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A Bivariate Replacement Policy for an Extreme Shock Maintenance Model under Partial Sum Process

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Abstract

A study on a degenerative simple repairable system under an extreme shock maintenance model has yielded explicit expressions for the long-run average cost using a bivariate replacement policies (T, N),(T^+,N),(U, N),(U^-,N).Moreover, the research has established the existence of an optimal conditions that minimizes costs.

Key Words: Partial Sum Process, Replacement Policy, Renewal Reward Process, Shock models.

AMS Classification: 60K10, 90B25.

1 Introduction

Reliability heavily relies on the study of maintenance problems. While most maintenance models focus solely on internal failure causes, they often overlook external factors. System failures can be triggered by external shocks. Shock models successfully applied in physics, communication, electronic engineering, and medicine, and also they are underutilized in analyzing deteriorating systems interrupted by random shocks. According to our model, system failure occurs when a single significant shock's damage surpasses a predetermined threshold. This type of shock is termed a 'deadly shock,' as its damage exceeds the threshold, leading to system

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failure. This concept is foundational to extreme shock models. Notably, Chen and Li (2008) explored maintenance strategies under the N-policy using an extreme shock model.

This study derives the long-run average costs for multistate degenerative systems under four bivariate replacement policies: (T,N) Policy: Replace the system after a fixed working age (T) or upon N-th failure, (U, N) Policy: Replace the system after cumulative repair time (U) or upon N-th failure,(T^+,N) Policy: Replace the system at the first failure point after cumulative operating time exceeds T or upon N-th failure, (U^-,N) Policy: Replace the system at the failure point just before total repair time exceeds U or upon N-th failure.

The study establishes the existence of optimal solutions for bivariate replacement policies, ensuring system efficiency. This research investigates the maintenance optimization problem using a bivariate replacement policy under an extreme shock model, demonstrating that the optimal bivariate policy outperforms univariate optimal policies .

The rest of the paper is organized as follows: In Section 2, A comprehensive overview of the model is provided. In Section 3,4,5 and 6, The long-run average cost per unit time is analytically derived for the model, exploring the impact of bivariate replacement policies. Finally the Conclusion is given in Section 7.

2 The Model

We make the following assumptions about the model for a simple degenerative reparable system subject to shocks.

Assumption 2.1 : At time $t = 0$, a new system is installed. Whenever the system fails, it will be repaired. The system will be replaced by an identical new one, some times later.

Assumption 2.2 : Once the system is operating, the shocks from the environment arrive according to a renewal process. Let X_{ni}, i = 1,2, … be the intervals between the (i − 1)-st and i-th shock, after the (n-1)-st repair. Let $E(X11) = \lambda$. Assume that X_{ni}, i = 1,2, …, are ii$d$ sequences, for all $n \in \mathbb{N}$.

Assumption 2.3 : Let Y_{ni}, i = 1,2, … be the sequence of the amount of shock damage produced by the i- th shock, after the $(n-1)$-st repair. Let $E(Y11) = \mu$. Then $\{Y_{ni}$, i = 1,2,3, … } are ii$d$ sequences, for all $n \in \mathbb{N}$. If the system fails, it is closed, so that the random shocks have no effect on the system during the repair time. In the n-th operating stage, that is, after the $(n-1)$-st repair, the system will fail, if the amount of the shock damage first exceed $(2^{n-1}\beta_0)M$, where $0 < \beta_0 \leq 1$ and M is a positive constant.

Assumption 2.4 : Let Z_n, n = 1,2, … be the repair time after the n-th repair and Z_n, n =1,2, … constitute a non decreasing partial sum process with $E(Z_1) = \delta$ and ratio β_0, such that

$0 < \beta_0 < 1$. $Nn(t)$ is the counting process denoting the number of shocks after the $(n-1)$-st repair. It is clear that $E(Z_n) = \delta\left(\frac{\mu}{2^{i-1}\beta_0}\right)$.

Assumption 2.5 : Let r be the reward rate per unit time of the system when it is operating and c be the repair cost rate per unit time of the system and the replacement cost is R. The replacement time is a random variable Z with $E(Z) = \tau$.

Assumption 2.6 : The sequences $\{X_{ni}, i = 1,2, \dots\}$, $\{Y_{ni}, i = 1,2, \dots\}$, $\{Z_n, n = 1,2, \dots\}$ and Z are independent.

Assumption 2.7 : The replacement policy (T, N) ,(T^+,N) , (U, N) , (U^-,N) is adapted.

3 The Bivariate Replacement Policy (T, N)

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (T, N) policy. Let

$$L_n = \min\{l;\ Y_{nl} > 2^{n-1}\beta_0 M\}$$

and

$$W_n = \sum_{i=1}^{l_n} X_{ni}$$

Thus , L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P\{L_n = k\} = p_n q_n^{k-1}$, $k=1,2,3\dots$ where $p_n=P(Y_{nl} > 2^{i-1}\beta_0 M)$ and $q_n=1-p_n$. We have $E(Ln)=\frac{1}{p_n}$. Since $\{X_{ni}, i = 1,2, \dots\}$, $\{Y_{ni}, i = 1,2, \dots\}$ are independent , it is clear that L_n are independent. Now

$$E(W_n) = E\left(\sum_{i=1}^{l_n} X_{ni}\right)$$

$$= E(l_n)E(X_{n1})$$

$$= \frac{\lambda}{p_n}$$

The distribution function of W_n is $F_n(.)$.

The Working age T of the system at time t is the cumulative life-time given by

$$T(t)=\begin{cases} t - V_n & ,U_n + V_n \le t \le U_{n+1} + V_n \\ U_{n+1} & ,U_{n+1} + V_n \le t \le U_{n+1} + V_{n+1} \end{cases}$$

Where, $U_n = \sum_{k=1}^{n} W_k$ and $V_n = \sum_{k=1}^{n} Z_k$ and $U_0 = V_0 = 0$.

