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Some Bivariate Replacement Policies Under Partial Product Process in an Alternative Repair Model

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Abstract

A common assumption in replacement problems is that the repair of a failed system may yield a functioning system, which may be either as good as new (complete repair) or as old as just prior to failure (partial repair). In this paper, we study the partial product process and replacement model for a deteriorating system under various conditions and a repairable system of an alternative repair model, called the Negligible Or Non-Negligible (NONN) repair times introduced by Thangaraj and Rizwan [2001] to develop a new repair model, some replacement policies (T,N) and (T^{+},N) with NONN repair times where T is the working age of the system and N is the number of failures of the system and T^{+} is the system replaced at the first failure point after the cumulative operating time exceeds T are studied. Furthermore, explicit expressions for the long-run average cost of the policies are derived. Optimal replacement policies for the deteriorating systems using partial product process is developed.

Key words: Partial product Process, Replacement policy, Extreme shock Maintenance model, Alternative Repair Times .

AMS classification: 60K10 , 90B25.

1 Introduction

The study of maintenance problem plays an important role in reliability. Most of the maintenance models just pay attention on the internal cause of the system failure, but do not an external cause of the system failure. A system failure may be caused by some external cause of the system failure. A system failure may be caused by some external cause, such as shock. The Shock models have been successfully applied to different fields, such as physics, communication, electronic engineering and medicine, etc. However, only a very few authors consider the deteriorating systems

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interrupted by random shocks. In our model, the system will fail, if the amount of shock damage by one big shock exceeds a specific threshold. In the model, the shock is called a deadly shock if the amount of damage of the shock to the system exceeds a specific threshold so that the system will fail. This kind of shock models is called an extreme shock model. Chen and Li [2008] have considered an extreme shock model. Also an alternative repair model called the Negligible Or Non-Negligible (NONN) repair times introduced by Thangaraj and Rizwan [2001] is incorporated in this paper to develop an extreme shock model for the maintenance problem some bivariate replacement policies. The long-run average cost for a degenerative system under the replacement policy just prior to failure (partial repair). In this paper, we study the partial product process and replacement model for a deteriorating system under various conditions and a repairable system of an alternative repair model, called the Negligible Or Non-Negligible (NONN) repair times introduced by Thangaraj and Rizwan [2001] to develop a new repair model, some replacement policies (T,N), and (T^+,N) with NONN repair times where T is the working age of the system and N is the number of failures of the system, T^+ is the system replaced at the first failure point after the cumulative operating time exceeds T. Existence of optimality under the aforesaid bivariate replacement policies are deduced. The rest of the paper is organized as follows: In Section 2, we give some basic preliminaries, In Section 3, we give some model assumptions. In Section 4, we derive explicit expressions for the long-run average cost per unit time for this model under the bivariate replacement policies (T,N) and (T^+,N). Finally, conclusion is given in Section 5.

2 Preliminaries

The preliminary definitions and results relevant to this are given below.

Definition 2.1 Given two random variables X and Y, X is said to be ***stochastically smaller than*** Y (or Y is ***stochastically greater than*** X), if $P(X > \alpha) \leqslant P(Y > \alpha)$ for all real α. This is written as $X \leqslant_{st} Y$ or $Y \geqslant_{st} X$.

Definition 2.2 A stochastic process $\{X_n, n = 1,2,3\ldots\}$ is said to be ***stochastically decreasing (increasing)*** if $X_n \geqslant_{st} (\leqslant_{st}) X_{n+1}, \quad forall \quad n = 1,2,3\ldots.$

Definition 2.3 A stochastic process $\{Yn, n = 1,2,3\ldots\}$ is said to be ***stochastically increasing (decreasing)*** if $Yn \leqslant_{st} (\geqslant_{st}) Y_{n+1}, \quad forall \quad n = 1,2,3\ldots.$

Definition 2.4 An integer valued random variable N is said to be a stopping time for the sequence of independent random variables $X_1, X_2, \ldots$, if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1, 2, \ldots$.

Definition 2.5 Let $\{X_n, n = 1, 2, 3, \ldots\}$ be a sequence of independent and non-negative random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_n, n = 1, 2, 3 \ldots\}$ is called partial product process, if the distribution function of X_{k+1} is $F(\alpha_k x)(k = 1, 2, 3 \cdots)$, where $\alpha_k > 0$ are real constants and $\alpha_k = \alpha_0 \alpha_1 \alpha_2 \ldots \alpha_{k-1}$. In what follows, $F(x)$ denotes the distribution function of non-negative random variable X_1.

Lemma 2.6 If $\alpha_k = \alpha_0 \alpha_1 \alpha_2 \ldots \alpha_{k-1}$, then $\alpha_k = \alpha_0^{2^{k-1}} \quad (k = 1, 2, 3, \ldots)$.

Lemma 2.7 The partial product process $\{X_n, n = 1, 2, 3 \cdots\}$ is

(i) stochastically decreasing, if $\alpha_0 > 1$
(ii) stochastically increacing, if $0 < \alpha_0 < 1$

Definition 2.8 A bivariate replacement policy (T, N)
It is a policy that replaces the system at T or at N-th failure since the last replacement, whichever occurs soon.

Definition 2.9 A bivariate replacement policy (T^+, N)
A bivariate replacement policy (T^+, N) is a replacement model under which the system is replaced at the first failure point after the cumulative operating time exceeds T or at the occurance of N-th failure, whichever occurs earlier.

