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Solving Linear Heat Equation And Wave Equation Using Homotopy Perturbation Laplace-Carson Transform Method

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Abstract

In this paper, we will study about the advanced method is an appropriate union of the new integral transform named as “Laplace-Carson transform" and the homotopy perturbation method. Also finding the analytical solution of the heat equation and wave equation using the homotopy perturbation Laplace-Carson transform method (HPLCTM) is merged form of Laplace-Carson transform,homotopy perturbation method and He’s polynomial.

Key words: Laplace-carson transform,Homotopy Perturbation Method

AMS classification: 35K05, 35L05

1 Introduction

In mathematics, a partial differential equation (PDE)[4] is an equation which imposes relations between the various partial derivatives of a multivariable function.

Partial differential equations are common in mathematical oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrödinger equation, etc).

The heat equation[1] is a certain partial differential equation. Solutions of the heat equation are sometimes known as caloric functions. The theory of the heat equation was first developed by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region.

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The (two-dimensional) wave equation is a second-order linear partial differential equation for the description of waves or standing wave fields –as they occur in classical physics-such as mechanical wave(e.g. water waves, sound waves and seismic waves) or electromagnetic waves (including light waves).

Laplace-carson transform[3] is derived from the classical Fourier integral. Based on the mathematical simplicity of the Laplace-carson Transform and its fundamental properties. Laplace–carson transform was named after Pierre Simon Laplace and Renshaw Carson, is an integral transform with significant applications in the field of physics and engineering, particularly in the field of railway engineering and to facilitate the process of solving ordinary and partial differential equations in the time domain.

Laplace-carson transform and some its fundamental properties are also used to solve differential equations. The main aim of the paper is to determine the solution of the linear heat equation and wave equation using the homotopy perturbation Laplace-carson transform method.

2 Laplace-carson Transform

Definition 2.1 The Laplace-carson transform(Mahgoub) transform of the function $U(t)$ for all t≥ 0 is defined as:

$$L[U(t)] = p\int_0^\infty U(t)e^{-pt}dt = u(p) \tag{1}$$

where L is Laplace-carson transform transform operator.

1. $U(t)$ is integrable in every interval I$\subset$R of finite length,
2. $U(t) \equiv 0$ for all $t < 0$,
3. There exists a real number $c > 0$ such that $|U(t)e^{-ct}| < 0$ for all values $t \geq 0$.

2.2 Properties of Laplace-Carson Transform:[4]

S.No.	Name of the property	Mathematical Form
1	Linearity	$L[aU_1(t)+bU_2(t)] = aL[U_1(t)]+bL[U_2(t)]$
2	change of Scale	$L[U(at)] = u(\frac{p}{a})$
3	Shifting	$L[e^{at}U(t)] = \frac{p}{(p-a)}u(p-a)$
4	First Derivative	$L[U'(t)] = pu(p) - pU(0)$
5	Second Derivative	$L[U''(t)] = p^2u(p) - p^2U(0) - pU'(0)$
6	nth Derivative	$L[U^n(t)] = p^nu(p) - p^nU(0) - p^{n-1}U'(0) - \ldots\ldots - pU^{n-1}(0)$
7	Convolution	$L[U_1(t) * U_2(t)] = \frac{1}{p}L[U_1(t)]L[U_2(t)]$

2.3 Laplace-Carson transforms of useful Mathematical Functions:[3]

S.No.	$U(t)$	$L[U(t)] = u(p)$
1	1	1
2	t	$\frac{1}{p}, p > 0$
3	t^2	$(\frac{2}{p^2}), p > 0$
4	t^n,n∈N	$(\frac{n!}{p^n}), p > 0$
5	$t^n, n > -1$	$\frac{1}{p^n}\Gamma(n+1), p > 0$
6	e^{at}	$\frac{p}{p-a}, p > a$
7	$sinat$	$\frac{ap}{p^2+a^2}, p > 0$
8	$cosat$	$\frac{p^2}{p^2+a^2}, p > 0$

2.4 Inverse Laplace-carson Transform :[3]

If $L[U(t)] = u(p)$ then $U(t)$ is called the inverse Laplace-carson transform of $u(p)$. Mathematically, it is represented as $U(t) = L^{-1}[u(p)]$, where the operator L^{-1} is called the inverse Laplace-carson transform operator. Inverse Laplace-carson transform of useful mathematical functions are presented in the below table.

S.No.	$u(p)$	$U(t) = L^{-1}[u(p)]$
1	1	1
2	$\frac{1}{p}$	t
3	$\frac{1}{p^2}$	$\frac{t^2}{2!}$
4	$\frac{1}{p^n}$,n∈N	$\frac{t^n}{n!}$
5	$\frac{1}{p^n}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6	$\frac{p}{p-a}$	e^{at}
7	$\frac{p}{p^2+a^2}$	$\frac{sinat}{a}$
8	$\frac{p^2}{p^2+a^2}$	$cosat$

3 Study of Heat equation with Laplace-carson Transform Homotopy Perturbation Method (LCTHPM)

In this section, we will study heat equation and its application by using homotopy perturbation Laplace-carson transform method.