The replacement process can be modelled as a renewal process, where $\{T_n, n=1,2,..\}$ represents the inter-replacement time between the (n-1)-st and n-th replacements.

Specifically, T_1 denotes the time to first replacement, while T_n (n ≥ 2) denotes the time between the (n-1)-st and n-th replacements. This sequence {T_n,n=1,2,..} forms a renewal process. Cycles are defined by the time between consecutive system replacements, spanning from initial installation to first replacement, and subsequent replacement intervals. Each replacement concludes one cycle and begins another. By combining the successive cycles with the costs associated with each, we establish a renewal reward process, facilitating the assessment of cumulative costs, rewards, and system efficiency.

The length of the cycle under the replacement policy (T,N) is

$$W = \left(T + \sum_{n=1}^{\eta} Z_n\right)\chi_{(U_N>T)} + \left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(U_N\le T)} + Z$$

where $\eta = 0,1,2, \dots , N-1$ is the number of failures before the working age of the system exceeds T and $\chi(A)$ denotes the indicator function.

The expected length of the cycle is

$$E(W)= E[\left(T + \sum_{n=1}^{\eta} Z_n\right)\chi_{(U_N>T)} + (\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n)\chi_{(U_N\le T)} + Z]$$

$$= E[\left(T + \sum_{n=1}^{\eta} Z_n\right)\chi_{(U_N>T)}] + E[(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n)\chi_{(U_N\le T)}] + E[Z]$$

$$= E[T\chi_{(U_N>T)}] + E[(\left(\sum_{n=1}^{\eta} Z_n\right)\chi_{(U_N>T)}]$$

$$+E\{E[(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n)\chi_{(U_N\le T)}/U_N = u]\}+E[Z]$$

$$= T\overline{F_N(T)} + \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)E(\chi_{(U_n\le T<U_N)}) + \int_0^T E(\sum_{n=1}^{N} W_n)udF_N(u)$$

$$+\int_0^T \sum_{n=1}^{N-1} E(Z_n)dF_N(u) + \tau$$

$$= T\overline{F_N(T)} + \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)(F_n(T) - F_N(T)) + \int_0^T \sum_{n=1}^{N}\frac{\lambda}{p_n}udF_N(u)$$

$$+ \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)F_N(T) + \tau$$

$$= T\overline{F_N(T)} + \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)(F_n(T)) + \int_0^T \sum_{n=1}^{N}\frac{\lambda}{p_n}udF_N(u) + \tau \qquad (3.1)$$

Let $C(T,N)$ be the long-run average cost per unit time under the bivariate replacement policy (T, N). By the renewal reward theorem, the long-run average cost per unit time under the replacement policy-(T, N) is given by

$$C(T,N) = \frac{The\ expected\ cost\ incurred\ in\ a\ cycle}{The\ ecpected\ length\ of\ a\ cycle}$$

$$= \frac{\left[E\left\{\left(c\sum_{n=1}^{\eta} Z_n - rT\right)\chi_{(U_N>T)}\right\} + c_p E(Z) + E\left\{\left(c\sum_{n=1}^{N-1} Z_n - r\sum_{n=1}^{N} W_n\right)\chi_{(U_N\le T)}\right\} + R\right]}{E(W)} \qquad (3.2)$$

Using the equation (3.1) in equation (3.2) we obtain the following result.

Theorem 3.1 For the model described in Section 2, under the assumptions 2.1 to 2.7, the long-run average cost per unit time under the bivariate replacement policy (T, N) for a simple degenerative repairable system is given by

$$\mathcal{C}(T,N) = \frac{c\delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)(F_n(T)) + c_p\,\tau + R - r\mathrm{T}\overline{F_N(T)} - r\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^T u\,dF_N(u)}{\mathrm{T}\overline{F_N(T)} + \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{i-1}\beta_0}\right)(F_n(T)) + \sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^T u\,dF_N(u) + \tau}$$

Deductions

The long-run average cost $\mathcal{C}(T,N)$ is a bivariate function in T and N. Obviously , when N is fixed, $\mathcal{C}(T,N)$ is a function of T. For fixed N = m , it can be written as $\mathcal{C}(T,N) = \mathcal{C}_m(T),\ m = 1,2,\ldots$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $\mathcal{C}_m(T_m^*)$ is minimized. That is, when N=1,2,3…m…,we can find $T_1^*, T_2^*, T_3^*, \ldots, T_m^*, \ldots$, respectively such that the corresponding $\mathcal{C}_1(T_1^*)$, $\mathcal{C}_2(T_2^*)$, $\mathcal{C}_3(T_3^*)$,… $\mathcal{C}_m(T_m^*), \ldots$ are minimized. Because the total life-time of a multistate degenerative system is limited ,the minimum of the long-run average cost per unit time exists . so we can determine the minimum of the long-run average cost per unit time based on $\mathcal{C}_1(T_1^*)$, $\mathcal{C}(T_2^*)$, $\mathcal{C}_3(T_3^*)$,… $\mathcal{C}_m(T_m^*), \ldots$

Then, if the minimum is denoted by $\mathcal{C}_n(T_n^*)$ we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$\mathcal{C}(T,N)^* = \min_N \mathcal{C}_n(T_n^*)$$

$$= \min_N[\min_T \mathcal{C}(T,N)]$$

$$\le \min_N \mathcal{C}(\infty,N) = \mathcal{C}(N^*)$$

The optimal policy $(T,N)^*$ is better than the optimal policy N^*, moreover, under some mild conditions the optimal replacement policy $N*$ is better than the optimal policy $T*$. So under the same conditions, an optimal policy $(T, N)*$ is better than the optimal replacement policies $N*$ and $T*$.

4 The Bivariate Replacement Policy *(T^+, N)*

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (T^{+},N) policy. Let L_n = min{l; Y_{nl} > 2^(n-1) β_0 M} and

$$W_n = \sum_{i=1}^{L_n} X_{ni}$$

Thus L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P\{L_n = k\} = p_n q_n^{k-1}$,

k=1,2,3… where p_n= P (Ynl > 2^{i-1} β_0M) and q_n=1-p_n. We have E(L_n)= $\frac{1}{p_n}$. Since {Xni, i = 1,2, … }, {$Y$$n$i, i = 1,2, … } are independent , it is clear that L_n and {Xni}are independent.