Definition 2.10 If the sequence of nonnegative random variables $\{X_1, X_2, \ldots\}$ is independent and identically distributed, then the counting process $\{N(t), t \geqslant 0\}$ is said to be a renewal process.

Definition 2.11 If a repair to a system after failure is done in negligible or non-negligible time, then it will be called a model with NONN repair times

In this case, whenever the system fails, two possibilities may arise: either, the repair takes Negligible time with probability p ; or Non-Negligible time with probability

$1-p$.

Lemma 2.12 Let $E(X_1)=\lambda$, $\text{var}(X_1)=\sigma^2$. Then for $k=1,2,3,\ldots$. $E\left(X_{k+1}\right)=\frac{\lambda}{\alpha_0^{2^{k-1}}}$ and $\text{var}(X_{k+1})=\frac{\sigma^2}{{\alpha_0}^{2^k}}$, where $\alpha_0>0$.

Theorem 2.13 (Wald's equation) If $X_1,X_2,\ldots$ are independent and identically distributed random variables having finite expectations and if N is the stopping time for $X_1,X_2,\ldots$ such that $E[N]<\infty$, then

$$E\left[\sum_{n=1}^{N} X_n\right]=E[N]E\left[X_1\right].$$

Theorem 2.14 (Wald's equation for partial product process) Suppose that $\{X_k, k=1,2,3,\ldots\}$ forms a partial product process with ratio α_0 and $E[X_1]=\mu<\infty$ and if $\omega(t)=\sup\limits_{k\in\mathbb{Z}^+}\{k:V_k\leq t\}$ and $V_k=\sum\limits_{i=1}^{k} X_i$. Then for $t>0$,

$$E\left[V_{\omega(t)+1}\right]=\mu E\left[1+\sum_{k=2}^{\omega(t)+1}\frac{1}{\alpha_0^{2^{k-2}}}\right].$$

Definition 2.15 (Renewal Reward Theorem) A cycle is completed every time a renewal occurs then If E[R] < ∞ and E[X] < ∞, then

(i) with probability 1, $\lim_{t\to\infty}\frac{R(t)}{t}=\frac{E[R]}{E[X]}$

(ii) $\lim_{t\to\infty}\frac{E[R]}{E[X]}$

the above states that in the long run average reward is just the expected reward earned during a cycle divided by the expected length of a cycle.

3 Model Assumptions

We make the following assumptions about the model for a simple degenerative reparable system subject to shocks.

Assumption 3.1 At time $t=0$, a new system is installed . Whenever the system fails it will be repaired. The System will be replaced by an identical new one, sometimes later.

Assumption 3.2 Once the system is operating , the shocks from the environment arrive according to a renewal process. Let $X_{ni},\ i=1,2,\dots$ be the intervals between the $(i-1)$-st and i-th shock, after the $(n-1)$-st repair. Let $E(X_{11})=\lambda$. Assume that $X_{ni},\ i$=1,2,.... are independent and identically distributed random variables, for all $n\in\mathbb{N}$.

Assumption 3.3 Let $Y_{ni}, i=1,2,\dots$ be the sequence of the amount of shock damage produced by the i-th shock, after the $(n-1)$-st repair. Let $E(Y_{11})=\mu$. Then $\{Y_{ni}, i=1,2,\dots\}$ are iid sequences, for all $n\in\mathbb{N}$. If the system fails, it is closed, so that the random shocks have no effect on the system during the repair time. In the n-th operating stage, that is, after the $(n-1)$-st repair, the system will fail, if the amount of the shock damage first exceed $\alpha_0^{2^{n-1}}M$, where $0<\alpha_0\leqslant 1$ and M is a positive constant.

Assumption 3.4 Let $Z_n,\ n=1,2,\dots$ be the repair time after the n-th repair and $\{Z_n, n=1,2,\dots\}$ constitute a non decreasing partial product process with $E(Z_1)=\delta$ and ratio α_0, such that $0<\alpha_0<1$. $N_n(t)$ is the counting process denoting the number of shocks after the $(n-1)$-st repair. It is clear that $E(Z_n)=\frac{\lambda}{\alpha_0^{2^{n-1}}}$.

Assumption 3.5 Let r be the reward rate per unit time of the system, when it is operating and c be the repair cost rate per unit time of the system and the replacement cost is R. The replacement time is a random variable Z with $E(Z)=\tau$.

Assumption 3.6 The sequences $\{X_{ni}, i=1,2,\dots\}$, $\{Y_{ni}, i=1,2,\dots\}$, $\{Z_n, n=1,2,\dots\}$ and Z are independent.

Assumption 3.7 Assume that $F_n(t)$ is the cumulative distribution of $U_n=\sum_{i=1}^{n} W_i$ and $G_n(t)$ is the cumulative distribution of $V_n=\sum_{i=1}^{n} Z_i$.

Assumption 3.8 Define $\xi_n=\begin{cases} Z_n & if\ \ Z_n>0 \\ 1 & if\ \ Z_n=0 \end{cases}$ for $n=1,2,\dots$

Assumption 3.9 The replacement policy (T,N) and (T^+,N) is adapted.

