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Some Results in Fixed Point Theorem using Weakly Compatible Mappings in Neutrosophic Metric Spaces

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Abstract

Neutrosophic metric space was introduced by Kirisci and Simsek in 2020. In this paper, the author introduced the concept of coupled fixed point, coupled coincidence point, common coupled fixed point. To define the above concept two mappings are used also, we prove some common fixed point using compatible and weakly compatible mappings in Neutrosophic metric space.

Key words: Neutrosophic metric space, Fixed point, Compatible, Weakly compatible.

H**AMS classification:** 47H10, 54H25.

1 Introduction

Zadeh [20] proposed the concept of fuzzy sets in 1965. The application of fuzzy set theory is crucial to all engineering and mathematical fields. In 1975, Kramosil and Michalek presented the idea of fuzzy metric space. George and Veeramani then reformulated the idea of fuzzy metric space [9, 10].In fuzzy metric space, only a membership value between 0 and 1 was defined. But in reality, this knowledge proved to be inadequate. In 1983, Atanassov [3, 4] proposed the Intuitionistic fuzzy set, which was an expansion of the fuzzy set. Non-membership value was specified in intuitive fuzzy sets.

Many authors have written on the idea of coupled fixed point, such as Lakshmikantham [5, 12], Sedghi et al. [14]. Using contractive criteria, Jin-xuan Fang[8] developed the idea of common fixed point theorems for compatible and weakly compatible systems. A common fixed point theorem for mappings in fuzzy metric spaces under contractive conditions was established by Xin-Qi Hu [11]. The fixed

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point theorem in metric spaces was proved by several authors [2] - [6].

Later, in 1995, Florentin Smarandache[17] defined a neutrosophic set, which was an extension of intuitionistic fuzzy sets. We include a new concept in the Neutrosophic set, namely indeterminacy. Thus, every neutrosophic set deals with three components, namely the truth value (T), the indeterminacy value (I) and the false value (F). Neutrosophic sets play a vital role in medical image processing. Jeyaraman et al. [18, 19] discussed fixed point results in neutrosophic metric spaces using contraction mapping and coupled coincidence point. Also, Shakila et.al.[15, 16] discussed about the fixed point results in Neutrosophic b-metric spaces.

In this paper, we have proved that there is a common fixed point in weakly compatible mappings in neutrosophic metric space using contractive conditions with an example.

2 Preliminaries

Definition 2.1 Let $*$ be a t-norm and let $*_n : [0,1] \times [0,1] \to [0,1], n \in \mathbb{N}$ be defined as follows: $*_1(\kappa) = *(\kappa,\kappa), *_{n+1}(\kappa) = *(*_n(\kappa),\kappa), (n \in \mathbb{N}, \kappa \in [0,1])$.
The t-norm $*$ is said to be of $\mathfrak{H}$ - type if t is continuous and the family $\{*_n(\kappa), n \in \mathbb{N}\}$ is equicontinuous at $\kappa = 1$.
The family of functions $\{*_n(\kappa), n \in \mathbb{N}\}$ is equicontinuous at $\kappa = 1$ if for every $\beta \in [0,1]$ there exists $\gamma(\beta) \in (0,1)$ such that the following condition is satisfied: $\kappa > 1 - \gamma(\beta)$ implies $*_n(\kappa) > 1 - \beta$ for all $n \in \mathbb{N}$.

Definition 2.2 Let $\Diamond$ and $\odot$ be t-conorms let $\Diamond_n : [0,1] \times [0,1] \to [0,1], \odot_n : [0,1] \times [0,1] \to [0,1], n \in \mathbb{N}$, be described as follows:

$$\Diamond_1(\kappa) = \Diamond(\kappa,\kappa), \Diamond_{n+1}(\kappa) = \Diamond(\Diamond_n(\kappa),\kappa), (n \in \mathbb{N}, \kappa \in [0,1])$$
$$\odot_1(\kappa) = \odot(\kappa,\kappa), \odot_{n+1}(\kappa) = \odot(\odot_n(\kappa),\kappa), (n \in \mathbb{N}, \kappa \in [0,1])$$

We say that the t-conorms $\Diamond$, $\odot$ are of $\mathfrak{H}$ - type if t is continuous and the family $\{\Diamond_n(\kappa), \odot_n(\kappa), n \in \mathbb{N}\}$ is equicontinuous at $\kappa = 0$.
The family of functions $\{\Diamond_n(\kappa), \odot_n(\kappa), n \in \mathbb{N}\}$ is equicontinuous at $\kappa = 0$ if for every $\beta \in [0,1]$ there exists $\gamma(\beta) \in (0,1)$ such that the following condition is satisfied: $\kappa < \gamma(\beta)$ implies $\Diamond_n(\kappa) < \beta, \odot_n(\kappa) < \beta$ for all $n \in \mathbb{N}$.

Definition 2.3 A Neutrosophic Metric Space(NMS) is a 7-tuple $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$, if Λ is an arbitrary set, $*$ is a continuous triangular norm, $\Diamond$, $\odot$ are continuous triangular conorms, $\daleth, \Theta, \Upsilon$ are nonempty sets on $\Lambda^2 \times [0, \infty)$ that fulfills the following requirements: For every $\varrho, \varsigma, \delta \in \Lambda$ and $\varphi, \lambda > 0$,

(i) $\daleth(\varrho, \varsigma, \varphi) + \Gamma(\varrho, \varsigma, \varphi) + Z(\varrho, \varsigma, \varphi) \leq 3$,
(ii) $0 \leq \daleth(\varrho, \varsigma, \varphi) \leq 1; 0 \leq \Gamma(\varrho, \varsigma, \varphi) \leq 1; 0 \leq Z(\varrho, \varsigma, \varphi) \leq 1$,
(iii) $\daleth(\varrho, \varsigma, \varphi) = 1$ iff $\varrho = \varsigma$,
(iv) $\daleth(\varrho, \varsigma, \varphi) = \daleth(\varsigma, \varrho, \varphi)$,
(v) $\daleth(\varrho, \varsigma, \varphi) * \daleth(\varsigma, \delta, \lambda) \leq \daleth(\varrho, \delta, \varphi + \lambda)$, for all $\varphi, \lambda > 0$,
(vi) $\daleth(\varrho, \varsigma, \cdot) : (0, \infty) \to [0, 1]$ is Neutrosophic continuous,
(vii) $\lim_{\varphi \to \infty} \daleth(\varrho, \varsigma, \varphi) = 1$, for all $\varphi > 0$,
(viii) $\Gamma(\varrho, \varsigma, \varphi) = 0$ iff $\varrho = \varsigma$,
(ix) $\Gamma(\varrho, \varsigma, \varphi) = \Gamma(\varsigma, \varrho, \varphi)$
(x) $\Gamma(\varrho, \varsigma, \varphi) \Diamond \Gamma(\varsigma, \delta, \lambda) \geq \Gamma(\varrho, \delta, \varphi + \lambda)$, for all $\varphi, \lambda > 0$,
(xi) $\Gamma(\varrho, \varsigma, \cdot) : (0, \infty) \to [0, 1]$ is continuous
(xii) $\lim_{\varphi \to \infty} \Gamma(\varrho, \varsigma, \varphi) = 0$ for all $\varphi > 0$,
(xiii) $Z(\varrho, \varsigma, \varphi) = 0$ iff $\varrho = \varsigma$,
(xiv) $Z(\varrho, \varsigma, \varphi) = Z(\varsigma, \varrho, \varphi)$,
(xv) $Z(\varrho, \varsigma, \varphi) \odot Z(\varsigma, \delta, \lambda) \geq Z(\varrho, \delta, \varphi + \lambda)$ for all $\varphi, \lambda > 0$,
(xvi) $Z(\varrho, \varsigma, \cdot) : (0, \infty) \to [0, 1]$ is continuous
(xvii) $\lim_{\varphi \to \infty} Z(\varrho, \varsigma, \varphi) = 0$ for all $\varphi > 0$,

Here, $\daleth(\varrho, \varsigma, \varphi), \Gamma(\varrho, \varsigma, \varphi)$ and $Z(\varrho, \varsigma, \varphi)$ signify the degree of nearness, the degree of neutralness and the degree of non-nearness between ϱ and ς with respect to φ respectively.