3.1 solution of the Heat equation

Heat equation is

$$u_t(x,t) = u_{xx}(x,t) - au(x,t) + g(x,t) \tag{2}$$

with conditions

$$u(x,0) = h(x), u_t(x,0) = f(x), u(0,t) = g(x) \tag{3}$$

Taking the Laplace-carson transform on both sides of equation(3.1),

$$L[u_t(x,t)] = L[u_{xx}(x,t) - au(x,t)] + L[g(x,t)] \tag{4}$$

Using the convolution property of Laplace-carson transform,we get

$$pu(p) - pu(0) = L[u_{xx}(x,t) - au(x,t)] + L[g(x,t)] \tag{5}$$

On simplifying and initial condition,we get

$$u(p) = f(x) + \frac{1}{p}L[u_{xx}(x,t) - au(x,t)] + \frac{1}{p}L[g(x,t)] \tag{6}$$

Taking inverse Laplace-carson transform on both sides,

$$u(x,t) = U(x,t) + L^{-1}\left[\frac{1}{p}L[u_{xx}(x,t) - au(x,t)]\right] \tag{7}$$

where $U(x,t)$ represents the term arising from the function and the specified initial conditions.

Using the HPM method, we get

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{8}$$

and the non-linear term can be written as

$$au(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{9}$$

where is $H_n(x,t)$ He's polynomials and given by

$$H_n(u_0,u_1,u_2,...,u_n) = \frac{1}{n!}\frac{\partial^2}{\partial p^2}\left[a\sum_{n=0}^{\infty} p^i u_i\right]_{p=0} \tag{10}$$

Substituting the equ.(3.8)and equ.(3.7) in equ.(3.6),we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = U(x,t) + L^{-1}\left[\frac{1}{p}L[\sum_{n=0}^{\infty} p^n u_n(x,t) - a\sum_{n=0}^{\infty} p^n H_n(u)]\right] \tag{11}$$

On collecting the coefficient of exponents of p

$$p^0 : u_0(x,t) = U(x,t) \tag{12}$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t)]\right] \tag{13}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t)]\right] \tag{14}$$

$$p^3 : u_3(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t)]\right] \tag{15}$$

$$\vdots$$

$$p^n : u_{n-1}(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}u_{n-1}(x,t) - u_{n-1}(x,t)]\right] \tag{16}$$

Hence, the solution is

$$u(x,t) = lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,t) \tag{17}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_n(x,t) \tag{18}$$

3.2 Example

Consider the heat equation, $u_t = u_{xx} - 3u + 3,\ t > 0$ with cnditions $u(0,t) = 1, u(x,0) = 1 + sinx, u_x(0,t) = e^{-4t}$.

Solution: Let

$$u_t = u_{xx} - 3u + 3, t > 0 \tag{19}$$

Taking Laplace-carson transform on both sides of equ.(3.18), we get

$$L[u_t(x,t)] = L[u_{xx} - 3u] + L[3] \tag{20}$$

Using the convolution property of Laplace-carson transform,we get

$$pu(p) - pu(0) = L[u_{xx} - 3u] + 3 \tag{21}$$

On simplifying and above conditions, we get

$$u(p) = 1 + sinx + \frac{3}{p} + \frac{1}{p}L[u_{xx} - 3u] \tag{22}$$

Taking inverse Laplace-carson transform on both sides of equ.(3.21),we get

$$u(x,t) = 1 + sinx + 3t + L^{-1}\left[\frac{1}{p}L[u_{xx} - 3u]\right] \tag{23}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + sinx + 3t + L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n(x,t) - 3\sum_{n=0}^{\infty} p^n H_n(x,t)]\right] \tag{24}$$

collecting the coefficients of exponents of p

$$p^0 : u_0(x,t) = 1 + sinx + 3t \tag{25}$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}(1 + sinx + 3t) - 3(1 + sinx + 3t]\right]$$

$$= -4tsinx - 3t - \frac{9t^2}{2!} \tag{26}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}(-4tsinx-3t-\frac{9t^2}{2!})-3(-4tsinx-3t-\frac{9t^2}{2!})]\right]$$

$$= \frac{16sinxt^2}{2!}+\frac{9t^2}{2!}+\frac{27t^3}{3!} \tag{27}$$

$$\vdots$$

Similarly, we can obtain further values. Hence the u(x,t) is

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+...+u_n(x,t) \tag{28}$$

$$u(x,t) = 1+sinx+3t-4tsinx-3t-\frac{9t^2}{2!}+\frac{16sinxt^2}{2!}+\frac{9t^2}{2!}+\frac{27t^3}{3!}-...$$
$$u(x,t) = 1+sinx(1-4t+\frac{16t^2}{2!}-...)$$
$$u(x,t) = 1+sinx(e^{-4t})$$

4 Study of Wave equation with Laplace-carson Transform Homotopy Perturbation Method (LCTHPM)

In this section, we will study wave equation and its application by using homotopy perturbation Laplace-carson transform method.