By Wald's equation ,

$$E(W_n) = E\left(\sum_{i=1}^{L_n} X_{ni}\right)$$

$$= E(L_n)E(X_{n1})$$

$$= \frac{\lambda}{p_n}$$

The distribution function of W_n is $F_n(.)$.

The Working age T of the system at time t is the cumulative life-time given by

$$T(t)= \begin{cases} t - V_n & , U_n + V_n \le t \le U_{n+1} + V_n \\ U_{n+1} & , U_{n+1} + V_n \le t \le U_{n+1} + V_{n+1} \end{cases}$$

where, $U_n = \sum_{k=1}^{n} W_k$ and $V_n = \sum_{k=1}^{n} Z_k$ and $U_0 = V_0 = 0$.

The distribution function of V_n is $G_n(.)$.

Let T_1 be the first replacement time; in general for n=2,3,.. let T_n be the time between the (n-1)-st replacement and the n-th replacement. Thus the sequence {T_n,n=1,2,..} forms a renewal process. A cycle is completed , if a replacement is done. A cycle is actually the time interval between the installation of the system and the first replacement or the time interval between two consecutive replacements. Finally, the successive cycles together with the cost incurred in each cycle will constitute a renewal reward process.

The length of the cycle under the replacement policy *(T^{+},N)* is

$$W=\{(\sum_{n=1}^{\eta} W_n + \sum_{n=1}^{\eta} Z_{n-1})\chi_{(U_N>T)}\}\{(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n)\chi_{(U_N\le T)}\} + Z$$

where η = 0,1,2, … , $N-1$ is the number of failures before the total operating time of the system exceeds T and $\chi(A)$ denotes the indicator function.
The random variable η has a geometric distribution given by

$$P(\eta = j) = P\big(W_1 \le T, W_2 \le T, \ldots W_{\eta-1} \le T, W_\eta > T\big);\ j = 1,2\ldots$$

Since η is a random variable,

$$E(\eta - 1) = \sum_{j=1}^{\infty}(j-1)\, P(\eta = j)$$

$$= \bar{F}(T)\sum_{j=1}^{\infty}(j-1)F^{j-1}(T)$$

$$= \frac{F(T)}{\bar{F}(T)}$$

The expected length of the cycle is

$$E(W) = E\left[\left(\sum_{n=1}^{\eta} W_n + \sum_{n=1}^{\eta} Z_{n-1}\right)\chi_{(U_N > T)}\right] + E\left[\left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(U_N \le T)}\right] + E[Z] \tag{4.1}$$

Consider,

$$E\left[\left(\sum_{n=1}^{N} W_n\right)\chi_{(U_N \le T)}\right] = E\left[E\left(\sum_{n=1}^{N} W_n \middle| U_N = u\right)\chi_{(U_N \le T)}\right]$$

$$= \int_0^T E\left(\sum_{n=1}^{N} W_n \middle| U_N = u\right) dF_N(u)$$

$$= \int_0^T E\left(\sum_{n=1}^{N} W_n\right)\ u\,dF_N(u)$$

$$= \int_0^T \sum_{n=1}^{N} \frac{\lambda}{p_n} u\,dF_N(u)$$

$$= \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u\,dF_N(u) \tag{4.2}$$

$$E\left[\left(\sum_{n=1}^{N-1} Z_n\right)\chi_{(U_N \le T)}\right] = E\left[E\left(\sum_{n=1}^{N-1} Z_n \mid U_N = u\right)\chi_{(U_N \le T)}\right]$$

$$= \int_0^T E\left(\sum_{n=1}^{N-1} Z_n \mid U_N = u\right)\ dF_N(u) = \int_0^T E\left(\sum_{n=1}^{N-1} Z_n\right) dF_N(u)$$

$$= \int_0^T \delta \sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right) dF_N(u)$$

$$= \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) \int_0^T dF_N(u)$$

$$= \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) \tag{4.3}$$

$$E\left[\left(\sum_{n=1}^{\eta} W_n\right)\chi_{(U_N>T)}\right] = E\left[\sum_{n=1}^{\eta} (W_n)\chi_{(U_N>T)}\right]$$

$$= \sum_{n=1}^{\eta} E(W_n \mid \eta = n-1) P(U_n \leq T < U_N)$$

$$= \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{\eta} \frac{\lambda}{p_n} \left[P(U_n \leq T < U_N)\right]$$

$$= \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n} \left[F_n(T) - F_N(T)\right] \tag{4.4}$$

$$E\left[\left(\sum_{n=1}^{\eta} Z_{n-1}\right)\chi_{(U_N>T)}\right] = E\left[\sum_{n=1}^{\eta} (Z_{n-1})\chi_{(U_N>T)}\right]$$

$$= \sum_{n=1}^{\eta} E(Z_{n-1} \mid \eta = n-1) P(U_n \leq T < U_N)$$

$$= \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) \left[F_n(T) - F_N(T)\right] \tag{4.5}$$

Using the equation (4.2),(4.3),(4.4),(4.5) in equation (4.1) we obtain the following

$$E(W) = E\left[\left(\sum_{n=1}^{\eta} W_n + \sum_{n=1}^{\eta} Z_{n-1}\right)\chi_{(U_N>T)}\right] + E\left[\left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(U_N\leq T)}\right] + E[Z]$$

$$= E\left[\left(\sum_{n=1}^{N} W_n\right)\chi_{(U_N\leq T)}\right] + E\left[\left(\sum_{n=1}^{N-1} Z_n\right)\chi_{(U_N\leq T)}\right] +$$

$$E\left[\left(\sum_{n=1}^{\eta} W_n\right)\chi_{(U_N>T)}\right] + E\left[\left(\sum_{n=1}^{\eta} Z_{n-1}\right)\chi_{(U_N>T)}\right] + E[Z]$$

$$= \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) + \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n} \left[P(U_n \leq T < U_N)\right]$$

$$+ \quad \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) \left[F_n(T) - F_N(T)\right] + \tau$$

$$= \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) + \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n} [F_n(T) - F_N(T)]$$
$$+ \quad \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [F_n(T) - F_N(T)] + \tau$$