Example 2.4 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a NMS. Define t-norm by $\mu_1 * \mu_2 = \mu_1 \cdot \mu_2$, t-conorm by $\mu_1 \Diamond \mu_2 = \mu_1 + \mu_2 - \mu_1 \cdot \mu_2$, $\mu_1 \odot \mu_2 = \mu_1 + \mu_2 - \mu_1 \cdot \mu_2$. For all $\varrho, \varsigma \in \Lambda, \varphi > 0$, define $\daleth(\varrho, \varsigma, \varphi) = \frac{\varphi}{\varphi + d(\varrho, \varsigma)}, \Gamma(\varrho, \varsigma, \varphi) = \frac{d(\varrho, \varsigma)}{\varphi + d(\varrho, \varsigma)}$ and $Z(\varrho, \varsigma, \varphi) = \frac{d(\varrho, \varsigma)}{\varphi}$. Then $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ is a NMS.

Definition 2.5 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a NMS. Then

(i) a sequence $\{\varrho_n\}$ in Λ is referred to as convergent to ϱ if $\lim_{n \to \infty} \daleth(\varrho_n, \varrho, \varphi) = 1$, $\lim_{n \to \infty} \Gamma(\varrho_n, \varrho, \varphi) = 0$ and $\lim_{n \to \infty} Z(\varrho_n, \varrho, \varphi) = 0$ for all $\varphi > 0$ and is denoted by

$\lim\limits_{n\to\infty} \varrho_n = \varrho$

(ii) a sequence $\{\varrho_n\}$ in Λ is referred to as Cauchy sequence if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $\daleth(\varrho_n, \varrho_m, \varphi) > 1-\varepsilon, \Gamma(\varrho_n, \varrho_m, \varphi) < \varepsilon$ and $Z(\varrho_n, \varrho_m, \varphi) < \varepsilon$ for all $n, m \geq n_0$.

(iii) a NMS is referred to as complete if and only if every Cauchy sequence in Λ is convergent.

Definition 2.6 Let $\Psi : R^+ \to R^+$ and for each $\psi \in \Psi$, where $R^+ \in [0, \infty)$ fulfills the following requirements:

(ψ-1) ψ is non-decreasing,
(ψ-2) ψ is upper semi continuous from the right,
(ψ-3) $\sum\limits_{n=0}^{\infty} \psi^n(\varphi) < +\infty$ for all $\varphi > 0$ where $\psi^{n+1}(\varphi) = \psi(\psi^n(\varphi)), n \in \mathbb{N}$,
(ψ-4) ψ is non-increasing,
(ψ-5) ψ is upper semi continuous from the left,
(ψ-6) $\sum\limits_{n=0}^{\infty} \psi^n(\varphi) > +\infty$ for all $\varphi > 0$ where $\psi^{n+1}(\varphi) = \psi(\psi^n(\varphi))$, $n \in \mathbb{N}$.

Lemma 2.7 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a NMS where $*$ is a continuous t-norm, $\Diamond, \odot$ are continuous t-conorms of $\mathfrak{H}$-type respectively. If there exists $\psi \in \Psi$ such that

$$\daleth(\varrho, \varsigma, \psi(\varphi)) \geq \daleth(\varrho, \varsigma, \varphi), \tag{1}$$

$$\Gamma(\varrho, \varsigma, \psi(\varphi)) \leq \Gamma(\varrho, \varsigma, \varphi) \text{ and} \tag{2}$$

$$Z(\varrho, \varsigma, \psi(\varphi)) \leq Z(\varrho, \varsigma, \varphi) \tag{3}$$

for all $\varphi > 0$ then $\varrho = \varsigma$.

Definition 2.8 An element $(\varrho, \varsigma) \in \Lambda \times \Lambda$ is referred to as a coupled fixed point of the mappings $\Omega : \Lambda \times \Lambda \to \Lambda$ if $\Omega(\varrho, \varsigma) = \varrho, \Omega(\varsigma, \varrho) = \varsigma$

Definition 2.9 An element $(\varrho, \varsigma) \in \Lambda \times \Lambda$ is referred to as a coupled coincidence point of the mappings $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ if

$$\Omega(\varrho, \varsigma) = \sigma(\varrho), \Omega(\varsigma, \varrho) = \sigma(\varsigma). \tag{4}$$

Definition 2.10 An element $(\varrho, \varsigma) \in \Lambda \times \Lambda$ is referred to as a common coupled fixed point of the mappings $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ if

$$\varrho = \Omega(\varrho, \varsigma) = \sigma(\varrho), \varsigma = \Omega(\varsigma, \varrho) = \sigma(\varsigma) \tag{5}$$

Definition 2.11 An element $\varrho \in \Lambda$ is referred to as a common fixed point of the mappings

$$\Omega : \Lambda \times \Lambda \to \Lambda \,\text{and}\, \sigma : \Lambda \to \Lambda \,\text{if}\, \varrho = \sigma(\varrho) = \Omega(\varrho, \varrho). \tag{6}$$

Definition 2.12 The mappings $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ are said to be compatible if

$$\lim_{n\to\infty} \daleth(\sigma\Omega(\varrho_n, \varsigma_n), \Omega(\sigma(\varrho_n), \sigma(\varsigma_n)), \varphi) = 1, \tag{7}$$

$$\lim_{n\to\infty} \daleth(\sigma\Omega(\varsigma_n, \varrho_n), \Omega(\sigma(\varsigma_n), \sigma(\varrho_n)), \varphi) = 1, \tag{8}$$

$$\lim_{n\to\infty} \Gamma(\sigma\Omega(\varrho_n, \varsigma_n), \Omega(\sigma(\varrho_n), \sigma(\varsigma_n)), \varphi) = 0, \tag{9}$$

$$\lim_{n\to\infty} \Gamma(\sigma\Omega(\varsigma_n, \varrho_n), \Omega(\sigma(\varsigma_n), \sigma(\varrho_n)), \varphi) = 0, \tag{10}$$

$$\lim_{n\to\infty} Z(\sigma\Omega(\varrho_n, \varsigma_n), \Omega(\sigma(\varrho_n), \sigma(\varsigma_n)), \varphi) = 0 \,\text{and} \tag{11}$$

$$\lim_{n\to\infty} Z(\sigma\Omega(\varsigma_n, \varrho_n), \Omega(\sigma(\varsigma_n), \sigma(\varrho_n)), \varphi) = 0, \tag{12}$$

for all $\varphi > 0$ whenever $\{\varrho_n\}$ and $\{\varsigma_n\}$ are sequences in Λ, such that

$$\lim_{n\to\infty} \Omega(\varrho_n, \varsigma_n) = \lim_{n\to\infty} \sigma(\varrho_n) = \varrho, \lim_{n\to\infty} \Omega(\varsigma_n, \varrho_n) = \lim_{n\to\infty} \sigma(\varsigma_n) = \varsigma \tag{13}$$

for all $\varrho, \varsigma \in \Lambda$ are satisfied.

