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Stability Analysis Of System Of Functional Equations From A Hotel Model

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Abstract

In this paper, we explore the Ulam - Hyers stability of system of additive functional equations from a hotel model in Banach Space via classical Hyers Method.

Keywords: Additive functional equations; Hyers-Ulam stability; Hyers-Ulam-Rassias stability; Gavruta stability, Banach space.

AMS classification: 39B52, 32B72, 32B82

1 Introduction

The stability problem of functional equations initiated from a question of S.M. Ulam [22] concerning the stability of group homomorphisms. D.H. Hyers [14] contributed a first positive partial reply to the question of Ulam for Banach spaces.

Hyers theorem was generalized by T. Aoki [2] for additive mappings, Th.M. Rassias [20] and J.M. Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization of all the overhead effects was achieved by P. Gavruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias method. In 2008, a special case of Gavrutas theorem for the unbounded Cauchy difference was obtained by Ravi etal., [21] by considering the summation of both the sum and the product of two norms in the sprit of Rassias approach.

The well-known additive functional equation is

$$H(u+v) = H(u) + H(v). \tag{1}$$

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In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called additive Cauchy functional equation in honor of Cauchy (see[17]).

The various forms of additive functional equations

$$f(2x-y)+f(x-2y)=3f(x)-3f(y) \tag{2}$$

$$f(2x \pm y \pm z)=f(x \pm y)+f(x \pm z) \tag{3}$$

$$f(x)+f(x)=f(2x) \tag{4}$$

$$f(y)=f\left(\frac{y+z}{2}\right)+f\left(\frac{y-z}{2}\right) \tag{5}$$

$$f(x)+f(y+z)-f(x+y)=f(z); ||f(x)+f(y+z)-f(x+y)|| \leq ||f(z)|| \tag{6}$$

$$f\left(ax+y\right)-f\left(x-ay\right)=\left(a-1\right)f\left(x\right)+\left(1+a\right)f\left(y\right) \tag{7}$$

were conferred by D.O. Lee [16], M. Arunkumar [3, 4, 5, 6, 19]. The solution and stability of several additive functional equations were discussed in [1, 7, 8, 9, 10, 12, 11, 15].

Usually, we take food on the hotels with my friends. One day we got a though that we want to find which hotel serves fine and tasty food and also we prefer to inform my friends about the hotels. The quality of the hotel observed by the quality of food, service of the weightier and some may give tips. We found four types of the following Hotels.

H_1 : We went to a first hotel to take food and ordered food. Servers first gave some water and a handful of chips to eat till arriving the food. After some initially time the food came and they served charmingly. The food was very tasty. So we decided to help for them and then gave some tips for them.

H_2 : We went to second hotel to take food and ordered food. The food was tasty and they served pleasantly. But the impression was not as in the first hotel. So we did not help them and we avoided to give some tips.

H_3 : Later we went to a third hotel to take food. But this one was self-serviced. So there were no servers and there was no need of giving tips. Here also the food was fine.

H_4 : Finally we went to fourth hotel. As usual we ordered food. Before arrival of food, the servers take care us. But the food was not tasty and we did not like it. Even though they served in good manner we did not gave any tips for them.

Based on the above data, let us have the following assumptions: u_{i1} denotes the foods; u_{i2} denotes quality; u_{i3} denotes service; u_{i4} denotes giving tips; respectively. If it is satisfactory it denotes + (PLUS) and not satisfactory it denotes (MINUS).

Now, we collect the following data, with the help of Binary digits, If good we give 1 (one) and Bad we give 0 (zero). (for food, quality, service and tips and $i = 1, 2, 3, 4$.)

Hotel H_i	Food (u_{i1})	Quality(u_{i2})	Service(u_{i3})	Tips(u_{i4})	Total Sum	Preference
H_1	1	1	1	1	4	1 *
H_2	1	1	1	0	3	2 *
H_3	1	1	0	0	2	3
H_4	1	0	1	0	2	4

Table 1.1

The above data can be transformed into system of additive functional equations as follows.

$$H_1(u_1 + u_2 + u_3 + u_4) = H_1(u_1) + H_1(u_2) + H_1(u_3) + H_1(u_4) \tag{8}$$
$$H_2(u_1 + u_2 + u_3 - u_4) = H_2(u_1) + H_2(u_2) + H_2(u_3) - H_2(u_4) \tag{9}$$
$$H_3(u_1 + u_2 - u_3 - u_4) = H_3(u_1) + H_3(u_2) - H_3(u_3) - H_3(u_4) \tag{10}$$
$$H_4(u_1 - u_2 + u_3 - u_4) = H_4(u_1) - H_4(u_2) + H_4(u_3) - H_4(u_4) \tag{11}$$

In this paper, we explore the Ulam - Hyers stability of system of additive functional equations (8), (9), (10), (11) in Banach Space via classical Hyers Method.