Let $C(T^+, N)$ be the long-run average cost per unit time under the bivariate replacement policy (T^+,N). By the renewal reward theorem, the long-run average cost per unit time under the replacement policy (T^+,N) is given by

$$C(T^+, N) = \frac{\text{the expected cost incurred in a cycle}}{\text{the ecpected length of a cycle}}$$

$$C(T^+, N) = \frac{\begin{bmatrix} E\left\{\left(c \sum_{n=1}^{\eta} Z_{n-1} - r \sum_{n=1}^{\eta} W_n\right) \chi_{(U_N > T)}\right\} + c_p E(Z) \\ +E\left\{\left(c \sum_{n=1}^{N-1} Z_n - r \sum_{n=1}^{N} W_n\right) \chi_{(U_N \le T)}\right\} + R \end{bmatrix}}{E(W)}$$

$$= \frac{\begin{bmatrix} \mathrm{E}\left[c \sum_{n=1}^{\eta} Z_{n-1}\right] \chi_{(U_N > T)} - E\left[r \sum_{n=1}^{\eta} W_n\right] \chi_{(U_N > T)} + c_p E(Z) \\ +E\left[E\left[c \sum_{n=1}^{N-1} Z_n\right] \chi_{(U_N \le T)}\right] - E\left[E\left[r \sum_{n=1}^{N} W_n\right] \chi_{(U_N \le T)}\right] + R \end{bmatrix}}{E(W)}$$

$$= \frac{\begin{bmatrix} c\delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [F_n(T) - F_N(T)] \quad - r \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{\eta} \frac{\lambda}{p_n} [F_n(T) - F_N(T)] \quad + \quad c_p \tau \\ + \ c\,\delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) \quad - \ r \ \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) \quad + R \end{bmatrix}}{E(W)}$$

Theorem 4.1 *For the model described in Section 2, under the assumptions 2.1 to 2.7, the long-run average cost per unit time under the bivariate replacement policy* (T^+,N) *for a simple degenerative repairable system is given by*

$$C(T^+, N) = \frac{\begin{bmatrix} c\delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [F_n(T) - F_N(T)] \quad - r \ \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{\eta} \frac{\lambda}{p_n} [F_n(T) - F_N(T)] \\ + \ c\,\delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) \qquad - \ r \ \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) \\ + \ R \qquad\qquad + \ c_p \tau \end{bmatrix}}{\begin{bmatrix} \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) F_N(T) \quad + \frac{F(T)}{\bar{F}(T)} \sum_{n=1}^{N-1} \frac{\lambda}{p_n} [F_n(T) - F_N(T)] \\ + \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [F_n(T) - F_N(T)] \qquad + \tau \end{bmatrix}}$$

Deductions

The long-run average cost $C(T^+, N)$ is a bivariate function in T^+ and N. Obviously , when N is fixed, $C(T^+, N)$ is a function of T^+. For fixed N=m , it can be written as

$$C(T^+, N) = C_m(T^+),\ m = 1,2,\dots$$

Thus, for a fixed m, we can find ${T^+}^*_m$ by analytical or numerical methods such that $C_m({T^+}^*_m)$ is minimized. That is , when N=1,2,3…m…, we can find ${T^+}^*_1, {T^+}^*_2, {T^+}^*_3, \dots, {T^+}^*_m, \dots,$ respectively such that the corresponding $C_1({T^+}^*_1)$, $C_2({T^+}^*_2)$, $C_3({T^+}^*_3)$,… $C_m({T^+}^*_m), \dots$ are minimized. Because the total life-time of a multistate degenerative system is limited ,the minimum of the long-run average cost per unit time exists . so we can determine the minimum of the long-run average cost per unit time based on $C_1({T^+}^*_1)$, $C_2({T^+}^*_2)$, $C_3({T^+}^*_3)$,… $C_m({T^+}^*_m), \dots$

Then, if the minimum is denoted by $C_n({T^+}^*_n)$ we obtain the bivariate optimal replacement policy $(T^+, N)^*$ such that

$$C(T^+, N)^* = \min_N C_n({T^+}^*_n)$$

$$= [\min_{T^+} C(T^+, N)]$$

$$\leq C(\infty, N) = C(N^*)$$

The optimal policy $(T^+, N)^*$ is better than the optimal policy N^*, moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy ${T^+}^*$. So under the same conditions, an optimal policy $(T^+, N)^*$ is better than the optimal replacement policies N^* and ${T^+}^*$.

5 The Bivariate Replacement Policy (U, N)

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (U, N) policy. Let $L_n = \min\{l;\ \ Y_{nl} > 2^{n-1}\beta_0 M\}$ and

$$W_n = \sum_{i=1}^{L_n} X_{ni}$$

Thus L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P\{L_n = k\} = p_n q_n^{k-1}$, $k = 1,2,3 \dots$ where $p_n = P\ (Y_{nl} > 2^{i-1}\beta_0 M)$ and $q_n = 1 - p_n$. We have E(L_n)= $\frac{1}{p_n}$

. Since $\{X_{ni}, i = 1,2, \dots\}$, $\{Y_{ni}, i = 1,2, \dots\}$ are independent , it is clear that L_n and $\{X_{ni}\}$are independent.

By Wald's equation ,

$$E(W_n) = E\left(\sum_{i=1}^{L_n} X_{ni}\right)$$

$$= E(L_n)E(X_{n1})$$

$$= \frac{\lambda}{p_n}$$

The Working age T of the system at time t is the cumulative life-time given by

$$T(t)=\begin{cases} t - V_n & , U_n + V_n \le t \le U_{n+1} + V_n \\ U_{n+1} & , U_{n+1} + V_n \le t \le U_{n+1} + V_{n+1} \end{cases}$$

where, $U_n = \sum_{k=1}^{n} W_k$ and $V_n = \sum_{k=1}^{n} Z_k$ and $U_0 = V_0 = 0$.