Definition 2.13 The mappings $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ are referred to as weakly compatible mappings if $\Omega(\varrho, \varsigma) = \sigma(\varrho), \Omega(\varsigma, \varrho) = \sigma(\varsigma)$ implies that $\sigma\Omega(\varrho, \varsigma) = \Omega(\sigma\varrho, \sigma\varsigma), \sigma\Omega(\varsigma, \varrho) = \Omega(\sigma\varsigma, \sigma\varrho)$ for every $\varrho, \varsigma \in \Lambda$.

Remark 2.14 If Ω and σ are compatible then they are weakly compatible, but the converse need not be true.

3 Main Results

Denote $[\daleth(\varrho,\varsigma,\varphi)]^n = \underbrace{\daleth(\varrho,\varsigma,\varphi) * \daleth(\varrho,\varsigma,\varphi) * \cdots * \daleth(\varrho,\varsigma,\varphi)}_{n}$ for all $n \in \mathbb{N}$.

Theorem 3.1 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a complete NMS, where $*$ is a continuous t-norm, $\Diamond$ and $\odot$ are continuous t-conorm of $\mathfrak{H}$ - type satisfying ((vii), (xii) and (xvii) of definition (2.3)). Let $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ be two mappings and there exists $\psi \in \Psi$ in such a way that

$$\daleth(\Omega(\varrho,\varsigma), \Omega(\rho,\tau), \psi(\varphi)) \geq \daleth(\sigma(\varrho), \sigma(\rho), \varphi) * \daleth(\sigma(\varsigma), \sigma(\tau), \varphi), \tag{14}$$

$$\Gamma(\Omega(\varrho,\varsigma), \Omega(\rho,\tau), \psi(\varphi)) \leq \Gamma(\sigma(\varrho), \sigma(\rho), \varphi) \Diamond \Gamma(\sigma(\varsigma), \sigma(\tau), \varphi), \tag{15}$$

$$Z(\Omega(\varrho,\varsigma), \Omega(\rho,\tau), \psi(\varphi)) \leq Z(\sigma(\varrho), \sigma(\rho), \varphi) \odot Z(\sigma(\varsigma), \sigma(\tau), \varphi) \tag{16}$$

for all $\varrho, \varsigma, \rho, \tau \in \Lambda, \varphi > 0$. Suppose that $\Omega(\Lambda \times \Lambda) \subseteq \sigma(\Lambda), \sigma$ is continuous, Ω and σ are compatible. Then there exists $\varrho, \varsigma \in \Lambda$ such that $\varrho = \sigma(\varrho) = \Omega(\varrho, \varrho)$; that is, Ω and σ have a unique common fixed point in Λ.

Theorem 3.2 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a NMS, where $*$ is a continuous t-norm, $\Diamond, \odot$ are continuous t-conorms of $\mathfrak{H}$ - type satisfying ((vii), (xii) and (xvii) of definition (2.3)). Let $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ be two weakly compatible mappings and there exists $\psi \in \Psi$ satisfying (14), (15) and (16). Suppose that $\Omega(\Lambda \times \Lambda) \subseteq \sigma(\Lambda)$ and $\Omega(\Lambda \times \Lambda)$ or $\sigma(\Lambda)$ is complete. Then Ω and σ have a unique common fixed point in Λ.

Proof: Let $\varrho_0, \varsigma_0 \in \Lambda$ be two arbitrary points in Λ. Since $\Omega(\Lambda \times \Lambda) \subseteq \sigma(\Lambda)$, we can choose $\varrho_1, \varsigma_1 \in \Lambda$ such that $\sigma(\varrho_1) = \Omega(\varrho_0, \varsigma_0)$ and $\sigma(\varsigma_1) = \Omega(\varsigma_0, \varrho_0)$. Continuing this method, we can two sequences $\{\varrho_n\}$ and $\{\varsigma_n\}$ in Λ such that

$$\sigma(\varrho_{n+1}) = \Omega(\varrho_n, \varsigma_n), \sigma(\varsigma_{n+1}) = \Omega(\varsigma_n, \varrho_n) \text{ for all} n \geq 0. \tag{17}$$

We will the prove the above result in 4 steps.

Step 1: First we shall demonstrate that $\{\sigma\varrho_n\}$ and $\{\sigma\varsigma_n\}$ are Cauchy sequences. Since $*$ is a t-norm, $\Diamond$ and $\odot$ are t-conorms of $\mathfrak{H}$ - type, for any $\beta > 0$, there exists $\nu > 0$ such that, $\underbrace{(1-\nu) * (1-\nu) * \cdots * (1-\nu)}_{q} \geq 1 - \beta$, $\underbrace{\nu \Diamond \nu \Diamond \ldots \Diamond \nu}_{q} \leq \beta$,

$\underbrace{\nu \odot \nu \odot \cdots \odot \nu}_{q} \leq \beta$, for all $q \in \mathbb{N}$. Since $\daleth(\varrho, \varsigma, \cdot), \Gamma(\varrho, \varsigma, \cdot)$ and $Z(\varrho, \varsigma, \cdot)$ is continuous and $\lim_{\varphi \to \infty} \daleth(\varrho, \varsigma, \varphi) = 1$, $\lim_{\varphi \to \infty} \Gamma(\varrho, \varsigma, \varphi) = 0$ and $\lim_{\varphi \to \infty} Z(\varrho, \varsigma, \varphi) = 0$ for all $\varrho, \varsigma \in \Lambda$, there exists $\varphi_0 > 0$ such that

$$\daleth(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) \geq 1 - \nu, \daleth(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) \geq 1 - \nu, \tag{18}$$

$$\Gamma(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) \leq \nu, \Gamma(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) \leq \nu, \tag{19}$$

$$Z(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) \leq \nu, Z(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) \leq \nu, \tag{20}$$

On the other hand, since $\psi \in \Psi$, by using (ψ-3), we have $\sum_{n=1}^{\infty} \psi^n(\varphi_0) < \infty$. Then for any $\varphi > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi > \sum_{q=n_0}^{\infty} \psi^q(\varphi_0) \tag{21}$$