4 The Bivariate Replacement Policies with NONN Repair Times

4.1 The Bivariate Replacement Policy (T,N) with NONN Repair Times

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (T,N) policy. Let $L_n = \min\{l; Y_{nl} > \alpha_0^{2^{n-1}} M\}$ and $W_n = \sum_{i=1}^{L_n} X_{ni}$. Thus, L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P[L_n = k] = p_n q_n^{k-1}$, $k = 1,2,\ldots$, where $p_n = P[Y_{nl} > \alpha_0^{2^{n-1}} M]$ and $q_n = 1 - p_n$. We have $E(L_n) = \dfrac{1}{p_n}$. Since $\{X_{ni},\ i = 1,2,\ldots\}$ and $\{Y_{ni},\ i = 1,2,\ldots\}$ are independent, it is clear that L_n and $\{X_{ni}\}$are independent. Now

$$\begin{aligned} E(W_n) &= E\left(\sum_{i=1}^{L_n} X_{ni}\right) \\ &= E(L_n)\ E(X_{n1}) \\ &= \frac{\lambda}{p_n}. \end{aligned}$$

The distribution function of W_n is $F_n(\,\cdot\,)$.
The working age T of the system at time t is the cumulative life time given by

$$T(t) = \begin{cases} t - V_n, & U_n + V_n \le t < U_{n+1} + V_n \\ U_{n+1}, & U_{n+1} + V_n \le t < U_{n+1} + V_{n+1}, \end{cases}$$

where $U_n = \sum_{i=1}^{n} W_i$ and $V_n = \sum_{i=1}^{n} Z_i$ and $U_0 = V_0 = 0$.
By assumption,

$$\begin{aligned} E(\xi_n) &= E(Z_n)P(Z_n > 0) + 1P(Z_n = 0) \\ &= \frac{\lambda}{\alpha_0^{2^{n-1}}}(1-p) + p. \end{aligned}$$

Let $W_{N-n} = \sum_{j=n+1}^{N} W_j$. Then $U_N = U_n + W_{N-n}$. Moreover U_n and W_{N-n} are independent , and

$$H_{N-n}(t) = \int_0^\infty H_{N-1-n}(a(t-y))\,dH(y),$$

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Where $H_{N-1-n}(t)$ is the distribution of $\sum_{j=n+1}^{N} X_j$. Since the distribution function of X_{n+1} is $H(t) = F(a^n(t))$, By induction $H_{N-n}(t) = F_{N-n}(a^n(t))$. Now,

$$\begin{aligned} E[\chi(U_n < T < U_N)] &= P(U_n < T < U_N + W_{N-n}) \\ &= \int_o^T \int_{T-u}^{\infty} dH_{N-n(t)} dF_n(u) \\ &= \int_0^T \bar{F}_{N-n}(a^n(T-u)) dF_n(u). \end{aligned}$$

Let T_1 be the replacement time; in general for $n = 2, 3, \ldots$, let T_n be the time between the $(n-1)$-st replacement and the n-th replacement. Thus the sequence $\{T_n,\ n = 1, 2, \ldots\}$ forms a renewal process. A cycle is completed, if a replacement is done. A cycle is actually the time interval between the installation of the system and the first replacement or the time interval between two consecutive replacements. Finally the successive cycles together with the cost incurred in each cycle will constitute a renewal reward process. The length of the cycle under the replacement policy (T, N) is

$$W = \left[T + \sum_{n=1}^{\eta} \xi_n\right] \chi_{(U_n > T)} + \left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{\eta-1} \xi_n\right] \chi_{(U_n \le T)} + Z,$$

where $\eta = 0, 1, 2, \ldots N-1$ is the number of failures before the working age of the system exceeds T and $\chi_{(A)}$ denotes the indicator function. The expected length of a cycle is

$$\begin{aligned} E(W) &= E\left[T + \left[\sum_{n=1}^{\eta} \xi_n\right] \chi_{(U_n > T)}\right] + E\left[\left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{\eta-1} \xi_n\right] \chi_{(U_n \le T)}\right] + E(Z) \\ &= E\left[T\chi_{(U_n > T)} + E\left[\sum_{n=1}^{\eta} \xi_n\right] \chi_{(U_n > T)}\right] + E\left[E\left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{\eta-1} \xi_n\right] \chi_{(U_n \le T)} \middle| U_n = u\right] + E(Z) \\ &= T\overline{F}_N(T) + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] E[\chi_{(U_n \le T < U_n)}] + \int_0^T E\left[\sum_{n=1}^{N} W_n\right] u dF_N(u) \end{aligned}$$

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$$
\begin{aligned}
&+\int_0^T\left[\sum_{n=1}^{\eta-1}E(\xi_n)\right]dF_N(u)+\tau\\
=\;& T\overline{F}_N(T)+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right]P(U_N\le T<U_n+W_{N-n})+\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^T udF_{N(u)}\\
&+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right]+\tau\\
=\;& T\overline{F}_N(T)+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right]\int_o^T\int_{T-u}^{\infty}dH_{N-n}(t)dF_n(u)+\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_o^T udF_N(u)\\
&+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right]+\tau\\
=\;& T\overline{F}_N(T)+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right]\int_0^T\bar{F}_{N-n}(a^n(T-u))dF_n(u)+\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_o^T udF_N(u)\\
&+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right]+\tau. \qquad (1)
\end{aligned}
$$

Let $\mathscr{C}(T,N)$ be the long run average cost per unit time under the bivariate replacement policy (T,N). By the renewal reward theorem , the long run average cost per unit time under the replacement policy - (T,N) is given by

$$
\begin{aligned}
\mathscr{C}(T,N) &= \frac{\text{expected cost incurred in a cycle}}{\text{expected length of a cycle}}\\
&= \frac{E\left[\left[c\sum_{n=1}^{\eta}\xi_n-rT\right]\chi_{(U_n>T)}\right]+c_pE(Z)+E\left[\left[c\sum_{n=1}^{\eta-1}\xi_n-r\sum_{n=1}^{N}W_n\right]\chi_{(U_n\le T)}\right]+R}{E(W)}. \qquad (2)
\end{aligned}
$$

