big(\frac{A(2^{i+1}h)}{2^{i+1}} - \frac{A(2^i h)}{2^i}\Big), \frac{r}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\right) \geq min \bigcup_{i=0}^{n-1}\left(N\left(\Big(\frac{A(2^{i+1}h)}{2^{i+1}} - \frac{A(2^i h)}{2^i}\Big), \frac{r}{2}\left(\frac{\tau}{2}\right)^i\right)\right) \tag{32}$$

$$N\left(\sum_{i=0}^{n-1}\Big(\frac{A(2^{i+1}h)}{2^{i+1}} - \frac{A(2^i h)}{2^i}\Big), \frac{r}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{33}$$

$$N\left(\Big(\frac{A(2^n h)}{2^n} - A(h)\Big), \frac{r}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{34}$$

for all $h \in V_1$ and all $r > 0$. Replace h as $2^m h$ in the above inequality, we obtain

$$N\left(\Big(\frac{A(2^{n+m}h)}{2^{n+m}} - \frac{A(2^m h)}{2^m}\Big), \frac{r}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\frac{1}{2^m}\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), \frac{r}{\tau^m}\Big) \tag{35}$$

for all $h \in V_1$ and all $r > 0$. Replace r as $\tau^m r$ in the above inequality, we obtain

$$N\left(\Big(\frac{A(2^{n+m}h)}{2^{n+m}} - \frac{A(2^m h)}{2^m}\Big), \frac{r}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\frac{\tau^m}{2^m}\right) \geq N^{'}\Big(\phi\big((-\gamma+2)h, h\big), r\Big) \tag{36}$$

for all $h \in V_1$ and all $r > 0$. Replace r as $\frac{r}{\frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\left(\frac{\tau}{2}\right)^m}$ in the above inequality

$$N\left(\left(\frac{A(2^{n+m}h)}{2^{n+m}} - \frac{A(2^m h)}{2^m}\right), r\right) \geq N'\left(\phi\big((-\gamma+2)h, h\big), \frac{r}{\frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{\tau}{2}\right)^i\left(\frac{\tau}{2}\right)^m}\right) \tag{37}$$

for all $h \in V_1$ and all $r > 0$ and all $m, n \geq 0$. since $0 < \tau < 2$ and $\sum_{i=0}^{n}\left(\frac{\tau}{2}\right)^i < \infty$, the cauchy criterion for convergence and(F5) implies that $\{\frac{A(2^m h)}{2^m}\}$ is Cauchy sequence in (V_2, N'). since (V_2, N') is a Fuzzy Banach space, this sequence is converges to some point $\mathcal{A} \in V_2$. Now we can define the mapping $\mathcal{A} : V_1 \to V_2$ by

$$\lim_{n\to\infty} N\left(\mathcal{A}(h) - \frac{A(2^{Pn}h)}{2^{Pn}}, r\right) = 1 \tag{38}$$

for all $h \in V_1$ and all $r > 0$. Let $m = 0$ and $n \to \infty$ in (37) and using F[6] , we get

$$N\Big(\big(\mathcal{A}(h) - A(h)\big), r\Big) \geq N'\Big(\phi\big((-\gamma+2)h, h\big), r(2-\tau)\Big) \tag{39}$$

for all $h \in V_1$ and all $r > 0$.To prove $\mathcal{A}$ satisfies (6). Replace g as $2^n g$ and h as $2^n h$ in 22, we obtain

$$N\left(\left(\frac{A\big(2^n(g+\gamma h)\big)}{2^n}\right) - \left(\frac{A\big(2^n(g+(\gamma-1)h)\big)}{2^n}\right) - \left(\frac{1!A(2^n h)}{2^n}\right), \frac{r}{2^n}\right) \geq N'\big(\phi(2^n g, 2^n h), r\big) \tag{40}$$

for all $g, h \in V_1$ and all $r > 0$.