By using condition (14), (15) and (16) we have

$$\begin{aligned}
&\daleth(\sigma\varrho_1, \sigma\varrho_2, \psi(\varphi_0)) = \daleth(\Omega(\varrho_0, \varsigma_0), \Omega(\varrho_1, \varsigma_1), \psi(\varphi_0)) \geq \daleth(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) * \daleth(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)\\
&\daleth(\sigma\varsigma_1, \sigma\varsigma_2, \psi(\varphi_0)) = \daleth(\Omega(\varsigma_0, \varrho_0), \Omega(\varsigma_1, \varrho_1), \psi(\varphi_0)) \geq \daleth(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) * \daleth(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)\\
&\Gamma(\sigma\varrho_1, \sigma\varrho_2, \psi(\varphi_0)) = \Gamma(\Omega(\varrho_0, \varsigma_0), \Omega(\varrho_1, \varsigma_1), \psi(\varphi_0)) \leq \Gamma(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) \Diamond \Gamma(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)\\
&\Gamma(\sigma\varsigma_1, \sigma\varsigma_2, \psi(\varphi_0)) = \Gamma(\Omega(\varsigma_0, \varrho_0), \Omega(\varsigma_1, \varrho_1), \psi(\varphi_0)) \leq \Gamma(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) \Diamond \Gamma(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)\\
&Z(\sigma\varrho_1, \sigma\varrho_2, \psi(\varphi_0)) = Z(\Omega(\varrho_0, \varsigma_0), \Omega(\varrho_1, \varsigma_1), \psi(\varphi_0)) \leq Z(\sigma\varrho_0, \sigma\varrho_1, \varphi_0) \odot Z(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)\\
&Z(\sigma\varsigma_1, \sigma\varsigma_2, \psi(\varphi_0)) = Z(\Omega(\varsigma_0, \varrho_0), \Omega(\varsigma_1, \varrho_1), \psi(\varphi_0)) \leq Z(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0) \odot Z(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\daleth(\sigma\varrho_2, \sigma\varrho_3, \psi^2(\varphi_0)) &= \daleth(\Omega(\varrho_1, \varsigma_1), \Omega(\varrho_2, \varsigma_2), \psi^2(\varphi_0))\\
&\geq \daleth(\sigma\varrho_1, \sigma\varrho_2, \psi(\varphi_0)) * \daleth(\sigma\varsigma_1, \sigma\varsigma_2, \psi(\varphi_0))\\
&\geq [\daleth(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)]^2 * [\daleth(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)]^2\\
\daleth(\sigma\varsigma_2, \sigma\varsigma_3, \psi^2(\varphi_0)) &= \daleth(\Omega(\varsigma_1, \varrho_1), \Omega(\varsigma_2, \varrho_2), \psi^2(\varphi_0))\\
&\geq [\daleth(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)]^2 * [\daleth(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)]^2\\
\Gamma(\sigma\varrho_2, \sigma\varrho_3, \psi^2(\varphi_0)) &= \Gamma(\Omega(\varrho_1, \varsigma_1), \Omega(\varrho_2, \varsigma_2), \psi^2(\varphi_0))\\
&\leq \Gamma(\sigma\varrho_1, \sigma\varrho_2, \psi(\varphi_0)) \Diamond \Gamma(\sigma\varsigma_1, \sigma\varsigma_2, \psi(\varphi_0))\\
&\leq [\Gamma(\sigma\varrho_0, \sigma\varrho_1, \varphi_0)]^2 \Diamond [\Gamma(\sigma\varsigma_0, \sigma\varsigma_1, \varphi_0)]^2
\end{aligned}$$

$$\begin{aligned}\Gamma(\sigma\varsigma_2,\sigma\varsigma_3,\psi^2(\varphi_0)) &= \Gamma(\Omega(\varsigma_1,\varrho_1),\Omega(\varsigma_2,\varrho_2),\psi^2(\varphi_0))\\ &\leq [\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^2\Diamond[\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^2\\ Z(\sigma\varrho_2,\sigma\varrho_3,\psi^2(\varphi_0)) &= Z(\Omega(\varrho_1,\varsigma_1),\Omega(\varrho_2,\varsigma_2),\psi^2(\varphi_0))\\ &\leq Z(\sigma\varrho_1,\sigma\varrho_2,\psi(\varphi_0))\odot Z(\sigma\varsigma_1,\sigma\varsigma_2,\psi(\varphi_0))\\ &\leq [Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^2\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^2\\ Z(\sigma\varsigma_2,\sigma\varsigma_3,\psi^2(\varphi_0)) &= Z(\Omega(\varsigma_1,\varrho_1),\Omega(\varsigma_2,\varrho_2),\psi^2(\varphi_0))\\ &\leq [Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^2\odot[Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^2\end{aligned}$$

By induction we get,

$$\begin{aligned}&\daleth(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))\geq[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}*[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}},\\ &\daleth(\sigma\varsigma_n,\sigma\varsigma_{n+1},\psi^n(\varphi_0))\geq[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}*[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\\ &\Gamma(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))\leq[\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\Diamond[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}},\\ &\Gamma(\sigma\varsigma_n,\sigma\varsigma_{n+1},\psi^n(\varphi_0))\leq[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}\Diamond[\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\\ &Z(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))\leq[Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}},\\ &Z(\sigma\varsigma_n,\sigma\varsigma_{n+1},\psi^n(\varphi_0))\leq[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}\odot[Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\end{aligned}$$

From (18), (19), (20) and (21) for $p>n\geq n_0$, we get

$$\begin{aligned}\daleth(\sigma\varrho_n,\sigma\varrho_p,\varphi) &\geq \daleth(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n_0}^{\infty}\psi^q(\varphi_0))\\ &\geq \daleth(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n}^{p-1}\psi^q(\varphi_0))\\ &\geq \daleth(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))*\daleth(\sigma\varrho_{n+1},\sigma\varrho_{n+2},\psi^{n+1}(\varphi_0))*\cdots*\\ &\daleth(\sigma\varrho_{p-1},\sigma\varrho_p,\psi^{p-1}(\varphi_0))\\ &\geq[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}*[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}*[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^n}\\ &*[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^n}*\cdots*[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-2}}*[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-2}}\\ &=[\daleth(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-1}-2^{n-1}}*[\daleth(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-1}-2^{n-1}}\\ &\geq\underbrace{(1-\nu)*(1-\nu)*\cdots*(1-\nu)}_{2^p-2^n}\geq1-\beta\end{aligned}$$

which implies

$$\daleth(\sigma\varrho_n,\sigma\varrho_p,\varphi)>1-\beta \tag{22}$$

$$\begin{aligned}
\Gamma(\sigma\varrho_n,\sigma\varrho_p,\varphi) &\leq \Gamma(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n_0}^{\infty}\psi^q(\varphi_0)) \\
&\leq \Gamma(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n}^{p-1}\psi^q(\varphi_0)) \\
&\leq \Gamma(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))\Diamond\Gamma(\sigma\varrho_{n+1},\sigma\varrho_{n+2},\psi^{n+1}(\varphi_0))\Diamond\ldots\Diamond \\
&\Gamma(\sigma\varrho_{p-1},\sigma\varrho_p,\psi^{p-1}(\varphi_0)) \\
&\leq [\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\Diamond[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}\Diamond[\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^n} \\
&\Diamond[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^n}\Diamond\ldots\Diamond[\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-2}}\Diamond[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-2}} \\
&= [\Gamma(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-1}-2^{n-1}}\Diamond[\Gamma(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-1}-2^{n-1}} \\
&\leq \underbrace{\nu\Diamond\nu\Diamond\ldots\Diamond\nu}_{2^p-2^n} \leq \beta
\end{aligned}$$

which implies

$$\Gamma(\sigma\varrho_n,\sigma\varrho_p,\varphi) < \beta \tag{23}$$

$$\begin{aligned}
Z(\sigma\varrho_n,\sigma\varrho_p,\varphi) &\leq Z(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n_0}^{\infty}\psi^q(\varphi_0)) \\
&\leq Z(\sigma\varrho_n,\sigma\varrho_p,\sum_{q=n}^{p-1}\psi^q(\varphi_0)) \\
&\leq Z(\sigma\varrho_n,\sigma\varrho_{n+1},\psi^n(\varphi_0))\odot Z(\sigma\varrho_{n+1},\sigma\varrho_{n+2},\psi^{n+1}(\varphi_0))\odot\cdots\odot \\
&Z(\sigma\varrho_{p-1},\sigma\varrho_p,\psi^{p-1}(\varphi_0)) \\
&\leq [Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{n-1}}\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{n-1}}\odot[Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^n} \\
&\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^n}\odot\cdots\odot[Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-2}}\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-2}} \\
&= [Z(\sigma\varrho_0,\sigma\varrho_1,\varphi_0)]^{2^{p-1}-2^{n-1}}\odot[Z(\sigma\varsigma_0,\sigma\varsigma_1,\varphi_0)]^{2^{p-1}-2^{n-1}} \\
&\leq \underbrace{\nu\odot\nu\odot\cdots\odot\nu}_{2^p-2^n} \leq \beta
\end{aligned}$$

which implies

$$Z(\sigma\varrho_n,\sigma\varrho_p,\varphi) < \beta \tag{24}$$

for all $p, n \in \mathbb{N}$ with $p > n \geq n_0$ and $\varphi > 0$.
Hence, $\{\sigma\varrho_n\}$ is a Cauchy sequence. Similarly, we can prove that $\{\sigma\varsigma_n\}$ is a Cauchy sequence.