Consider

$$
\begin{aligned}
E\left[\sum_{n=1}^{\eta}\xi_n\chi_{(U_n>T)}\right] &= E\left[E\left(\sum_{k=1}^{\eta}Z_k\middle|\eta=n\right)\chi_{(U_n>T)}\right]\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\eta}E\left(Z_k\right)\right)P(\eta=n)\chi_{(U_n>T)}\\
&= \sum_{n=1}^{\infty}\left[E\left(Z_1\right)+\sum_{k=2}^{\eta}E\left(Z_k\right)\right]P(\eta=n)\chi_{(U_n>T)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\left[E\left(Z_1\right)+\sum_{k=2}^{\eta} E\left(Z_k\right)\right] P(\eta=n)\chi_{(U_n>T)} \\
&= E\left(Z_1\right)\sum_{n=1}^{\infty} P(\eta=n)+\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n} E\left(Z_k\right)\right) P(\eta=n)\chi_{(U_n>T)} \\
&= \lambda\sum_{n=1}^{\infty} P(\eta=n)+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} E\left(Z_{k+1}\right)\right) P(\eta=n)\chi_{(U_n>T)}
\end{aligned}
$$

$$
\begin{aligned}
E\left[\sum_{n=1}^{\eta}\xi_n\chi_{(U_n>T)}\right] &= \lambda\sum_{n=1}^{\infty} P(\eta=n)+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\frac{\lambda}{\alpha_0^{2^{k-1}}}\right) P(\eta=n)\chi_{(U_n>T)} \\
&= \lambda P(n=1)+\lambda\sum_{n=2}^{\infty}\left[1+\sum_{k=1}^{n-1}\frac{1}{\alpha_0^{2^{k-1}}}\right] P(\eta=n) \\
&= \lambda\left(F_1(T)-F_2(T)\right)+\lambda\sum_{n=2}^{\infty}\left[1+\sum_{k=1}^{n-1}\frac{1}{\alpha_0^{2^{k-1}}}\right]\left(F_n(T)-F_{n+1}(T)\right) \\
&= \lambda F_1(T)+\lambda\sum_{n=2}^{\infty}\frac{1}{\alpha_0^{2^{n-1}}}F_{n+1}(T) \\
&= \lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right]. \qquad (3)
\end{aligned}
$$

Now,

$$
\begin{aligned}
E\left[\sum_{n=1}^{\eta-1}\xi_n\right] &= E\left[E\left(\sum_{n=1}^{\eta-1} Z_n \mid \eta=n\right)\right] \\
&= \sum_{n=1}^{\infty}\left(\sum_{n=1}^{\eta-1} E\left(Z_n\right)\right) P(\eta=n) \\
&= \sum_{n=1}^{\infty}\left[E\left(Z_1\right)+\sum_{k=2}^{\eta-1} E\left(Z_k\right)\right] P(\eta=n) \\
&= E\left(Z_1\right)\sum_{n=1}^{\infty} P(\eta=n)+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{\eta-1} E\left(Z_k\right)\right) P(\eta=n)
\end{aligned}
$$

$$\begin{aligned}
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} E\left(Z_{k+1}\right)\right) P(\eta = n) \\
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} \frac{\lambda(1-p)}{\alpha_0^{2^{k-1}}} + p\right) P(\eta = n) \\
&= \lambda P(\eta = 1) + \lambda \sum_{n=2}^{\infty}\left[1 + \sum_{k=1}^{n-1} \frac{1-p}{\alpha_0^{2^{k-1}}} + p\right] P(\eta = n) \\
&= \lambda\left(F_1 - F_2\right) + \lambda \sum_{n=2}^{\infty}\left[1 + \sum_{k=1}^{n-1} \frac{1-p}{\alpha_0^{2^{k-1}}} + p\right] F_n(T) - F_{n+1}(T) \\
&= \lambda F_1(T) + \lambda \sum_{n=2}^{\infty} \frac{1-p}{\alpha_0^{2^{n-1}}} F_{n+1}(T) + p
\end{aligned}$$

$$E\left[\sum_{n=1}^{\eta-1} \xi_n\right] = \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right]. \tag{4}$$

On substituting (1), (3) and (4) in equation (2), we obtain the following

Theorem 4.1 For the model described in section 2, under the assumptions 2.1 to 2.9, the long run average cost per unit time under the bivariate replacement policy (T,N) for a simple degenerative repairable system is given by

$$\mathscr{C}(T,N) = \frac{\left[\begin{array}{c} c\lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \int_0^T \bar{F}_{N-n}(a^n(T-u))dF_n(u) + c_p\tau + R \\ -rT\overline{F}_N(T) - r\sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right] + \tau \end{array}\right]}{\left[\begin{array}{c} T\overline{F}_N(T) + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-1}}}\right] \int_0^T \bar{F}_{N-n}(a^n(T-u))dF_n(u) \\ + \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u) + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right] + \tau \end{array}\right]}$$