$$\begin{aligned} &N\big(\mathcal{A}(g+\gamma h) - \mathcal{A}(g+(\gamma-1)h) - 1!\mathcal{A}(h), r\big) \\ &\quad \geq min\Bigg\{N\Big(\mathcal{A}(g+\gamma h) - \frac{A(2^n(g+\gamma h))}{2^n}, \frac{r}{4}\Big) \\ &\qquad + N\Big(-\mathcal{A}(g+(\gamma-1)h) + \frac{A(2^n(g+(\gamma h-1))}{2^n}, \frac{r}{4}\Big) + N\Big(-1!\mathcal{A}(h) + 1!\frac{A(2^n h)}{2^n}, \frac{r}{4}\Big) \\ &\qquad\quad + N\Big(\frac{A(2^n(g+\gamma h))}{2^n} - \frac{A(2^n(g+(\gamma h-1))}{2^n} - 1!\frac{A(2^n h)}{2^n}\Big)\Bigg\} \\ &\quad \geq min\{1, 1, 1, 1\} \end{aligned}$$

we obtain that $\mathcal{A}$ satisfies the functional equation (6).

To prove that $\mathcal{A}$ is unique. Let $\mathcal{A}^{'}$ be the another additive mapping satisfying (6) and (39). Now,

$$\begin{aligned} & N(\mathcal{A}(h)-\mathcal{A}^{'}(h),r) \\ & \quad = N\Big(\Big\{\Big(\frac{\mathcal{A}(2^n h)}{2^n}-\frac{A(2^n h)}{2^n}\Big)+\Big(\frac{\mathcal{A}^{'}(2^n h)}{2^n}-\frac{A(2^n h)}{2^n}\Big)\Big\},\frac{r}{2}+\frac{r}{2}\Big) \\ & \quad \geq min\Big\{N\Big(\frac{\mathcal{A}(2^n h)}{2^n}-\frac{A(2^n h)}{2^n},\frac{r}{2}\Big),N\Big(-\frac{\mathcal{A}^{'}(2^n h)}{2^n}+\frac{A(2^n h)}{2^n},\frac{r}{2}\Big)\Big\} \\ & \quad \geq min\Big\{N^{'}\Big(\phi((-\gamma+2)h,h),\frac{r(2-\tau)2^n}{2.\tau^n}\Big),N^{'}\Big(\phi((-\gamma+2)h,h),\frac{r(2-\tau)2^n}{2.\tau^n}\Big)\Big\} \\ & \quad = N^{'}\Big(\phi((-\gamma+2)h,h),\frac{r(2-\tau)2^n}{2.\tau^n}\Big) \end{aligned}$$

for all $h\in V_1$ and all $r>0$. Since, $\lim_{n\to\infty}\frac{r(2-\tau)2^n}{2.\tau^n}=\infty$. we obtain $\mathcal{A}(h)$ is unique. Hence the theorem is true for $P=1$. Now we have to prove for $P=-1$. Replace h as $\frac{h}{2}$ in (25), we get

$$N\Big(A(h)-2A\Big(\frac{h}{2}\Big),r\Big)\geq N^{'}\Big(\phi\Big((-\gamma+2)\frac{h}{2},\frac{h}{2}\Big),r\Big) \tag{41}$$

for all $g,h\in V_1$ and all $r>0$. The remaining proof is similar to that of above case. Hence, the theorem is true for P=-1. Finally the proof of the theorem is now completed.

Corollary 3.2 Let c and d be non negative real numbers. Let a function $A:V_1\to V_2$ satisfies the inequality

$$N\big(A(g+\gamma h)-A(g+(\gamma-1)h)-1!A(h),r\big)\geq\begin{cases} c, \\ c\{||g||^d+||h||^d\}, \\ c\{||g||^d||h||^d\}, \\ c\{||g||^{d_1}+||h||^{d_2}\} \end{cases} \tag{42}$$

for all $g,h\in V_1$ and all $r>0$. Then there exists a unique additive function $\mathcal{A}:V_1\to$