2 Solution of The Functional Equations

In this section, the authors derived the general solution of additive functional equation (8), (9), (10), (11) by assuming N and M as real vector space.

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Theorem 2.1 If $H : N \to M$ bring about the functional equation (1) if and only if $H : N \to M$ brings the functional equations (8), (9), (10) and (11) respectively.

Proof: For all
$u, v, u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23}, u_{24}, u_{31}, u_{32}, u_{33}, u_{34}, u_{41}, u_{42}, u_{43}, u_{44} \in N$.
Suppose $H : N \to M$ fulfilling the additive functional equation (1). Alternating (u, v) by $(0,0), (u,u), (u,2u)$ and $(-u,u)$ in (1) and for any $c > 0$, we obtain

$$H(0) = 0; H(2u) = 2H(u); H(3u) = 3H(u); H(-u) = -H(u); H(cu) = cH(u);\ \forall u \in N. \tag{12}$$

Let us take u by $u_{11} + u_{12}$ and v by $u_{13} + u_{14}$ in (1) and take $H = H_1$, we obtain (8).

Put $(u_{11}, u_{12}, u_{13}, u_{14})$ by $(0,0,0,u_{24})$ in (8) and take $H_1 = H_2$, we get

$$H_2(-u_{24}) = -H_2(u_{24});\quad \forall u_{24} \in N. \tag{13}$$

Let us take$(u_{11}, u_{12}, u_{13}, u_{14})$ by $(u_{21}, u_{22}, u_{23}, -u_{24})$ in (8), using (13) and take $H_1 = H_2$, we arrive (9).

Substituting $(u_{21}, u_{22}, u_{23}, -u_{24})$ by $(0,0,-u_{33},0)$, $(0,0,0,u_{34})$ in (9) and take $H_2 = H_3$, we have

$$H_3(-u_{33}) = -H_3(u_{33}), H_3(-u_{44}) = -H_3(u_{44});\quad \forall u_{33}, u_{44} \in N. \tag{14}$$

Let us take $(u_{21}, u_{22}, u_{23}, u_{24})$ by $(u_{31}, u_{32}, u_{33}, -u_{34})$ in (9), using (14) and take $H_2 = H_3$, we arrive (10).

Substituting $(u_{31}, u_{32}, u_{33}, -u_{34})$ by $(0,-u_{42},0,0)$, $(0,0,-u_{43},0)$, $(0,0,0,u_{44})$ in (10) and take $H_3 = H_4$, we get

$$H_4(-u_{42}) = -H_4(u_{42}), H_4(-u_{43}) = -H_4(u_{43}), H_4(-u_{44}) = -H_4(u_{44}); \forall u_{42}, u_{43}, u_{44} \in N. \tag{15}$$

Let us take$(u_{31}, u_{32}, u_{33}, u_{34})$ by $(u_{41}, -u_{42}, -u_{43}, -u_{44})$ in (10). Using (15) and take $H_3 = H_4$, we obtain (11).

Putting $(u_{41}, u_{42}, u_{43}, u_{44})$ as $(u, 0, v, 0)$ and H_4 by H in (11), we arrive (1) as desired.

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Remark 2.2 From the above theorem, we see that all the functional equation (1), (8), (9), (10) and (11) are equivalent with respect to its solutions.

3 Stability Results: Hyers Direct Method

To provide stability results for (8), (9), (10), (11) assume that R and S be Banach spaces for derive the stability results.

From Table 1.1, the quantity of the hotel observed by the quality of food, service of the weightier and some give tips. If we are not satisfy in tips, service of weightier, quality of food we take that variable as zero (0) and if we satisfy all we give equal preference to all variables to prove the stability results.