Deductions

The long run average cost $\mathscr{C}(T,N)$ is a bivariate function in T and N. Obviously, when N is fixed, $\mathscr{C}(T,N)$ is a function of T. For fixed $N = m$, it can be written as

$$\mathscr{C}(T,N) = C_m(T), m = 1, 2, \ldots.$$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $C_m(T_m^*)$ is minimized. That is, when $N = 1, 2, \ldots, m, \ldots$, we can find T_1^*, T_2^*, T_3^*,... T_m^* ,... respectively, such that $C_1(T_1^*), C_2(T_2^*), \ldots, C_m(T_m^*), \ldots$ are minimized. Because the total life -time of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $C_1(T_1^*), C_2(T_2^*), \ldots, C_m(T_m^*), \ldots$.
Then, if the minimum is denoted by $C_n(T_n^*)$, we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$\begin{aligned}\mathscr{C}((T,N)^*) &= \min_N C_n(T_n^*)\\ &= \min_N[\min_T C(T,N)]\\ &\leq C(\infty,N)\\ &\equiv C(N^*)\end{aligned}$$

the optimal policy $(T,N)^*$ is better than the optimal policy N^*. Moreover, under some mild conditions, Stadje and Zukerman [1990] showed that an optimal replacement policy N^* is better than the optimal policy T^*. so under the same conditions, an optimal policy $(T,N)^*$ is better than the optimal replacement policies N^* and T^*.

4.2 The Bivariate Replacement Policy (T^+,N) with NONN Repair Times

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (T,N) policy. Let $L_n = \min\{l; Y_{nl} > \alpha_0^{2^{n-1}} M\}$ and $W_n = \sum_{i=1}^{L_n} X_{ni}$. Thus, L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P[L_n = k] = p_n q_n^{k-1},\ k = 1, 2, \ldots$, where $p_n = P[Y_{nl} > \alpha_0^{2^{n-1}} M]$ and $q_n = 1 - p_n$. We have $E(L_n) = \frac{1}{p_n}$. Since $\{X_{ni},\ i = 1, 2, \ldots\}$ and $\{Y_{ni},\ i = 1, 2, \ldots\}$ are independent, it is clear that L_n and $\{X_{ni}\}$are independent. Now

$$\begin{aligned}E(W_n) &= E\left(\sum_{i=1}^{L_n} X_{ni}\right)\\ &= E(L_n)\ E(X_{n1})\\ &= \frac{\lambda}{p_n}.\end{aligned}$$

The distribution function of W_n is $F_n(\cdot)$.
The working age T of the system at time t is the cumulative life time given by

$$T(t) = \begin{cases} t - V_n, & U_n + V_n \le t < U_{n+1} + V_n \\ U_{n+1}, & U_{n+1} + V_n \le t < U_{n+1} + V_{n+1}, \end{cases}$$

where $U_n = \sum_{i=1}^{n} W_i$ and $V_n = \sum_{i=1}^{n} Z_i$ and $U_0 = V_0 = 0$.
By assumption,

$$\begin{aligned} E(\xi_n) &= E(Z_n)P(Z_n > 0) + 1P(Z_n = 0) \\ &= \frac{\lambda}{\alpha_0^{2^{n-1}}}(1-p) + p. \end{aligned}$$

Let T_1 be the replacement time; in general for $n = 2, 3, \ldots$, let T_n be the time between the $(n-1)$-st replacement and the n-th replacement. Thus the sequence $\{T_n,\ n = 1, 2, \ldots\}$ forms a renewal process. A cycle is completed, if a replacement is done. A cycle is actually the time interval between the installation of the system and the first replacement or the time interval between two consecutive replacements. Finally the successive cycles together with the cost incurred in each cycle will constitute a renewal reward process. The length of the cycle under the replacement policy (T^+, N) is

$$W = \left[\left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{\eta-1} \xi_n \right] \chi_{(U_N \leqslant T)} \right] + \left[\left[\sum_{n=1}^{\eta} W_n + \sum_{n=1}^{\eta} \xi_{n-1} \right] \chi_{(U_N > T)} \right] + Z,$$

where $\eta = 1, 2, \ldots N-1$ is the number of failures before the total operating time of the system exceeds T and $\chi_{(A)}$ denotes the indicator function. The random variable η has a geometric distribution given by

$$\begin{aligned} P(\eta = j) &= P(W_1 \leqslant T, W_2 \leqslant T \ldots, W\eta - 1 \leqslant T, W_\eta > T); \quad \text{j=1,2,...} \\ &= F^{j-1}(T)\bar{F}(T). \end{aligned}$$

Since η is a random variable,

$$\begin{aligned} E(\eta-1) &= \sum_{j=1}^{\infty} P(\eta=j) \\ &= \bar{F}(T)\sum_{j=1}^{\infty}(j-1)F^{j-1}(T) \\ &= \frac{F(T)}{\bar{F}(T)} \end{aligned}$$

The expected length of a cycle is

$$E(W) = E\left[\left[\sum_{n=1}^{N} W_n + \sum_{n=1}^{\eta-1}\xi_n\right]\chi_{(U_N \leqslant T)}\right] + E\left[\left[\sum_{n=1}^{\eta} W_n + \sum_{n=1}^{\eta}\xi_{n-1}\right]\chi_{(U_N > T)}\right] + E(Z) \tag{5}$$

Consider, $\left[\left(\sum_{n=1}^{N} W_n\right)\chi_{(U_N \leq T)}\right] = E\left[E\left(\sum_{n=1}^{N} W_n\right)\middle| U_N = u\chi_{(U_N \leq T)}\right]$

$$\begin{aligned} &= \int_o^T E\left(\sum_{n=1}^{N} W_n \middle| U_N = u\right) dF_N(u) \\ &= \int_o^T u dF_N(u) E(W_n) \\ &= \sum_{n=1}^{N} \frac{\lambda}{p_n}\int_o^T u dF_N(u) \end{aligned}$$