The replacement times are represented by the sequence $\{U_n, n=1,2,..\}$, where U_1 denotes the time to the first replacement, and U_n (n ≥ 2) represents the time between the (n-1)-st and n-th replacements. This sequence forms a renewal process, where each replacement restarts the cycle, ensuring independent and identically distributed inter-replacement times. A cycle is defined as the time interval between the installation of the system and its first replacement, or the time interval between two consecutive replacements. Upon completion of a cycle, a replacement is performed, marking the beginning of a new cycle. Successive cycles, coupled with the costs incurred in each cycle, form a renewal reward process. This process enables the analysis of long-term system performance and associated costs, leveraging the renewal theory framework.
The length of the cycle under the replacement policy (U, N) is

$$W = \left(U + \sum_{n=1}^{\eta} Z_n\right)\chi_{(V_N > U)} + \left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(V_N \le U)} + Z$$

where $\eta = 0,1,2, \dots, N-1$ is the number of failures before the total repair time of the system exceeds U and $\chi(A)$ denotes the indicator function.

The expected length of the cycle is

$$E(W) = E\left[\left(U + \sum_{n=1}^{\eta} Z_n\right)\chi_{(V_N > U)} + \left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(V_N \le U)} + Z\right]$$

$$= \mathrm{E}\left[\left(U+\sum_{n=1}^{\eta} Z_n\right)\chi_{(V_N>U)}\right]+\mathrm{E}\left[\left(\sum_{n=1}^{N} W_n+\sum_{n=1}^{N-1} Z_n\right)\chi_{(V_N\le U)}\right]+\ \mathrm{E}[\mathrm{Z}]$$

$$= E[U\chi_{(V_N>U)}]+E\left[\left(\sum_{n=1}^{\eta} Z_n\right)\chi_{(V_N>U)}\right]$$

$$+E\left\{E\left[\left(\sum_{n=1}^{N} W_n+\sum_{n=1}^{N-1} Z_n\right)\chi_{(V_N\le U)}\Big/ V_N=u\right]\right\}+\mathrm{E}[\mathrm{Z}]$$

$$= U\overline{G_N(U)}\ ++\delta\mu\left[G_2(U)+\sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]E(\chi_{(V_n\le U<V_N)})$$

$$+\int_0^U E(\sum_{n=1}^{N} W_n)udG_N(u)\ +\int_0^U\sum_{n=1}^{N-1} Z_n\, dG_N(u)$$

$$+\tau$$

$$= U\overline{G_N(U)}+\delta\mu\left[G_2(U)+\sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]P(V_n\le U<V_N)+$$

$$\int_0^T\sum_{n=1}^{N}\frac{\lambda}{p_n}udG_N(u)+\ \delta\,\mu\left[1\ +\sum_{n=2}^{N-1}\left(\frac{1}{2^{n-2}\beta_0}\right)\right]G_N(U)+\ \ \tau$$

$$= U\overline{G_N(U)}\ +\delta\mu\left[G_2(U)+\sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]G_N(U)+\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^U udG_N(U)$$

$$+\,\delta\,\mu\left[1\ +\sum_{n=2}^{N-1}\left(\frac{1}{2^{n-2}\beta_0}\right)\right]G_N(U)\ +\ \tau \tag{5.1}$$

Let $\mathcal{C}(U,N)$ be the long-run average cost per unit time under the bivariate replacement policy (U,N). By the renewal reward theorem, the long-run average cost per unit time under the replacement policy-(U,N) is given by

$$\mathcal{C}(U,N)=\frac{\text{the expected cost incurred in a cycle}}{\text{the ecpected length of a cycle}}$$

$$\mathcal{C}(U,N)=\frac{\begin{bmatrix}\mathrm{E}\left\{\left(\mathrm{c}\sum_{n=1}^{\eta}Z_n-rU\right)\chi_{(V_N>U)}\right\}+E\left\{\left(c\sum_{n=1}^{N-1}Z_n-r\sum_{n=1}^{N}W_n\right)\chi_{(V_N\le U)}\right\}\\+R+c_pE(Z)\end{bmatrix}}{E(W)}$$

(5.2)

Consider,

$$\mathrm{E}\left[\left(\mathrm{c}\sum_{n=1}^{\eta}Z_n\right)\chi_{(V_N>U)}\right]=\mathrm{E}\left[\left(\mathrm{c}\sum_{n=1}^{\eta}Z_n\right)\chi_{(V_n\le U<V_N)}\right]$$

$$=\mathrm{c}\sum_{n=1}^{\eta}E(Z_n)E[\chi_{(V_n\le U<V_N)}]$$

$$=\mathrm{c}\sum_{n=1}^{\eta}E(Z_n)P(V_n\le U<V_N)$$

$$=\delta\mathrm{c}\,\mu\left[G_2(U)+\sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]G_N(U) \quad (5.3)$$