Theorem 3.1 Let $p = \pm 1$. If $H_1, H_2, H_3, H_4 : R \to S$ are functions with inequalities

$$\begin{aligned}
&\|H_1(u_1+u_2+u_3+u_4) - H_1(u_1) - H_1(u_2) - H_1(u_3) - H_1(u_4)\| \\
&\le \tau_1(u_{11}, u_{12}, u_{13}, u_{14}) \qquad (16) \\
&\|H_2(u_{21}+u_{22}+u_{23}-u_{24}) - H_2(u_{21}) - H_2(u_{22}) - H_2(u_{23}) + H_2(u_{24})\| \\
&\le \tau_2(u_{21}, u_{22}, u_{23}, u_{24}) \qquad (17) \\
&\|H_3(u_{31}+u_{32}-u_{33}-u_{34}) - H_3(u_{31}) - H_3(u_{32}) + H_3(u_{33}) + H_3(u_{34})\| \\
&\le \tau_3(u_{31}, u_{32}, u_{33}, u_{34}) \qquad (18) \\
&\|H_4(u_{41}-u_{42}+u_{43}-u_{44}) - H_4(u_{41}) + H_4(u_{42}) - H_4(u_{43}) + H_4(u_{44})\| \\
&\le \tau_4(u_{41}, u_{42}, u_{43}, u_{44}) \qquad (19)
\end{aligned}$$

where $\tau_1, \tau_2, \tau_3, \tau_4 : R \to S$ are function fulfilling the conditions

$$\begin{aligned}
&\lim_{s\to\infty} \frac{\tau_1(4^{sp}u_{11}, 4^{sp}u_{12}, 4^{sp}u_{13}, 4^{sp}u_{14})}{4^{sp}} = 0; \lim_{s\to\infty} \frac{\tau_2(3^{sp}u_{21}, 3^{sp}u_{22}, 3^{sp}u_{23}, 3^{sp}u_{24})}{3^{sp}} = 0; \\
&\lim_{s\to\infty} \frac{\tau_3(2^{sp}u_{31}, 2^{sp}u_{32}, 2^{sp}u_{33}, 2^{sp}u_{34})}{2^{sp}} = 0; \lim_{s\to\infty} \frac{\tau_4(2^{sp}u_{41}, 42^{sp}u_{42}, 2^{sp}u_{43}, 2^{sp}u_{44})}{2^{sp}} = 0
\end{aligned} \qquad (20)$$

for all $u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23}, u_{24}, u_{31}, u_{32}, u_{33}, u_{34}, u_{41}, u_{42}, u_{43}, u_{44} \in R$. Then there exists a unique additive mappings $M_1, M_2, M_3, M_4 : R \to S$ fulfilling the functional equations (8), (9), (10), (11) and

$$\|H_1(u_1) - M_1(u_1)\| \leq \frac{1}{4} \sum_{q=\frac{1-p}{2}}^{\infty} \frac{\tau_1\left(4^{qp}u_1, 4^{qp}u_1, 4^{qp}u_1, 4^{qp}u_1\right)}{4^{qp}}; \forall\ u_1 \in R \tag{21}$$

$$\|H_2(u_2) - M_2(u_2)\| \leq \frac{1}{3} \sum_{q=\frac{1-p}{2}}^{\infty} \frac{\tau_2\left(3^{qp}u_2, 3^{qp}u_2, 3^{qp}u_2, 0\right)}{3^{qp}}; \forall\ u_2 \in R \tag{22}$$

$$\|H_3(u_3) - M_3(u_3)\| \leq \frac{1}{2} \sum_{q=\frac{1-p}{2}}^{\infty} \frac{\tau_3\left(2^{qp}u_3, 2^{qp}u_3, 0, 0\right)}{2^{qp}}; \forall\ u_3 \in R \tag{23}$$

$$\|H_4(u_4) - M_4(u_4)\| \leq \frac{1}{2} \sum_{q=\frac{1-p}{2}}^{\infty} \frac{\tau_4\left(2^{qp}u_4, 0, 2^{qp}u_4, 0\right)}{2^{qp}}; \forall\ u_4 \in R; \tag{24}$$

where

$$M_1(u_1) = \lim_{s\to\infty} \frac{H_1\left(4^{sp}u_1\right)}{4^{sp}}; \forall\ u_1 \in R; \qquad M_2(u_2) = \lim_{s\to\infty} \frac{H_2\left(3^{sp}u_2\right)}{3^{sp}}; \forall\ u_2 \in R$$
$$M_3(u_3) = \lim_{s\to\infty} \frac{H_3\left(2^{sp}u_3\right)}{2^{sp}}; \forall\ u_3 \in R; \qquad M_4(u_4) = \lim_{s\to\infty} \frac{H_4\left(2^{sp}u_4\right)}{2^{sp}}; \forall\ u_4 \in R;$$

respectively.