V_2 such that

$$N\Big(\big(\mathcal{A}(h)-A(h)\big),r\Big) \geq \begin{cases} N^{'}(c,r|1|), \\ N^{'}\big(c\{||h||^{d}((-\gamma+2)^{d}+1)\},r|2-2^{d}|\big),d\neq 1 \\ N^{'}\big(c\{||h||^{2d}((-\gamma+2)^{d}+1)\},r|2-2^{2d}|\big),d\neq \frac{1}{2} \\ N^{'}\big(c\{||h||^{d_1+d_2}((-\gamma+2)^{d_1}+1)\},r|2-2^{d_1+d_2}|\big),d_1+d_2\neq 1 \end{cases} \tag{43}$$

for all $h\in V_1$ and all $r>0$.

4.Stability Results in Fuzzy Banach Space Space :Fixed Point Method

In this part, the generalised Ulam-Hyers stability of a functional equation (6) is analyzed in Fuzzy Banach space applying the Fixed Point method.

Theorem 4.1 [27] (The alternative of fixed point) Let (X,d) be a complete generalized metric space and Let $J:X\to Y$ be a strictly contractive mapping with Lipschitz constant $L<1$.Then for each given element $x\in X$, either $d\big(J^n x, J^{(n+1)}x\big)=\infty$ for all non negative integers n or there exists positive integers n_0 such that

(FP1) $d\big(J^n x, J^{(n+1)}x\big)<\infty$ for all $n\geq n_0$;
(FP2) The sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(FP3) y^* is the unique fixed point of J in the set $Y=\{y\in X/d(J^{(n_0)}x,y<\infty)$;
(FP4) $d(y,y^*)\leq\Big(\frac{1}{1-L}\Big)d(y,Jy)$ for all $y\in X$.

Theorem 4.2 Assume that $A:V_1\to V_2$ be a mapping and there exist a function $\phi:V_1^2\to(0,\infty]$ is a function fulfilling the inequality

$$N\big(A(g+\gamma h)-A(g+(\gamma-1)h)-1!A(h),r\big)\geq N^{'}(\phi(g,h),r) \tag{44}$$

with the condition

$$\lim_{n\to\infty} N^{'}\Big(\phi\big(\mathcal{J}_i^n g,\mathcal{J}_i^n h\big),\mathcal{J}_i^n r\Big)=1 \tag{45}$$

for all $g, h \in V_1$ and for all $r > 0$. In the case there exists $L = L(i)$ such that

$$\mathcal{J}_i^n = \begin{cases} 2; i = 0 \\ \frac{1}{2}; i = 1 \end{cases} \tag{46}$$

has the property

$$h \to \mathcal{B}(h) = \phi\Big(\big(-\gamma + 2\big)\frac{h}{2}, \frac{h}{2}\Big) \tag{47}$$

$$N'\Big(L\frac{\mathcal{B}(\mathcal{J}_i h)}{\mathcal{J}_i}, r\Big) = N'\big(\mathcal{B}(h), r\big) \tag{48}$$

for all $h \in V_1$. There exists a unique mapping $\mathcal{A} : V_1 \to V_2$ which satisfying (6) and

$$N\Big(\mathcal{A}(h) - A(h), r\Big) \geq N'\Big(\frac{L^{1-i}}{1-L}\mathcal{B}(h), r\Big) \tag{49}$$

for all $h \in V_1$ and for all $r > 0$.

Proof. Define a set $\Psi = \{a | a : V_1 \to V_2, a(0) = 0\}$ and introduce the generalized metric on Ψ by

$$d(a, b) = \inf\{k \in [o, \infty)/N\Big(a(h) - b(h), r\Big) \geq N'\big(\mathcal{B}(h), kr\big) \tag{50}$$