Now,

$$E\left[\left(c\sum_{n=1}^{N-1}Z_n\right)\chi_{(V_N\le U)}\right]=E\left[E\left[\left(c\sum_{n=1}^{N-1}Z_n\mid V_N=U\right)\chi_{(V_N\le U)}\right]\right]$$

$$=\int_0^U cE\left(\sum_{n=1}^{N-1}Z_n\mid V_N=U\right)dG_N(U)$$

$$=\int_0^U c\left(\sum_{n=1}^{N-1}E(Z_n)\right)dG_N(U)$$

$$=\sum_{n=1}^{N-1}E(Z_n)\int_0^U c\,dG_N(U)$$

$$=\delta c\mu\left[1+\sum_{n=2}^{N-1}\left(\frac{1}{2^{n-2}\beta_0}\right)\right]G_N(U) \quad (5.4)$$

$$E\left[r\left(\sum_{n=1}^{N}W_n\right)\chi_{(V_N\le U)}\right]=E\left[E\left[\left(r\sum_{n=1}^{N}W_n\mid V_N=U\right)\chi_{(V_N\le U)}\right]\right]$$

$$=\int_0^U rE\left(\sum_{n=1}^{N}W_n\mid V_N=U\right)dG_N(U)$$

$$=rE\left(\sum_{n=1}^{N}W_n\right)\int_0^U dG_N(U)$$

$$=r\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^U dG_N(U)$$

$$=r\sum_{n=1}^{N}\frac{\lambda}{p_n}G_N(U) \quad (5.5)$$

And

$$E[rU\chi_{(V_N>U)}] = rE[U\chi_{(V_N>U)}]$$

$$= rUE[\chi_{(V_N>U)}]$$

$$= rU\overline{G_N(U)} \quad (5.6)$$

Using the equation (5.1),(5.3),(5.4),(5.5),(5.6) in equation (5.2) we obtain the following result.

Theorem 5.1 For the model described in Section 2, under the assumptions 2.1 to 2.7, the long-run average cost per unit time under the bivariate replacement policy (U, N) for a simple degenerative repairable system is given by

$$C(T,N) = \frac{\begin{bmatrix} \delta c\,\mu\left[G_2(U) + \sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]G_N(U) - rU\overline{G_N(U)} \\ +c\delta\mu\left[1 + \sum_{n=2}^{N-1}\left(\frac{1}{2^{n-2}\beta_0}\right)\right]G_N(U) - r\sum_{n=1}^{N}\frac{\lambda}{p_n}G_N(U) + R + c_p\tau \end{bmatrix}}{\begin{bmatrix} U\overline{G_N(U)} + \delta\mu\left[G_2(U) + \sum_{n=2}^{\infty}\left(\frac{G_{n+1}(U)}{2^{n-1}\beta_0}\right)\right]G_N(U) \\ + \sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^U u\,dG_N(U) + \delta\mu\left[1 + \sum_{n=2}^{N-1}\left(\frac{1}{2^{n-2}\beta_0}\right)\right]G_N(U) + \tau \end{bmatrix}} \quad (5.7)$$

Deductions

The long-run average cost $C(U,N)$ is a bivariate function in U and N. Obviously , when N is fixed, $C(U,N)$ is a function of U. For fixed $N = m$, it can be written as $C(U,N) = C_m(U),\ m = 1,2,\ldots$

Thus, for a fixed m, we can find U_m^* by analytical or numerical methods such that $C_m(U_m^*)$ is minimized. That is, when N=1,2,3…m…, we can find $U_1^*, U_2^*, U_3^*, \ldots, U_m^*, \ldots$, respectively such that the corresponding $C_1(U_1^*)$, $C_2(U_2^*)$, $C_3(U_3^*)$,… $C_m(U_m^*)$, ... are minimized. Because the total life-time of a multistate degenerative system is limited ,the minimum of the long-run average cost per unit time exists . so we can determine the minimum of the long-run average cost per unit time based on $C_1(U_1^*)$, $C_2(U_2^*)$, $C_3(U_3^*)$,… $C_m(U_m^*)$, ...

Then, if the minimum is denoted by $C_n(U_n^*)$ we obtain the bivariate optimal replacement policy $(U,N)^*$ such that

$$C(U,N)^* = \min_N C_n(U_n^*)$$

$$= [\min_{U} C(U,N)]$$

$$\leq C(\infty, N) = C(N^*)$$

The optimal policy $(U,N)^*$ is better than the optimal policy N^*, moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy U^*. So under the same conditions, an optimal policy $(U, N)^*$ is better than the optimal replacement policies N^* and U^*.

6 The Bivariate Replacement Policy (U^-,N)

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (U^-,N) policy. Let $L_n = \min\{l; Y_{nl} > 2^{n-1}\beta_0 M\}$

and

$$W_n = \sum_{i=1}^{L_n} X_{ni}$$

Thus L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$-st failure and L_n has a geometric distribution with $P\{L_n = k\} = p_n q_n^{k-1}$, $k = 1,2,3 \ldots$ where $p_n = P\,(Y_{nl} > 2^{i-1}\beta_0 M)$ and $q_n = 1 - p_n$. We have E(L_n)= $\frac{1}{p_n}$. Since $\{X_{ni},\ i = 1,2, \ldots\}$, $\{Y_{ni},\ i = 1,2, \ldots\}$ are independent , it is clear that L_n and $\{X_{ni}\}$are independent.

By Wald's equation ,

$$E(W_n) = E\left(\sum_{i=1}^{L_n} X_{ni}\right)$$

$$= E(L_n)E(X_{n1})$$

$$= \frac{\lambda}{p_n}$$

The distribution function of W_n is $F_n(.)$.

The Working age T of the system at time t is the cumulative life-time given by

$$T(t)=\begin{cases} t-V_n & ,U_n+V_n \le t \le U_{n+1}+V_n \\ U_{n+1} & ,U_{n+1}+V_n \le t \le U_{n+1}+V_{n+1} \end{cases}$$

where, $U_n = \sum_{k=1}^{n} W_k$ and $V_n = \sum_{k=1}^{n} Z_k$ and $U_0 = V_0 = 0$.

The distribution function of V_n is $G_n(.)$.
The replacement times are represented by the sequence $\{U_n$,n=1,2,..$\}$, where U_1 denotes the time to the first replacement, and U_n (n ≥ 2) represents the time between the (n-1)-st and n-th replacements. This sequence forms a renewal process, where each replacement restarts the cycle, ensuring independent and identically distributed inter-replacement times. In the context of renewal theory, a cycle is formally defined as the interval between system installation and first replacement or between successive replacements. Cycle completion coincides with replacement.

The resulting sequence of cycles and associated costs forms a renewal reward process, characterized by independent and identically distributed cycles and rewards, enabling analysis of system performance and cost optimization.