Proof. Alternating

$$(u_{11}, u_{12}, u_{13}, u_{14}) = (u_1, u_1, u_1, u_1) \text{ in } (16)\ ;$$
$$(u_{21}, u_{22}, u_{23}, u_{24}) = (u_2, u_2, u_2, 0) \text{ in } (17);$$
$$(u_{31}, u_{32}, u_{33}, u_{34}) = (u_3, u_3, 0, 0) \text{ in } (18);$$
$$(u_{41}, u_{42}, u_{43}, u_{44}) = (u_4, 0, u_4, 0) \text{ in } (19),$$

we obtain

$$\|H_1(4u_1) - 4H_1(u_1)\| \leq \tau_1(u_1, u_1, u_1, u_1); u_1 \in R; \tag{25}$$
$$\|H_2(3u_2) - 3H_2(u_2)\| \leq \tau_2(u_2, u_2, u_2, 0); u_2 \in R; \tag{26}$$
$$\|H_3(2u_3) - 2H_3(u_3)\| \leq \tau_3(u_3, u_3, 0, 0); u_3 \in R; \tag{27}$$
$$\|H_4(2u_4) - 2H_4(u_4)\| \leq \tau_4(u_4, 0, u_4, 0); u_4 \in R. \tag{28}$$

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The above inequalities can be rewritten as

$$\left\|\frac{H_1(4u_1)}{4}-H_1(u_1)\right\|\leq\frac{\tau_1(u_1,u_1,u_1,u_1)}{4};u_1\in R; \tag{29}$$

$$\left\|\frac{H_2(3u_2)}{3}-H_2(u_2)\right\|\leq\frac{\tau_2(u_2,u_2,u_2,0)}{3};u_2\in R; \tag{30}$$

$$\left\|\frac{H_3(2u_3)}{2}-H_3(u_3)\right\|\leq\frac{\tau_3(u_3,u_3,0,0)}{2};u_3\in R; \tag{31}$$

$$\left\|\frac{H_4(2u_4)}{2}-H_4(u_4)\right\|\leq\frac{\tau_4(u_4,0,u_4,0)}{4};u_4\in R. \tag{32}$$

Again alternating, u_1 by $4u_1$ and divide 4; u_2 by $3u_2$ and divide 3; u_3 by $2u_3$ and divide 2; u_4 by $2u_4$ and divide 2 in above inequalities, they becomes

$$\left\|\frac{H_1(4^2u_1)}{4^2}-\frac{H_1(4u_1)}{4}\right\|\leq\frac{\tau_1(4u_1,4u_1,4u_1,4u_1)}{4^2};u_1\in R; \tag{33}$$

$$\left\|\frac{H_2(2^2u_2)}{3^2}-\frac{H_2(3u_2)}{3}\right\|\leq\frac{\tau_2(3u_2,3u_2,3u_2,0)}{3^2};u_2\in R; \tag{34}$$

$$\left\|\frac{H_3(2^2u_3)}{2^2}-\frac{H_3(2u_3)}{2}\right\|\leq\frac{\tau_3(2u_3,2u_3,0,0)}{2^2};u_3\in R; \tag{35}$$

$$\left\|\frac{H_4(2^2u_4)}{2^2}-\frac{H_4(2u_4)}{2}\right\|\leq\frac{\tau_4(2u_4,0,2u_4,0)}{2^2};u_4\in R. \tag{36}$$