Where $F_n(.)$ is the n-fold convolution of $F(.)$ with itself

$$\begin{aligned} E\left[\sum_{n=1}^{\eta-1}\xi_n\right] &= E\left[E\left(\sum_{n=1}^{\eta-1} Z_n \mid \eta = n\right)\right] \\ &= \sum_{n=1}^{\infty}\left(\sum_{n=1}^{\eta-1} E(Z_n)\right) \quad P(\eta=n) \\ &= \sum_{n=1}^{\infty}\left[E(Z_1) + \sum_{k=2}^{\eta-1} E(Z_k)\right] P(\eta=n) \\ &= E(Z_1)\sum_{n=1}^{\infty} P(\eta=n) + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{\eta-1} E(Z_k)\right) P(\eta=n) \end{aligned}$$

$$
\begin{aligned}
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} E\left(Z_{k+1}\right) \right) P(\eta = n) \\
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{\lambda}{\alpha_0^{2^{k-1}}} \right) P(\eta = n) \\
&= \lambda P(\eta = 1) + \lambda \sum_{n=2}^{\infty} \left[1 + \sum_{k=1}^{n-1} \frac{1-p}{\alpha_0^{2^{k-1}}} + p \right] P(\eta = n) \\
&= \lambda \left(F_1 - F_2\right) + \lambda \sum_{n=2}^{\infty} \left[1 + \sum_{k=1}^{n-1} \frac{1-p}{\alpha_0^{2^{k-1}}} + p \right] F_n(T) - F_{n+1}(T) \\
&= \lambda F_1(T) + \lambda \sum_{n=2}^{\infty} \frac{1-p}{\alpha_0^{2^{n-1}}} F_{n+1}(T) + p
\end{aligned}
$$

$$
E\left[\sum_{n=1}^{\eta-1} \xi_n\right] = \lambda \left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p \right]. \tag{6}
$$

$$
\begin{aligned}
E\left[\left(\sum_{n=1}^{\eta} W_n\right) \chi_{(U_n > T)}\right] &= E\left[\left(\sum_{n=1}^{\eta} W_n \chi_{(U_N > T)}\right)\right] \\
&= \left(\sum_{n=1}^{N-1} E(W_n \big| \eta = n-1) P(U_n \leqslant T < U_N) \right) \\
&= \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n} [F_n(T) - F_N(T)]
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
E\left[\sum_{n=1}^{\eta} \xi_{n-1}\right] &= E\left[E\left(\sum_{n=1}^{\eta} Z_{n-1} \mid \eta = n \right) \right] \\
&= \sum_{n=1}^{\infty} \left(\sum_{n=1}^{\eta} E\left(Z_{n-1}\right) \right) \quad P(\eta = n) \\
&= \sum_{n=1}^{\infty} \left[E\left(Z_1\right) + \sum_{k=2}^{\eta} E\left(Z_{k+1}\right) \right] P(\eta = n) \\
&= E\left(Z_1\right) \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{\eta} E\left(Z_{k+1}\right) \right) P(\eta = n) \\
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} E\left(Z_{k+2}\right) \right) P(\eta = n)
\end{aligned}
$$

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$$
\begin{aligned}
&= \lambda \sum_{n=1}^{\infty} P(\eta = n) + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\frac{\lambda}{\alpha_0^{2^{k-2}}}\right)P(\eta = n)\\
&= \lambda P(\eta = 1) + \lambda \sum_{n=2}^{\infty}\left[1+\sum_{k=1}^{n-1}\frac{1}{\alpha_0^{2^{k-2}}}\right]P(\eta = n)\\
&= \lambda\,(F_1 - F_2) + \lambda \sum_{n=2}^{\infty}\left[1+\sum_{k=1}^{n-1}\frac{1}{\alpha_0^{2^{k-2}}}\right]F_n(T) - F_{n+1}(T)\\
&= \lambda F_1(T) + \lambda \sum_{n=2}^{\infty}\frac{1}{\alpha_0^{2^{n-2}}}F_{n+1}(T)\\
E\left[\sum_{n=1}^{\eta}\xi_{n-1}\right] &= \lambda\left[F_1(T) + \sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right]. \qquad (8)
\end{aligned}
$$