Step 2: Here, we will prove that Ω and σ have a coupled coincidence point.
We can assume that $\sigma(\Lambda)$ is complete and hence there exists $\varrho, \varsigma \in \sigma(\Lambda)$ and $\omega, \varsigma \in \Lambda$ such that

$$\lim_{n\to\infty} \sigma(\varrho_n) = \lim_{n\to\infty} \Omega(\varrho_n, \varsigma_n) = \sigma(\omega) = \varrho, \tag{25}$$

$$\lim_{n\to\infty} \sigma(\varsigma_n) = \lim_{n\to\infty} \Omega(\varsigma_n, \varrho_n) = \sigma(\varsigma) = \varsigma \tag{26}$$

By using (14), (15) and (16) we get,

$$\begin{aligned}
&\daleth(\Omega(\varrho_n, \varsigma_n), \Omega(\omega, \varsigma), \psi(\varphi)) \geq \daleth(\sigma\varrho_n, \sigma(\omega), \varphi) * \daleth(\sigma\varsigma_n, \sigma(\varsigma), \varphi),\\
&\Gamma(\Omega(\varrho_n, \varsigma_n), \Omega(\omega, \varsigma), \psi(\varphi)) \leq \Gamma(\sigma\varrho_n, \sigma(\omega), \varphi)\Diamond\Gamma(\sigma\varsigma_n, \sigma(\varsigma), \varphi),\\
&Z(\Omega(\varrho_n, \varsigma_n), \Omega(\omega, \varsigma), \psi(\varphi)) \leq Z(\sigma\varrho_n, \sigma(\omega), \varphi) \odot Z(\sigma\varsigma_n, \sigma(\varsigma), \varphi).
\end{aligned}$$

Since $\daleth, \Gamma$ and Z is continuous, taking limit as $n \to \infty$, we get
$\daleth(\sigma(\omega), \Omega(\omega, \varsigma), \psi(\varphi)) = 1$, $\Gamma(\sigma(\omega), \Omega(\omega, \varsigma), \psi(\varphi)) = 0$, $Z(\sigma(\omega), \Omega(\omega, \varsigma), \psi(\varphi)) = 0$, which shows that $\Omega(\omega, \varsigma) = \sigma(\omega) = \varrho$.
Similarly, we can prove that $\Omega(\varsigma, \omega) = \sigma(\varsigma) = \varsigma$. Since Ω and σ are weakly compatible, we get that $\sigma\Omega(\omega, \varsigma) = \Omega(\sigma(\omega), \sigma(\varsigma))$ and $\sigma\Omega(\varsigma, \omega) = \Omega(\sigma(\varsigma), \sigma(\omega))$, which implies that $\sigma(\varrho) = \Omega(\varrho, \varsigma)$ and $\sigma(\varsigma) = \Omega(\varsigma, \varrho)$.
Step 3: Next, we prove that $\sigma(\varrho) = \varsigma$ and $\sigma(\varsigma) = \varrho$.
Since $*$ is a t-norm $\Diamond$ and $\odot$ are t-conorms of $\mathfrak{H}$ - type, for any $\beta > 0$, there exists a $\nu > 0$ such that $\underbrace{(1-\nu) * (1-\nu) * \cdots * (1-\nu)}_{q} \geq 1 - \beta$, $\underbrace{\nu\Diamond\nu\Diamond\ldots\Diamond\nu}_{q} \leq \beta$, $\underbrace{\nu \odot \nu \odot \cdots \odot \nu}_{q} \leq \beta$, for all $q \in \mathbb{N}$.

Since $\daleth(\varrho, \varsigma, \cdot), \Gamma(\varrho, \varsigma, \cdot)$ and $Z(\varrho, \varsigma, \cdot)$ is continuous and
$\lim_{\varphi\to\infty} \daleth(\varrho, \varsigma, \varphi) = 1$, $\lim_{\varphi\to\infty} \Gamma(\varrho, \varsigma, \varphi) = 0$ and $\lim_{\varphi\to\infty} Z(\varrho, \varsigma, \varphi) = 0$
for all $\varrho, \varsigma \in \Lambda$, there exists $\varphi_0 > 0$ such that
$\daleth(\sigma\varrho, \varsigma, \varphi_0) \geq 1 - \nu$, $\daleth(\sigma\varsigma, \varrho, \varphi_0) \geq 1 - \nu$,
$\Gamma(\sigma\varrho, \varsigma, \varphi_0) \leq \nu, \Gamma(\sigma\varsigma, \varrho, \varphi_0) \leq \nu$ and
$Z(\sigma\varrho, \varsigma, \varphi_0) \leq \nu, Z(\sigma\varsigma, \varrho, \varphi_0) \leq \nu$.
Also, since $\psi \in \Psi$, by using (ψ-3), we have $\sum_{n=1}^{\infty} \psi^n(\varphi_0) < \infty$.
Then for any $\varphi > 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi > \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)$.
Since

$$\daleth(\sigma\varrho, \sigma\varsigma_{n+1}, \psi(\varphi_0)) = \daleth(\Omega(\varrho, \varsigma), \Omega(\varsigma_n, \varrho_n), \psi(\varphi_0))$$

$$\begin{aligned}
&\geq \daleth(\sigma\varrho,\sigma\varsigma_n,\varphi_0) * \daleth(\sigma\varsigma,\sigma\varrho_n,\varphi_0),\\
\Gamma(\sigma\varrho,\sigma\varsigma_{n+1},\psi(\varphi_0)) &= \Gamma(\Omega(\varrho,\varsigma),\Omega(\varsigma_n,\varrho_n),\psi(\varphi_0))\\
&\leq \Gamma(\sigma\varrho,\sigma\varsigma_n,\varphi_0)\Diamond\Gamma(\sigma\varsigma,\sigma\varrho_n,\varphi_0)\text{and}\\
Z(\sigma\varrho,\sigma\varsigma_{n+1},\psi(\varphi_0)) &= Z(\Omega(\varrho,\varsigma),\Omega(\varsigma_n,\varrho_n),\psi(\varphi_0))\\
&\leq Z(\sigma\varrho,\sigma\varsigma_n,\varphi_0)\odot Z(\sigma\varsigma,\sigma\varrho_n,\varphi_0),
\end{aligned}$$

Letting $n \to \infty$, we get

$$\daleth(\sigma\varrho,\varsigma,\psi(\varphi_0)) \geq \daleth(\sigma\varrho,\varsigma,\varphi_0) * \daleth(\sigma\varsigma,\varrho,\varphi_0) \tag{27}$$
$$\Gamma(\sigma\varrho,\varsigma,\psi(\varphi_0)) \leq \Gamma(\sigma\varrho,\varsigma,\varphi_0)\Diamond\Gamma(\sigma\varsigma,\varrho,\varphi_0) \tag{28}$$
$$Z(\sigma\varrho,\varsigma,\psi(\varphi_0)) \leq Z(\sigma\varrho,\varsigma,\varphi_0)\odot Z(\sigma\varsigma,\varrho,\varphi_0) \tag{29}$$