From (29) and (33) ;(30) and (34); (31) and (35) ; (32) and (36) using triangular inequality, we get

$$\left\|\frac{H_1(4^2u_1)}{4^2}-H_1(u_1)\right\|\leq\frac{1}{4}\left(\frac{\tau_1(4u_1,4u_1,4u_1,4u_1)}{4}+\tau_1(u_1,u_1,u_1,u_1)\right);u_1\in R; \tag{37}$$

$$\left\|\frac{H_2(3^2u_2)}{3^2}-H_2(u_2)\right\|\leq\frac{1}{3}\left(\frac{\tau_2(3u_2,3u_2,3u_2,0)}{3}+\tau_2(u_2,u_2,u_2,0)\right);u_2\in R; \tag{38}$$

$$\left\|\frac{H_3(2^2u_3)}{2^2}-H_3(u_3)\right\|\leq\frac{1}{2}\left(\frac{\tau_3(2u_3,2u_3,0,0)}{2}+\tau_3(u_3,u_3,0,0)\right);u_3\in R; \tag{39}$$

$$\left\|\frac{H_4(2^2u_4)}{2^2}-H_4(u_4)\right\|\leq\frac{1}{2}\left(\frac{\tau_4(2u_4,0,2u_4,0)}{2}+\tau_4(u_4,0,u_4,0)\right);u_4\in R. \tag{40}$$

The inequalities (37) , (38) , (39) and (40) generalized as follows

$$\left\|\frac{H_1\left(4^s u_1\right)}{4^s}-H_1\left(u_1\right)\right\| \leq \frac{1}{4} \sum_{q=0}^{s-1} \frac{\tau_1(4^q u_1, 4^q u_1, 4^q u_1, 4^q u_1)}{4^q}; u_1 \in R; \tag{41}$$

$$\left\|\frac{H_2\left(3^s u_2\right)}{3^s}-H_2\left(u_2\right)\right\| \leq \frac{1}{3} \sum_{q=0}^{s-1} \frac{\tau_2(3^q u_2, 3^q u_2, 3^q u_2, 0)}{3^q}; u_2 \in R; \tag{42}$$

$$\left\|\frac{H_3\left(2^s u_3\right)}{3^s}-H_3\left(u_3\right)\right\| \leq \frac{1}{2} \sum_{q=0}^{s-1} \frac{\tau_3(2^q u_3, 2^q u_3, 0, 0)}{2^q}; u_3 \in R; \tag{43}$$

$$\left\|\frac{H_4\left(2^s u_4\right)}{2^s}-H_4\left(u_4\right)\right\| \leq \frac{1}{2} \sum_{q=0}^{s-1} \frac{\tau_4(2^q u_4, 0, 2^q u_4, 0)}{2^q}; u_4 \in R. \tag{44}$$

By considering u_1 as $4^l u_1$ and dividing by 4^l in (41) , u_2 as $3^l u_2$ and dividing by 3^l in (42) , u_3 as $2^l u_3$ and dividing by 2^l in (43) , u_4 as $2^l u_4$ and dividing by 2^l in (44) , one can obtained that the sequences

$$\left\{\frac{H_1(4^s u_1)}{4^s}\right\}, \left\{\frac{H_2(3^s u_2)}{3^s}\right\}, \left\{\frac{H_3(2^s u_3)}{2^s}\right\}, \left\{\frac{H_4(2^s u_4)}{2^s}\right\},$$

are Cauchy sequences in R and S is complete, there exists a mapping $M_1, M_2, M_3, M_4 : R \to S$ such that

$$\begin{aligned} M_1(u_1) &= \lim_{s\to\infty} \frac{H_1\left(4^{sp} u_1\right)}{4^{sp}}; \\ M_2(u_2) &= \lim_{s\to\infty} \frac{H_2\left(3^{sp} u_2\right)}{3^{sp}}; \\ M_3(u_3) &= \lim_{s\to\infty} \frac{H_3\left(2^{sp} u_3\right)}{2^{sp}}; \\ M_4(u_4) &= \lim_{s\to\infty} \frac{H_4\left(2^{sp} u_4\right)}{2^{sp}} \end{aligned}$$

for all $u_1, u_2, u_3, u_4 \in R$. Letting s tends to infinity in (41); (42); (43); (44), we see that (21); (22); (23); (24), holds for $p = 1$. Assuming