It is easy to see that (Ψ, d) is complete. Define a function $T : \Psi \to \Psi$ by

$$Ta(h) = \frac{a(\mathcal{J}_i h)}{\mathcal{J}_i} \tag{51}$$

for all $h \in V_1$. For $a, b \in \Psi$

$$\begin{aligned} d(a, b) = k &\Rightarrow N\Big(a(h) - b(h), r\Big) \geq N'\big(\mathcal{B}(h), kr\big) \\ &\Rightarrow N\Big\{\mathcal{J}_i\Big(\frac{a(\mathcal{J}_i h)}{\mathcal{J}_i} - \frac{b(\mathcal{J}_i h)}{\mathcal{J}_i}\Big), r\Big\} \geq N'\big(\mathcal{B}(h), kr\big) \\ &\Rightarrow N\big\{(Ta(h) - Tb(h)), r\big\} \geq kL \\ &\Rightarrow d(Ta, Tb) \leq Ld(a, b). \end{aligned}$$

This implies that

$$d\big(Ta, Tb\big) \leq Ld(a, b) \tag{52}$$

(ie) L is a strictly contractive mapping in Ψ with lipschitz constant L.
Using (47) in (41) for the case $i = 1$ it reduces to

$$N\Big(A(h) - 2A\Big(\frac{h}{2}\Big), r\Big) \geq N'\Big(\mathcal{B}(h), r\Big)$$

$$d(A, TA) \leq 1 = L^{(1-i)} \tag{53}$$

Using (47) in (27) for the case $i = 0$ it reduces to

$$N\Big(\Big(\frac{A(2h)}{2} - A(h)\Big), r\Big) \geq N'\Big(\mathcal{B}(h), 2r\Big)$$

$$d(TA, A) \leq L = L^{(1-i)} \tag{54}$$

From (53) and (54), we reach

$$d(A, TA) \leq L^{1-i} < \infty.$$

Hence, $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$ [FP1] holds.
By fixed point theorem [FP2],the sequence $(T^n x)$ convergent to a fixed point y^* of T.
It follows that there exists a fixed point $\mathcal{A}$ of T in Ψ such that

$$\mathcal{A}(h) = N - \lim_{n\to\infty} \frac{A(\mathcal{J}_i h)}{\mathcal{J}_i} \tag{55}$$

In order to prove $\mathcal{A} : V_1 \to V_2$ is additive. The rest of the proof is similar tracing to that of Theorem 3.1 . By using fixed point theorem[FP3], define $\mathcal{A}$ is the unique fixed point of T in the set

$$\Delta = \{A \in \Psi : d(a, \mathcal{A}) < \infty\}$$

therefore $\mathcal{A}$ is a unique function such that by [FP4]

$$N\Big(\mathcal{A}(h) - A(h), r\Big) \geq N'\Big(\frac{L^{1-i}}{1-L}\mathcal{B}(h), r\Big)$$

for all $h \in V_1$ and all $r > 0$.

This completes the proof of the theorem.

Corollary 4.3 Let c and d be non negative real numbers. Let a function $A : V_1 \to V_2$ satisfies the inequality

$$N\big(A(g+\gamma h)-A(g+(\gamma-1)h)-1!A(h),r\big) \geq \begin{cases} c, \\ c\{||g||^d+||h||^d\}, \\ c\{||g||^d||h||^d\}, \\ c\{||g||^{d_1}+||h||^{d_2}\} \end{cases} \tag{56}$$

for all $g, h \in V_1$ and all $r > 0$. Then there exists a unique additive function $\mathcal{A} : V_1 \to V_2$ such that

$$N\Big(\big(\mathcal{A}(h)-A(h)\big),r\Big) \geq \begin{cases} N'(c, r|1|), \\ N'\big(c\{||h||^d((-\gamma+2)^d+1)\}, r|2-2^d|\big), d \neq 1 \\ N'\big(c\{||h||^{2d}((-\gamma+2)^d+1)\}, r|2-2^{2d}|\big), d \neq \frac{1}{2} \\ N'\big(c\{||h||^{d_1+d_2}((-\gamma+2)^{d_1}+1)\}, r|2-2^{d_1+d_2}|\big), d_1+d_2 \neq 1 \end{cases} \tag{57}$$

for all $h \in V_1$ and all $r > 0$.

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