Similarly, we can get

$$\daleth(\sigma\varsigma,\varrho,\psi(\varphi_0)) \geq \daleth(\sigma\varrho,\varsigma,\varphi_0) * \daleth(\sigma\varsigma,\varrho,\varphi_0) \tag{30}$$
$$\Gamma(\sigma\varsigma,\varrho,\psi(\varphi_0)) \leq \Gamma(\sigma\varrho,\varsigma,\varphi_0)\Diamond\Gamma(\sigma\varsigma,\varrho,\varphi_0) \tag{31}$$
$$Z(\sigma\varsigma,\varrho,\psi(\varphi_0)) \leq Z(\sigma\varrho,\varsigma,\varphi_0)\odot Z(\sigma\varsigma,\varrho,\varphi_0) \tag{32}$$

From (27),(28), (29), (30), (31) and (32) we have

$$\begin{aligned}
&\daleth(\sigma\varrho,\varsigma,\psi(\varphi_0)) * \daleth(\sigma\varsigma,\varrho,\psi(\varphi_0)) \geq [\daleth(\sigma\varrho,\varsigma,\varphi_0)]^2 * [\daleth(\sigma\varsigma,\varrho,\varphi_0)]^2\\
&\Gamma(\sigma\varrho,\varsigma,\psi(\varphi_0))\Diamond\Gamma(\sigma\varsigma,\varrho,\psi(\varphi_0)) \leq [\Gamma(\sigma\varrho,\varsigma,\varphi_0)]^2\Diamond[\Gamma(\sigma\varsigma,\varrho,\varphi_0)]^2\\
&Z(\sigma\varrho,\varsigma,\psi(\varphi_0))\odot Z(\sigma\varsigma,\varrho,\psi(\varphi_0)) \leq [Z(\sigma\varrho,\varsigma,\varphi_0)]^2\odot[Z(\sigma\varsigma,\varrho,\varphi_0)]^2
\end{aligned}$$

Proceeding like this n times we get,

$$\begin{aligned}
\daleth(\sigma\varrho,\varsigma,\psi^n(\varphi_0)) * \daleth(\sigma\varsigma,\varrho,\psi^n(\varphi_0)) &\geq [\daleth(\sigma\varrho,\varsigma,\psi^{n-1}(\varphi_0))]^2 * [\daleth(\sigma\varsigma,\varrho,\psi^{n-1}(\varphi_0))]^2\\
&\geq [\daleth(\sigma\varrho,\varsigma,\varphi_0)]^{2^n} * [\daleth(\sigma\varsigma,\varrho,\varphi_0)]^{2^n},\\
\Gamma(\sigma\varrho,\varsigma,\psi^n(\varphi_0))\Diamond\Gamma(\sigma\varsigma,\varrho,\psi^n(\varphi_0)) &\leq [\Gamma(\sigma\varrho,\varsigma,\psi^{n-1}(\varphi_0))]^2\Diamond[\Gamma(\sigma\varsigma,\varrho,\psi^{n-1}(\varphi_0))]^2\\
&\leq [\Gamma(\sigma\varrho,\varsigma,\varphi_0)]^{2^n}\Diamond[\Gamma(\sigma\varsigma,\varrho,\varphi_0)]^{2^n},\\
Z(\sigma\varrho,\varsigma,\psi^n(\varphi_0))\odot Z(\sigma\varsigma,\varrho,\psi^n(\varphi_0)) &\leq [Z(\sigma\varrho,\varsigma,\psi^{n-1}(\varphi_0))]^2\odot[Z(\sigma\varsigma,\varrho,\psi^{n-1}(\varphi_0))]^2\\
&\leq [Z(\sigma\varrho,\varsigma,\varphi_0)]^{2^n}\odot[Z(\sigma\varsigma,\varrho,\varphi_0)]^{2^n} \text{ for all } n\in\mathbb{N}.
\end{aligned}$$

Since $\varphi > \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)$ we get

$$\begin{aligned}
\daleth(\sigma\varrho, \varsigma, \varphi) * \daleth(\sigma\varsigma, \varrho, \varphi) &\geq \daleth(\sigma\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) * \daleth(\sigma\varsigma, \varrho, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) \\
&\geq \daleth(\sigma\varrho, \varsigma, \psi^{n_0}(\varphi_0)) * \daleth(\sigma\varsigma, \varrho, \psi^{n_0}(\varphi_0)) \\
&\geq [\daleth(\sigma\varrho, \varsigma, \varphi_0)]^{2^{n_0}} * [\daleth(\sigma\varsigma, \varrho, \varphi_0)]^{2^{n_0}} \\
&\geq \underbrace{(1-\nu) * (1-\nu) * \cdots * (1-\nu)}_{2^{2n_0}} \geq 1 - \beta,
\end{aligned}$$

$$\begin{aligned}
\Gamma(\sigma\varrho, \varsigma, \varphi) \Diamond \Gamma(\sigma\varsigma, \varrho, \varphi) &\leq \Gamma(\sigma\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) \Diamond \Gamma(\sigma\varsigma, \varrho, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) \\
&\leq \Gamma(\sigma\varrho, \varsigma, \psi^{n_0}(\varphi_0)) \Diamond \Gamma(\sigma\varsigma, \varrho, \psi^{n_0}(\varphi_0)) \\
&\leq [\Gamma(\sigma\varrho, \varsigma, \varphi_0)]^{2^{n_0}} \Diamond [\Gamma(\sigma\varsigma, \varrho, \varphi_0)]^{2^{n_0}} \\
&\leq \underbrace{\nu \Diamond \nu \Diamond \ldots \Diamond \nu}_{2^{2n_0}} \leq \beta \text{ and}
\end{aligned}$$

$$\begin{aligned}
Z(\sigma\varrho, \varsigma, \varphi) \odot Z(\sigma\varsigma, \varrho, \varphi) &\leq Z(\sigma\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) \odot Z(\sigma\varsigma, \varrho, \sum_{q=n_0}^{\infty} \psi^q(\varphi_0)) \\
&\leq Z(\sigma\varrho, \varsigma, \psi^{n_0}(\varphi_0)) \odot Z(\sigma\varsigma, \varrho, \psi^{n_0}(\varphi_0)) \\
&\leq [Z(\sigma\varrho, \varsigma, \varphi_0)]^{2^{n_0}} \odot [Z(\sigma\varsigma, \varrho, \varphi_0)]^{2^{n_0}} \\
&\leq \underbrace{\nu \odot \nu \odot \cdots \odot \nu}_{2^{2n_0}} \leq \beta.
\end{aligned}$$

Hence, for any $\beta > 0$, we get
$\daleth(\sigma\varrho, \varsigma, \varphi) * \daleth(\sigma\varsigma, \varrho, \varphi) \geq 1 - \beta \quad \Gamma(\sigma\varrho, \varsigma, \varphi) \Diamond \Gamma(\sigma\varsigma, \varrho, \varphi) \leq \beta$ and
$Z(\sigma\varrho, \varsigma, \varphi) \odot Z(\sigma\varsigma, \varrho, \varphi) \leq \beta$ for all $\varphi > 0$.
Hence we conclude that $\sigma\varrho = \varsigma$ and $\sigma\varsigma = \varrho$.
Step: 4 Next, we prove that $\varrho = \varsigma$
Since $*$ is a t-norm and $\Diamond$ is a t-conorm of $\mathfrak{H}$ - type, for any $\beta > 0$, there exists a $\nu > 0$ such that
$\underbrace{(1-\nu) * (1-\nu) * \cdots * (1-\nu)}_{q} \geq 1-\beta$, $\underbrace{\nu \Diamond \nu \Diamond \ldots \Diamond \nu}_{q} \leq \beta$ and $\underbrace{\nu \odot \nu \odot \cdots \odot \nu}_{q} \leq \beta$
for all $q \in \mathbb{N}$.
Since $\daleth(\varrho, \varsigma, \cdot), \Gamma(\varrho, \varsigma, \cdot)$ and $Z(\varrho, \varsigma, \cdot)$ is continuous and $\lim_{\varphi \to \infty} \daleth(\varrho, \varsigma, \varphi) = 1$, $\lim_{\varphi \to \infty} \Gamma(\varrho, \varsigma, \varphi) = 0$ and $\lim_{\varphi \to \infty} Z(\varrho, \varsigma, \varphi) = 0$ for all $\varrho, \varsigma \in \Lambda$, there exists $\varphi_0 > 0$ such that $\daleth(\varrho, \varsigma, \varphi_0) \geq 1 - \nu$, $\Gamma(\varrho, \varsigma, \varphi_0) \leq \nu$ and $Z(\varrho, \varsigma, \varphi_0) \leq \nu$.