The length of the cycle under the replacement policy (U^-,N) is

$$W = \left\{ \left(\sum_{n=1}^{\eta} W_n + \sum_{n=0}^{\upsilon} Z_n \right) \chi_{(V_N > U)} \right\} + \left\{ \left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n \right) \chi_{(V_N \le U)} \right\} + Z$$

where η = 0,1,2, … , $N-1$ is the number of failures before the total operating time of the system exceeds U and $\upsilon = 0,1,2, \dots, N-1$ is the number of repairs before the total repair time is expected to exceeds U . If $V_n \le U < V_{n+1}$ *for* n=0,1,2,.. $N-1$ then $U - V_n$ will be virtual repair time.

$\chi(A)$ denotes the indicator function.

The random variable η has a geometric distribution given by

$$P(\eta = j) = P\big(W_1 \le T, W_2 \le T, \dots W_{\eta-1} \le T, W_\eta > T\big);\ j = 1,2..$$
$$= G^{j-1}(U)\bar{G}(U)$$

Since η is a random variable,

$$E(\upsilon - 1) = \sum_{j=1}^{\infty} (j-1)\, P(\eta = j)$$
$$= \bar{G}(U) \sum_{j=1}^{\infty} (j-1) G^{j-1}(U)$$
$$= \frac{G(U)}{\bar{G}(U)}$$

The expected length of the cycle is

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$$E(W) = E\left[\left\{\left(\sum_{n=1}^{\eta} W_n + \sum_{n=0}^{\upsilon} Z_n\right)\chi_{(V_N>U)}\right\} + \left\{\left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right)\chi_{(V_N\leq U)}\right\}\right] + E[Z]$$

(6.1)

Consider,

$$E\left[(\textstyle\sum_{n=1}^{N} W_n)\chi_{(V_N\leq U)}\right] = E\left[E(\textstyle\sum_{n=1}^{N} W_n|(V_N = u)\chi_{(V_N\leq U)}\right]$$

$$= \int_0^U E(\textstyle\sum_{n=1}^{N} W_n|(V_N = u)dG_N(u)$$

$$= \int_0^U E(\textstyle\sum_{n=1}^{N} W_n)\ udG_N(u)$$

$$= \int_0^U \textstyle\sum_{n=1}^{N} \frac{\lambda}{p_n} udG_N(u)$$

$$= \textstyle\sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^U udG_N(u) \quad (6.2)$$

$$E\left[(\textstyle\sum_{n=1}^{N-1} Z_n)\chi_{(V_N\leq U)}\right] = E\left[E(\textstyle\sum_{n=1}^{N-1} Z_n \mid (V_N = u)\chi_{(V_N\leq U)}\right]$$

$$= \int_0^U E(\textstyle\sum_{n=1}^{N-1} Z_n \mid (V_N = u)\ dG_N(u)$$

$$= \int_0^U E(\textstyle\sum_{n=1}^{N-1} Z_n)\,dG_N(u)$$

$$= \int_0^U \delta \textstyle\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right) dG_N(u)$$

$$= \delta \textstyle\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right) \int_0^U dG_N(u)$$

$$= \delta \textstyle\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right) G_N(u) \quad (6.3)$$

$$E\left[(\textstyle\sum_{n=1}^{\eta} W_n)\chi_{(U_N>U)}\right] = E\left[\textstyle\sum_{n=1}^{\eta}(W_n)\chi_{(V_N>U)}\right]$$

$$=\textstyle\sum_{n=1}^{\eta} E(W_n \mid \eta = n-1)P(V_n \leq U < V_N)$$

$$= \textstyle\sum_{n=1}^{N-1} \frac{\lambda}{p_n}\left[P(V_{n-1} \leq U < V_N)\right]$$

$$= \textstyle\sum_{n=1}^{N-1} \frac{\lambda}{p_n}\left[G_{n-1}(U) - G_N(u)\right] \quad (6.4)$$

$$E\left[(\textstyle\sum_{n=1}^{\eta} Z_n)\chi_{(U_N>U)}\right] = E\left[\textstyle\sum_{n=1}^{\eta}(Z_n)\chi_{(V_N>U)}\right]$$

$$= \sum_{n=1}^{\eta} E(Z_n \mid v) P(V_n \leq U < V_N)$$

$$= \sum_{n=1}^{N-1} E(Z_n) E(v-1) P(V_n \leq U < V_N)$$

$$= \frac{G(U)}{\bar{G}(U)} \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [G_n(U) - G_N(u)] \qquad (6.5)$$

Using the equation (6.2),(6.3),(6.4),(6.5) in equation (6.1) we obtain the following

$$E(W) = E\left[\left(\sum_{n=1}^{\eta} W_n + \sum_{n=0}^{v} Z_n\right) \chi_{(V_N > U)}\right] + E\left[\left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Z_n\right) \chi_{(V_N \leq U)}\right] + E[Z]$$

$$= E\left[\left(\sum_{n=1}^{\eta} W_n\right) \chi_{(V_N \leq U)}\right] + E\left[\left(\sum_{n=1}^{N-1} Z_n\right) \chi_{(V_N \leq U)}\right] + E\left[\left(\sum_{n=1}^{N} W_n\right) \chi_{(V_N > U)}\right] + E\left[\left(\sum_{n=0}^{v} Z_n\right) \chi_{(V_N > U)}\right] + E[Z]$$

$$= \sum_{n=1}^{N} \frac{\lambda}{p_n} \int_0^U u dG_N(u) + \delta \sum_{n=1}^{N-1} \left(\frac{\mu}{2^{n-1}\beta_0}\right) G_N(u) + \sum_{n=1}^{N-1} \frac{\lambda}{p_n} [G_{n-1}(U) - G_N(u)] + \frac{G(U)}{\bar{G}(U)} \delta \sum_{n=2}^{N-1} \left(\frac{\mu}{2^{n-2}\beta_0}\right) [G_n(U) - G_N(u)] + \tau$$

$$E(W)=\sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^U u\,dG_N(u)+\delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right)G_N(u)\quad+\ \tau$$

$$+\sum_{n=1}^{N-1}\frac{\lambda}{p_n}[G_{n-1}(U)-G_N(u)]$$

$$+\frac{G(U)}{\bar{G}(U)}\delta\sum_{n=2}^{N-1}\left(\frac{\mu}{2^{n-2}\beta_0}\right)[G_n(U)-G_N(u)]$$