On substituting (6), (7), (8) and (9) in equation (5), we obtain the following

$$
\begin{aligned}
E(W) &= E\left[\left[\sum_{n=1}^{N}W_n + \sum_{n=1}^{\eta-1}\xi_n\right]\chi_{(U_N\leqslant T)}\right] + E\left[\left[\sum_{n=1}^{\eta}W_n + \sum_{n=1}^{\eta}\xi_{n-1}\right]\chi_{(U_N>T)}\right] + E(Z)\\
&= E\left[E\left[\left(\sum_{n=1}^{N}W_n\right)\chi_{(U_N\leqslant T)}\right] + \left[E\left(\sum_{n=1}^{\eta-1}\xi_n\right)\chi_{(U_N\leqslant T)}\right]\right]\\
&\qquad + E\left[\left[\left(\sum_{n=1}^{\eta}W_n\right)\chi_{(U_N>T)}\right] + \left[\left(\sum_{n=1}^{\eta}\xi_{n-1}\right)\chi_{(U_N>T)}\right]\right] + E(Z)\\
&= E\left[\left[E\left(\sum_{n=1}^{N}W_n\right)\middle|U_N = u\chi_{(U_N\leq T)}\right] + \left[E\left(\sum_{n=1}^{\eta-1}\xi_n\right)\middle|U_N = u\chi_{(U_N\leq T)}\right]\right]\\
&\qquad + E\left[\left[\left(\sum_{n=1}^{\eta}W_n\right)\chi_{(U_N>T)}\right] + \left[\left(\sum_{n=1}^{\eta}\xi_{n-1}\right)\chi_{(U_N>T)}\right]\right] + E(Z)\\
&= \left[\left[\int_o^T E\left(\sum_{n=1}^{N}W_n\middle|U_N = u\right)dF_N(u)\right] + \left[\int_o^T\left(\sum_{n=1}^{\eta-1}E(Z_n)\right)dF_N(u)\right]\right]
\end{aligned}
$$

$$+\left[\left[\left(\sum_{n=1}^{N-1} E\left(W_n \middle| \eta = n-1\right) P(U_n \leqslant T < U_N)\right)\right]\right]$$

$$+\left[\left(\sum_{n=1}^{\eta} E(Z_{n-1}) P(U_n \leqslant T < U_N)\right)\right] + E(Z)$$

$$= \left[\int_o^T u dF_N(u) E(W_n)\right] + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right]$$

$$+E\left[\left[\left(\sum_{n=1}^{N-1} E\left(W_n \middle| \eta = n-1\right)(U_n \leqslant T < U_N)\right)\right]\right]$$

$$+\left[\left(\sum_{n=1}^{\eta} E(Z_{n-1}) P(U_n \leqslant T < U_N)\right)\right] + E(Z)$$

$$= \left[\sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u)\right] + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right]$$

$$+\left[\frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n}[F_n(T) - F_N(T)]\right] + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right] + E(Z)$$

$$= \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_o^T u dF_N(u) + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}} + p\right]$$

$$+\left[\frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n}[F_n(T) - F_N(T)]\right] + \lambda\left[F_1(T) + \sum_{n=2}^{\infty} \frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right] + \tau \qquad (9)$$

Let $\mathscr{C}(T^+,N)$ be the long run average cost per unit unit per time under the bivariate replacement policy (T^+,N). By the elementary renewal theorem , the long run average cost per unit time under the replacement policy (T^+,N) is given by

$$\mathscr{C}(T^+,N) = \frac{\text{expected cost incurred in a cycle}}{\text{expected length of a cycle}}$$

$$= \frac{E\left[\left(c\sum_{n=1}^{\eta-1}\xi_n - r\sum_{n=1}^{N} W_n\right)\chi_{(U_N \leqslant T)}\right] + R + E\left[\left(c\sum_{n=1}^{\eta}(\xi_{n-1}) - r\sum_{n=1}^{\eta} W_n\right)\chi_{(U_N > T)}\right] + c_p E(Z)}{E(W)}.$$

$$= \frac{\begin{bmatrix} E\left[E\left(c\sum_{n=1}^{\eta-1}\xi_n\right)\chi_{(U_N\leqslant T)}\right] - E\left[E\left(r\sum_{n=1}^{N}W_n\right)\chi_{(U_N\leqslant T)}\right] + R + E\left[\left(c\sum_{n=1}^{\eta}(\xi_{n-1})\right)\chi_{(U_N>T)}\right] \\ -E\left[\left(r\sum_{n=1}^{\eta}W_n\right)\chi_{(U_N>T)}\right] + c_pE(Z) \end{bmatrix}}{E(W)}$$

$$= \frac{\begin{bmatrix} c\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right] - r\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_o^T udF_N(u)+R+ \\ \lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right] - r\frac{F(T)}{\bar{F}(T)}\sum_{n=1}^{N-1}\frac{\lambda}{p_n}[F_n(T)-F_N(T)]+c_p\tau \end{bmatrix}}{E(W)} \quad (7)$$

Theorem 4.2 For the model described in section 2 , the long run average cost per unit time under the bivariate replacement policy (T^+,N) for a simple degenerative repairable is given by

$$\mathscr{C}(T^+,N) = \frac{\begin{bmatrix} \left[c\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right]\right] - \left[r\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_o^T udF_N(u)\right] + \\ \lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right] - r\frac{F(T)}{\bar{F}(T)}\sum_{n=1}^{N-1}\frac{\lambda}{p_n}[F_n(T)-F_N(T)]+R+c_p\tau \end{bmatrix}}{\begin{bmatrix} \sum_{n=1}^{N}\frac{\lambda}{p_n}\int_o^T udF_N(u)+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)(1-p)}{\alpha_0^{2^{n-1}}}+p\right]+ \\ \frac{F(T)}{\bar{F}(T)}\sum_{n=1}^{N-1}\frac{\lambda}{p_n}[F_n(T)-F_N(T)]+\lambda\left[F_1(T)+\sum_{n=2}^{\infty}\frac{F_{n+1}(T)}{\alpha_0^{2^{n-2}}}\right]+\tau \end{bmatrix}} \quad (8)$$