But $\psi \in \Psi$, by using (ψ-3), we have $\sum_{n=1}^{\infty} \psi^{n}(\varphi_0) < \infty$. Then for any $\varphi > 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi > \sum_{q=n_0}^{\infty} \psi^{q}(\varphi_0)$.
By using (14), (15) and (16) we get,

$$\begin{aligned}
\daleth(\sigma\varrho_{n+1}, \sigma\varsigma_{n+1}, \psi(\varphi_0) &= \daleth(\Omega(\varrho_n, \varsigma_n), \Omega(\varsigma_n, \varrho_n), \psi(\varphi_0))) \\
&\geq \daleth(\sigma\varrho_n, \sigma\varsigma_n, \varphi_0) * \daleth(\sigma\varsigma_n, \sigma\varrho_n, \varphi_0), \\
\Gamma(\sigma\varrho_{n+1}, \sigma\varsigma_{n+1}, \psi(\varphi_0) &= \Gamma(\Omega(\varrho_n, \varsigma_n), \Omega(\varsigma_n, \varrho_n), \psi(\varphi_0))) \\
&\leq \Gamma(\sigma\varrho_n, \sigma\varsigma_n, \varphi_0) \Diamond \Gamma(\sigma\varsigma_n, \sigma\varrho_n, \varphi_0) \text{and} \\
Z(\sigma\varrho_{n+1}, \sigma\varsigma_{n+1}, \psi(\varphi_0) &= Z(\Omega(\varrho_n, \varsigma_n), \Omega(\varsigma_n, \varrho_n), \psi(\varphi_0))) \\
&\leq Z(\sigma\varrho_n, \sigma\varsigma_n, \varphi_0) \odot Z(\sigma\varsigma_n, \sigma\varrho_n, \varphi_0)
\end{aligned}$$

As $n \to \infty$ we get

$$\begin{aligned}
&\daleth(\varrho, \varsigma, \psi(\varphi_0)) \geq \daleth(\varrho, \varsigma, \varphi_0) * \daleth(\varsigma, \varrho, \varphi_0), \\
&\Gamma(\varrho, \varsigma, \psi(\varphi_0)) \leq \Gamma(\varrho, \varsigma, \varphi_0) \Diamond \Gamma(\varsigma, \varrho, \varphi_0) \text{and} \\
&Z(\varrho, \varsigma, \psi(\varphi_0)) \leq Z(\varrho, \varsigma, \varphi_0) \odot Z(\varsigma, \varrho, \varphi_0).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\daleth(\varrho, \varsigma, \varphi) &\geq \daleth(\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^{q}(\varphi_0)) \\
&\geq \daleth(\varrho, \varsigma, \psi^{n_0}(\varphi_0)) \\
&\geq [\daleth(\varrho, \varsigma, \varphi_0)]^{2^{n_0-1}} * [\daleth(\varsigma, \varrho, \varphi_0)]^{2^{n_0-1}} \\
&\geq \underbrace{(1-\nu) * (1-\nu) * \cdots * (1-\nu)}_{2^{2n_0-2}} \geq 1-\beta \\
\Gamma(\varrho, \varsigma, \varphi) &\leq \Gamma(\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^{q}(\varphi_0)) \\
&\leq \Gamma(\varrho, \varsigma, \psi^{n_0}(\varphi_0)) \\
&\leq [\Gamma(\varrho, \varsigma, \varphi_0)]^{2^{n_0-1}} \Diamond [\Gamma(\varsigma, \varrho, \varphi_0)]^{2^{n_0-1}} \\
&\leq \underbrace{\nu \Diamond \nu \Diamond \ldots \Diamond \nu}_{2^{2n_0-2}} \leq \beta \\
Z(\varrho, \varsigma, \varphi) &\leq Z(\varrho, \varsigma, \sum_{q=n_0}^{\infty} \psi^{q}(\varphi_0))
\end{aligned}$$

$$\begin{aligned}
&\leq Z(\varrho, \varsigma, \psi^{n_0}(\varphi_0))\\
&\leq [Z(\varrho, \varsigma, \varphi_0)]^{2^{n_0-1}} \odot [Z(\varsigma, \varrho, \varphi_0)]^{2^{n_0-1}}\\
&\leq \underbrace{\nu \odot \nu \odot \cdots \odot \nu}_{2^{2n_0-2}} \leq \beta
\end{aligned}$$

which shows that $\varrho = \varsigma$.
We have therefore demonstrated that there is a shared fixed point in Λ for Ω and σ. Using the same method as previously, the fixed point's uniqueness can be demonstrated. This completes the proof.

Corollary 3.3 Let $(\Lambda, \daleth, \Gamma, Z, *, \Diamond, \odot)$ be a NMS, where $*$ is a continuous t-norm, $\Diamond$ and $\odot$ are continuous t-conorms of $\mathfrak{H}$ - type satisfying (vii, xii and xvii of definition (2.3)).
Let $\Omega : \Lambda \times \Lambda \to \Lambda$ and there exists $\psi \in \Psi$ such that

$$\daleth(\Omega(\varrho, \varsigma), \Omega(\rho, \tau), \psi(\varphi)) \geq \daleth(\varrho, \rho, \varphi) * \daleth(\varsigma, \tau, \varphi), \tag{33}$$
$$\Gamma(\Omega(\varrho, \varsigma), \Omega(\rho, \tau), \psi(\varphi)) \leq \Gamma(\varrho, \rho, \varphi) \Diamond \Gamma(\varsigma, \tau, \varphi), \tag{34}$$
$$Z(\Omega(\varrho, \varsigma), \Omega(\rho, \tau), \psi(\varphi)) \leq Z(\varrho, \rho, \varphi) \odot Z(\varsigma, \tau, \varphi) \tag{35}$$

for all $\varrho, \varsigma, \rho, \tau \in \Lambda, \varphi > 0$. Then $\Omega(\Lambda)$ is complete. Then there exists $\varrho \in \Lambda$ such that $\varrho = \Omega(\varrho, \varrho)$, ie. Ω admits a unique fixed point in Λ.