4.1 solution of wave equation

Wave equation is

$$u_{tt}(x,t) = u_{xx}(x,t)-au(x,t)+g(x,t) \tag{29}$$

with initial conditions

$$u(x,0) = h(x), u_t(x,0) = f(x) \tag{30}$$

Taking the Laplace-carson transform on both sides of equation(4.1),

$$L[u_{tt}(x,t)] = L[u_{xx}(x,t)-au(x,t)]+L[g(x,t)] \tag{31}$$

Using the convolution property of Laplace-carson transform,we get

$$p^2u(p) - p^2u(0) - pu'(0) = L[u_{xx}(x,t) - au(x,t)] + L[g(x,t)] \tag{32}$$

On simplifying and initial condition,we get

$$u(p) = f(x) + \frac{1}{p^2}L[u_{xx}(x,t) - au(x,t)] + \frac{1}{p}L[g(x,t)] \tag{33}$$

Taking inverse Laplace-carson transform on both sides,

$$u(x,t) = U(x,t) + L^{-1}\left[\frac{1}{p^2}L[u_{xx}(x,t) - au(x,t)]\right] \tag{34}$$

where $U(x,t)$ represents the term arising from the function and the specified initial conditions.

Using the HPM method, we get

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{35}$$

and the non-linear term can be written as

$$au(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{36}$$

where is $H_n(x,t)$ He's polynomials and given by

$$H_n(u_0, u_1, u_2, ..., u_n) = \frac{1}{n!}\frac{\partial^2}{\partial p^2}\left[a\sum_{n=0}^{\infty} p^i u_i\right]_{p=0} \tag{37}$$

Substituting the equ.(4.8)and equ.(4.7) in equ.(4.6),we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = U(x,t) + L^{-1}\left[\frac{1}{p^2}L[\sum_{n=0}^{\infty} p^n u_n(x,t) - a\sum_{n=0}^{\infty} p^n H_n(u)]\right] \tag{38}$$

On collecting the coefficient of exponents of p

$$p^0 : u_0(x,t) = U(x,t) \tag{39}$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t)]\right] \quad (40)$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t)]\right] \quad (41)$$

$$p^3 : u_3(x,t) = L^{-1}\left[\frac{1}{p}L[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t)]\right] \quad (42)$$

$$\vdots$$

$$p^n : u_{n-1}(x,t) = L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}u_{n-1}(x,t) - u_{n-1}(x,t)]\right] \quad (43)$$

Hence, the solution is

$$u(x,t) = lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,t) \quad (44)$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_n(x,t) \quad (45)$$

4.2 Example

Consider the wave equation, $u_{tt} = u_{xx} - 3u$, $t > 0$ with conditions $u(0,t) = 0, u_t(0,t) = 2cosx$

Solution: Let

$$u_{tt} = u_{xx} - 3u, t > 0 \quad (46)$$

Taking Laplace-carson transform on both sides of equ.(4.18), we get

$$L[u_t(x,t)] = L[u_{xx} - 3u] \quad (47)$$

Using the convolution property of Laplace-carson transform,we get

$$p^2u(p) - p^2u(0) - pu'(0) = L[u_{xx} - 3u] \quad (48)$$

On simplifying and above conditions, we get

$$u(p) = \frac{2cosx}{p} + \frac{1}{p^2}L[u_{xx} - 3u] \quad (49)$$

Taking inverse Laplace-carson transform on both sides of equ.(4.21),we get

$$u(x,t) = 2tcosx + L^{-1}\left[\frac{1}{p^2}L[u_{xx} - 3u]\right] \tag{50}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 2tcosx + L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n(x,t) - 3\sum_{n=0}^{\infty} p^n H_n(x,t)]\right] \tag{51}$$

collecting the coefficients of exponents of p

$$p^0 : u_0(x,t) = 2tcosx \tag{52}$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}(2tcosx) - 3(2tcosx)]\right] = \frac{-8cosxt^3}{3!} \tag{53}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{p^2}L[\frac{\partial^2}{\partial x^2}(\frac{-8cosxt^3}{3!}) - 3(\frac{-8cosxt^3}{3!})\right] = \frac{16t^5tcosx}{5!} \tag{54}$$

$$\vdots$$

Similarly, we can obtain further values. Hence the u(x,t) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_n(x,t) \tag{55}$$

$$u(x,t) = 2tcosx - \frac{-8cosxt^3}{3!} + \frac{16cosxt^5}{5!} - ...$$
$$u(x,t) = cosx(2t - \frac{-8t^3}{3!} + \frac{16t^5cosx}{5!} - ...)$$
$$u(x,t) = cosxsin(2t)$$

5 Conclusion

In this paper, successfully determined the analytical solution of the linear heat equation and wave equation with conditions by using Homotopy perturbation Laplace-carson transform method. This result shows that this method is very effective and simple to solve.In this, LCTHPM may be considered as a nice simplification in numerical techniques and might find wide applications.

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