Deductions

The long run average cost $\mathscr{C}(T^+,N)$ is a bivariate function in T^+ and N. Obviously, when N is fixed, $\mathscr{C}(T^+,N)$ is a function of T^+.
For fixed $N=m$, it can be written as

$$\mathscr{C}(T^+,N) = C_m(T^+), \qquad m=1,2,\ldots.$$

Thus, for a fixed m, we can find ${T^+}^*_m$ by analytical or numerical methods such that $C_m({T^+}^*_m)$ is minimized. That is, when $N = 1, 2, \ldots, m, \ldots,$ we can find ${T^+}^*_1$, ${T^+}^*_2$, ${T^+}^*_3, \ldots {T^+}^*_m, \ldots$ respectively, such that $C_1({T^+}^*_1), C_2({T^+}^*_2), \ldots, C_m({T^+}^*_m), \ldots$ are minimized. Because the total life -time of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $C_1({T^+}^*_1), C_2({T^+}^*_2), \ldots, C_m({T^+}^*_m), \ldots$. Then, if the minimum is denoted by $C_n({T^+}^*_n)$, we obtain the bivariate optimal replacement policy $(T^+, N)^*$ such that

$$\begin{aligned}
\mathscr{C}((T^+, N)^*) &= \min_N C_n({T^+}^*_n) \\
&= \min_N [\min_{T^+} C(T^+, N)] \\
&\leq C(\infty, N) \\
&\equiv C(N^*)
\end{aligned}$$

the optimal policy $(T^+, N)^*$ is better than the optimal policy N^*. Moreover, under some mild conditions, an optimal replacement policy N^* is better than the optimal policy ${T^+}^*$. so under the same conditions, an optimal policy $(T^+, N)^*$ is better than the optimal replacement policies N^* and ${T^+}^*$.

5 Conclusion

In this paper, we have considered an extreme shock maintenance model for a degenerative simple repairable system. Explicit expression for the long run average cost under the bivariate replacement policy (T, N) and (T^+, N) is derived. The existence of optimal value of (T, N) and (T^+, N) is deduced.

References

[1] Babu D, Govindaraju P and Rizwan U, Partial Product Processes and Replacement Problem, International Journal of Current Advanced Research,7.1,139-142,(2018).

[2] Babu D, Govindaraju P and Rizwan U , A Maintenance Model for a Deteriorating System with Imperfect Delayed Repair under Partial Product Process, Journal of Applied Science and Computations, 6(4),795 - 800(2019).

[3] Babu D, Govindaraju P and Rizwan U, A δ-Shock Maintenance Model for a Deteriorating System with Imperfect Delayed Repair under Partial Product Process, Journal of Mathematics and Computer Science, 9(5), 571 - 581 (2019).

[4] Barlow RE and Proschan F, Mathematical Theory of Reliability, John Wiley, New York (1965).

[5] Barlow RE and Proschan F, Statistical Theory of Reliability and Life Testing, John Wiley, New York (1975).

[6] Chen J and Li Z , An Extended Extreme Shock Maintenance Model for a Deteriorating System, Reliability Engineering and System Safety, 93, 1123- 1129(2008).

[7] Feller W, An Introduction to Probability Theory and its Applications, John Wiley and Sons, New York (1965).

[8] Govindaraju P, Rizwan U and Thagaraj V, An extreme shock maintenance model under a Bivariate Replacement Policy,Research Methods in Mathematical Sciences, Ed. 1-10 (2011).

[9] Govindaraju P and Rajendiran R , A Study on an Optimal Replacement Policy for a Deteriorating System under Partial Product Process, Journal of Computational Mathematica, 4(1), 48-54(2020).

[10] Lam Y, A note on the Optimal Replacement Problem, Advances in Applied Probability,20, 479-482(1988).

[11] Lam Y, A Repair Replacement Model, Advances in Applied Probability, 22, 494-497(1990).

[12] Lam Y, An Optimal Repairable Replacement Model for Deteriorating Systems, Journal of Applied Probability, 28, 843-851(1991).

[13] Lam Y and Zhang YL, A Shock Model for the Maintenance Problem of a Repairable System, Computers and Operations Research,31, 1807-1820 (2004).

[14] Leung KNG, A note on A Bivariate Optimal Replacement Policy for a Repairable System , Engineering Optimization, 38 , 621 - 625 (2006).

[15] Muth EJ, An Optimal Decision rule for Repair vs. Replacement,IEEE Transactions on Reliability,3,179-181(1977).

[16] Raajpandiyan TR, Syed Tahir Hussainy and Rizwan U , Optimal Replacement Model under Partial Product Process, Stochastic Modeling and Applications, 26(3),170- 176 (2022).

[17] Raajpandiyan TR, Syed Tahir Hussainy and Rizwan U, A Bivariate Replacement Policy (U, N) under Partial Product Process, Asia Pacific Journal of Mathematics, 11(10),(2024).

[18] Raajpandiyan TR, Syed Tahir Hussainy and Rizwan U, A Bivariate Replacement Policy (T, N) under Partial Product Process,The Scientific Temper, 15(02),2065-2069 (2024).

[19] Ross SM, Stochastic Processes, (2nd ed), John Wiley and Sons, New York (1996).

[20] Thangaraj V and Revathy Sundarajan, Multivariate Optimal Replacement Policies for a System Subject to Shocks, Optimization, 41, 173-195(1997).

[21] Thangaraj V and Rizwan U, Optimal Replacement Policies in burn-in Process for an Alternative Repair Model, International Journal of Information Management Sciences, 12(3), 43-56 (2001).

[22] Zhang YL, A Bivariate Optimal Replacement Policy for a Repairable System, Journal of Applied Probability, 31, 1123-1127(1994).