The following example supports Theorem (3.2)

Example 3.4 Let $\Lambda = \{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\}, * = \min, \Diamond = \max, \odot = \max, \daleth(\varrho, \varsigma, \varphi) = \frac{\varphi}{|\varrho-\varsigma|+\varphi}, \Gamma(\varrho, \varsigma, \varphi) = \frac{|\varrho-\varsigma|}{|\varrho-\varsigma|+\varphi}, Z(\varrho, \varsigma, \varphi) = \frac{|\varrho-\varsigma|}{\varphi}$ for all $\varrho, \varsigma \in \Lambda, \varphi > 0$.
Then $\daleth(\varrho, \varsigma, \varphi), \Gamma(\varrho, \varsigma, \varphi)$ and $Z(\varrho, \varsigma, \varphi)$ is a NMS.
Let $\psi(\varphi) = \frac{\varphi}{2}$. Let $\Omega : \Lambda \times \Lambda \to \Lambda$ and $\sigma : \Lambda \to \Lambda$ be defined as

$$\sigma(\varrho) = \begin{cases} 0, & \varrho = 0 \\ 1, & \varrho = \frac{1}{2n+1} \\ \frac{1}{2n+1}, & \varrho = \frac{1}{2n} \end{cases}, \quad \Omega(\varrho, \varsigma) = \begin{cases} \frac{1}{(2n+1)^4}, & (\varrho, \varsigma) = (1/2n, 1/2n) \\ 0, & \text{others} \end{cases}$$

Let $\varrho_n, \varsigma_n = 1/2n$. We have $\sigma\varrho_n = \frac{1}{2n+1} \to 0, \Omega(\varrho_n, \varsigma_n) = \frac{1}{(2n+1)^4} \to 0$, but

$$\daleth(\Omega(\sigma\varrho_n, \sigma\varsigma_n), \sigma\Omega(\varrho_n, \varsigma_n), \varphi) = \daleth(0, 1, \varphi) \nrightarrow 0,$$

$$\Gamma(\Omega(\sigma\varrho_n,\sigma\varsigma_n),\sigma\Omega(\varrho_n,\varsigma_n),\varphi)=\Gamma(1,0,\varphi)\nrightarrow 1 \text{ and}$$
$$Z(\Omega(\sigma\varrho_n,\sigma\varsigma_n),\sigma\Omega(\varrho_n,\varsigma_n),\varphi)=Z(1,0,\varphi)\nrightarrow 1.$$

So Ω and σ are not compatible.
From $\Omega(\varrho,\varsigma)=\sigma(\varrho),\Omega(\varsigma,\varrho)=\sigma(\varsigma)$ we get $(\varrho,\varsigma)=(0,0)$ and $\sigma\Omega(0,0)=\Omega(\sigma 0,\sigma 0)$ which implies that Ω and σ are weakly compatiable. It is clear that
$\frac{\varphi}{\Lambda+\varphi}\geq\min\left\{\frac{\varphi}{H+\varphi},\frac{\varphi}{K+\varphi}\right\}\Leftrightarrow\Lambda\leq\max\{H,K\}$, for all $\Lambda,H,K\geq 0,\varphi>0$.
As we already defined $\daleth,\Gamma,Z,\psi$ and by the above result we get the inequality (14), (15) and (16).

$$\daleth(\Omega(\varrho,\varsigma),\Omega(\rho,\tau),\psi(\varphi))\geq\daleth(\sigma(\varrho),\sigma(\rho),\varphi)*\daleth(\sigma(\varsigma),\sigma(\tau),\varphi),$$
$$\Gamma(\Omega(\varrho,\varsigma),\Omega(\rho,\tau),\psi(\varphi))\leq\Gamma(\sigma(\varrho),\sigma(\rho),\varphi)\Diamond\Gamma(\sigma(\varsigma),\sigma(\tau),\varphi) \text{ and}$$
$$Z(\Omega(\varrho,\varsigma),\Omega(\rho,\tau),\psi(\varphi))\leq Z(\sigma(\varrho),\sigma(\rho),\varphi)\odot Z(\sigma(\varsigma),\sigma(\tau),\varphi)$$

which is equivalent to the following results:

$$2|\Omega(\varrho,\varsigma)-\Omega(\rho,\tau)|\leq\max\{|\sigma(\varrho)-\sigma(\rho)|,|\sigma(\varsigma)-\sigma(\tau)|\}, \tag{36}$$

We will verify the above inequality (36).
Let $T=\left\{\frac{1}{2n},n\in\mathbb{N}\right\},\Upsilon=\Lambda-T$.
Now, we discuss the following possibilities for (ϱ,ς) and (ρ,τ).
Case 1: $(\varrho,\varsigma)\in\Upsilon\times\Upsilon,(\rho,\tau)\in\Upsilon\times\Upsilon$. It is clear that (36) is true.
Case 2: $(\varrho,\varsigma)\in\Upsilon\times\Upsilon,(\rho,\tau)\in\Upsilon\times T$. It is clear that (36) is true.
Case 3: $(\varrho,\varsigma)\in\Upsilon\times\Upsilon,(\rho,\tau)\in T\times T$. If $\rho\neq\tau$, (36) is true. If $\rho=\tau$ then take $\rho=\tau=1/2n$. Then we get
$2|\Omega(\varrho,\varsigma)-\Omega(\rho,\tau)|=\frac{2}{(2n+1)^4},\max\{|\sigma(\varrho)-\sigma(\rho)|,|\sigma(\varsigma)-\sigma(\tau)|\}=\frac{2n}{2n+1}$
which shows that (36) is true.
Case 4: $(\varrho,\varsigma)\in\Upsilon\times T,(\rho,\tau)\in\Upsilon\times T$. It is clear that (36) is true.
Case 5: $(\varrho,\varsigma)\in\Upsilon\times T,(\rho,\tau)\in T\times T$. If $\rho\neq\tau$, (36) is true. If $\rho=\tau$ take $\varrho\in\Upsilon,\varsigma=\frac{1}{2r},\rho=\tau=\frac{1}{2n}$ then we get $2|\Omega(\varrho,\varsigma)-\Omega(\rho,\tau)|=\frac{2}{(2n+1)^4}$
$\max\{|\sigma(\varrho)-\sigma(\rho)|,|\sigma(\varsigma)-\sigma(\tau)|\}=\max\left\{\frac{1}{2n+1},\left|\frac{1}{2r+1}-\frac{1}{2n+1}\right|\right\}$, or
$\max\left\{|\sigma(\varrho)-\sigma(\rho)|,|\sigma(\varsigma)-\sigma(\tau)|\right\}=\max\left\{\frac{2n}{2n+1},\left|\frac{1}{2r+1}-\frac{1}{2n+1}\right|\right\}$ implies (36) is true.
Case 6: $(\varrho,\varsigma)\in T\times T,(\rho,\tau)\in T\times T$.
If $\varrho\neq\varsigma,\rho\neq\tau$, (36) is true.
If $\varrho\neq\varsigma,\rho=\tau$ take $\varrho=\frac{1}{2s},\varsigma=\frac{1}{2r},s\neq r,\rho=\tau=\frac{1}{2n}$ then $2|\Omega(\varrho,\varsigma)-\Omega(\rho,\tau)|=\frac{2}{(2n+1)^4}$, $\max\{|\sigma(\varrho)-\sigma(\rho)|,|\sigma(\varsigma)-\sigma(\tau)|\}=\max\left\{\left|\frac{1}{2s+1}-\frac{1}{2n+1}\right|,\left|\frac{1}{2r+1}-\frac{1}{2n+1}\right|\right\}$, which shows

that (36) is true. If $\varrho = \varsigma, \rho = \tau$ take $\varrho = \varsigma = 1/2p, \rho = \tau = 1/2q$. Then $2|\Omega(\varrho, \varsigma) - \Omega(\rho, \tau)| = 2\left|\frac{1}{(2s+1)^4} - \frac{1}{(2n+1)^4}\right|$, $\max\{|\sigma(\varrho) - \sigma(\rho)|, |\sigma(\varsigma) - \sigma(\tau)|\} = \left|\frac{1}{2s+1} - \frac{1}{2n+1}\right|$, which shows that (36) is true. Therefore 0 is the only comon fixed point of Ω and σ and all the criteria in Theorem (3.2) are met.

4 Conclusion

In this paper, we establish a new idea namely Neutrosophic Metric Space (NMS). In NMS, Compatible mappings and weakly compatible mappings are defined. Further, we have proved fixed point result for the above mappings with an example.

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