Let $\mathcal{C}(U^-,N)$ be the long-run average cost per unit time under the bivariate replacement policy (U^-,N). By the renewal reward theorem, the long-run average cost per unit time under the replacement policy (U^-,N) is given by

$$\mathcal{C}(U^-,N)=\frac{\text{the expected cost incurred in a cycle}}{\text{the ecpected length of a cycle}}$$

$$=\frac{\begin{bmatrix}E\left\{\left(c\sum_{n=0}^{\upsilon}Z_n-r\sum_{n=1}^{\eta}W_n\right)\chi_{(V_N>U)}\right\}+c_pE(Z)\\ +E\left\{\left(c\sum_{n=1}^{N-1}Z_n-r\sum_{n=1}^{N}W_n\right)\chi_{(V_N\le U)}\right\}+R\end{bmatrix}}{E(W)}$$

$$=\frac{\begin{bmatrix}\mathrm{E}[c\sum_{n=0}^{\upsilon}Z_n]\chi_{(V_N>U)}-E\left[r\sum_{n=1}^{\eta}W_n\right]\chi_{(V_N>U)}+c_pE(Z)\\ +E\left[E[c\sum_{n=1}^{N-1}Z_n]\chi_{(V_N\le U)}\right]-E\left[E[r\sum_{n=1}^{N}W_n]\chi_{(V_N\le U)}\right]+R\end{bmatrix}}{E(W)}$$

$$\mathcal{C}(U^-,N)=\frac{\begin{bmatrix}c\frac{G(U)}{\bar{G}(U)}\delta\sum_{n=2}^{N-1}\left(\frac{\mu}{2^{n-2}\beta_0}\right)[G_n(U)-G_N(u)] & -r\ \sum_{n=1}^{N-1}\frac{\lambda}{p_n}[G_{n-1}(U)-G_N(u)]\\ +\ c\ \delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right)G_N(u) & -\ r\ \sum_{n=1}^{N}\frac{\lambda}{p_n}\int_0^U u\,dG_N(u)\\ +\ c_p\tau & +\ R\end{bmatrix}}{E(W)}$$

Theorem 6.1 For the model described in Section 2, under the assumptions 2.1 to 2.7, the long-run average cost per unit time under the bivariate replacement policy (U⁻,N) for a simple degenerative repairable system is given by

$$C(U^-,N) = \frac{\left[\begin{array}{ll} c\frac{G(U)}{\bar{G}(U)}\delta\sum_{n=2}^{N-1}\left(\frac{\mu}{2^{n-2}\beta_0}\right)[G_n(U)-G_N(u)] & -r\ \sum_{n=1}^{N-1}\frac{\lambda}{p_n}[G_{n-1}(U)-G_N(u)] \\ +\ c\,\delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right)G_N(u) & -\ r\ \sum_{n=1}^{N-1}\frac{\lambda}{p_n}\int_0^U u dG_N(u) \\ +\ c_p\tau & +\ R \end{array}\right]}{\left[\begin{array}{ll} \sum_{n=1}^{N-1}\frac{\lambda}{p_n}\int_0^U u dG_N(u)+\delta\sum_{n=1}^{N-1}\left(\frac{\mu}{2^{n-1}\beta_0}\right)G_N(u) & +\sum_{n=1}^{N}\frac{\lambda}{p_n}[G_{n-1}(U)-G_N(u)] \\ +\frac{G(U)}{\bar{G}(U)}\delta\sum_{n=2}^{N-1}\left(\frac{\mu}{2^{n-2}\beta_0}\right)[G_n(U)-G_N(u)] & +\tau \end{array}\right]}$$

Deductions

The long-run average cost $C(U^-,N)$ is a bivariate function in U^- and N. Obviously , when N is fixed, $C(U^-,N)$ is a function of U^-. For fixed N=m, it can be written as

$$C(U^-,N) = C_m(U^-),\ m = 1,2,\ldots$$

Thus, for a fixed m, we can find U^{-*}_m by analytical or numerical methods such that $C_m(U^{-*}_m)$ is minimized. That is , when N=1,2,3…m…, we can find $U^{-*}_1, U^{-*}_2, U^{-*}_3, \ldots, U^{-*}_m, \ldots,$ respectively such that the corresponding $C_1(U^{-*}_1)$, $C_2(U^{-*}_2)$, $C_3(U^{-*}_3)$,… $C_m(U^{-*}_m), \ldots$ are minimized. Because the total life-time of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists . so we can determine the minimum of the long-run average cost per unit time based on $C_1(U^{-*}_1)$, $C_2(U^{-*}_2)$, $C_3(U^{-*}_3)$,… $C_m(U^{-*}_m), \ldots$

Then, if the minimum is denoted by $C_n(U^{-*}_n)$ we obtain the bivariate optimal replacement policy $(U^-,N)^*$ such that

$$C(U^-,N)^* = \min_N C_n(U^{-*}_n)$$

$$= [\min_{U^-} C(U^-,N)]$$

$$\leq C(\infty,N) = C(N^*)$$

The optimal policy $(U^-,N)^*$ is better than the optimal policy N^*, moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy U^{-*}. So under the same conditions, an optimal policy $(U^-, N)^*$ is better than the optimal replacement policies N^* and U^{-*}.

7 Conclusion

Considering an extreme shock maintenance model for a degenerative simple repairable system, explicit expressions for the long-run average cost under the bivariate replacement policies: (T, N), (U, N), (T^+,N), (U^-,N) has been obtained.

Existence of optimal value of has been deduced. This result enables optimization of maintenance strategies for degenerative repairable systems under extreme shocks.

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