$$\begin{aligned} &(u_{11}, u_{12}, u_{13}, u_{14}) \text{ as } (4^s u_{11}, 4^s u_{12}, 4^s u_{13}, 4^s u_{14}) \text{ and divide by } 4^s \text{ in } (16) \\ &(u_{21}, u_{22}, u_{23}, u_{24}) \text{ as } (3^s u_{21}, 3^s u_{22}, 3^s u_{23}, 3^s u_{24}) \text{ and divide by } 3^s \text{ in } (17) \\ &(u_{31}, u_{32}, u_{33}, u_{34}) \text{ as } (2^s u_{31}, 2^s u_{32}, 2^s u_{33}, 2^s u_{34}) \text{ and divide by } 2^s \text{ in } (18) \\ &(u_{41}, u_{42}, u_{43}, u_{44}) \text{ as } (2^s u_{41}, 2^s u_{42}, 2^s u_{43}, 2^s u_{44}) \text{ and divide by } 2^s \text{ in } (19), \end{aligned}$$

we arrive the following

$$\begin{aligned}\Bigg\| & \frac{H_1\left(4^s u_{11}+4^s u_{12}+4^s u_{13}+4^s u_{14}\right)}{4^s} \\ & -\left\{\frac{H_1\left(4^s u_{11}\right)}{4^s}+\frac{H_1\left(4^s u_{12}\right)}{4^s}+\frac{H_1\left(4^s u_{13}\right)}{4^s}+\frac{H_1\left(4^s u_{14}\right)}{4^s}\right\}\Bigg\| \\ & \qquad \leq \frac{\tau_1\left(4^s u_{11}, 4^s u_{12}, 4^s u_{13}, 4^s u_{14}\right)}{4^s};\end{aligned}$$

$$\begin{aligned}\Bigg\| & \frac{H_2\left(3^s u_{21}+3^s u_{22}+3^s u_{23}-3^s u_{24}\right)}{3^s} \\ & -\left\{\frac{H_2\left(3^s u_{21}\right)}{3^s}+\frac{H_2\left(3^s u_{22}\right)}{3^s}+\frac{H_2\left(3^s u_{23}\right)}{3^s}-\frac{H_2\left(3^s u_{24}\right)}{3^s}\right\}\Bigg\| \\ & \qquad \leq \frac{\tau_2\left(3^s u_{21}, 3^s u_{22}, 3^s u_{23}, 3^s u_{24}\right)}{3^s};\end{aligned}$$

$$\begin{aligned}\Bigg\| & \frac{H_3\left(2^s u_{31}+2^s u_{32}-2^s u_{33}-2^s u_{34}\right)}{2^s} \\ & -\left\{\frac{H_3\left(2^s u_{31}\right)}{2^s}+\frac{H_3\left(2^s u_{32}\right)}{2^s}-\frac{H_3\left(2^s u_{33}\right)}{2^s}-\frac{H_3\left(2^s u_{34}\right)}{2^s}\right\}\Bigg\| \\ & \qquad \leq \frac{\tau_3\left(2^s u3_1, 2^s u_{32}, 2^s u_{33}, 2^s u_{34}\right)}{2^s};\end{aligned}$$

$$\begin{aligned}\Bigg\| & \frac{H_4\left(2^s u_{41}-2^s u_{42}+2^s u_{43}-2^s u_{44}\right)}{2^s} \\ & -\left\{\frac{H_4\left(2^s u_{41}\right)}{2^s}-\frac{H_4\left(2^s u_{42}\right)}{2^s}+\frac{H_4\left(2^s u_{43}\right)}{2^s}-\frac{H_4\left(2^s u_{44}\right)}{2^s}\right\}\Bigg\| \\ & \qquad \leq \frac{\tau_4\left(2^s u_{41}, 2^s u_{42}, 2^s u_{43}, 2^s u_{44}\right)}{2^s}\end{aligned}$$

and let as taking $\lim s \to \infty$ on both sides of the above inequalities, we see that

$$\begin{aligned}M_1\left(u_{11}+u_{12}+u_{13}+u_{14}\right) &= M_1\left(u_{11}\right)+M_1\left(u_{12}\right)+M_1\left(u_{13}\right)+M_1\left(u_{14}\right); \\ M_2\left(u_{21}+u_{22}+u_{23}-u_{24}\right) &= M_2\left(u_{21}\right)+M_2\left(u_{22}\right)+M_2\left(u_{23}\right)-M_2\left(u_{24}\right); \\ M_3\left(u_{31}+u_{32}-u_{33}-u_{34}\right) &= M_3\left(u_{31}\right)+M_3\left(u_{32}\right)-M_3\left(u_{33}\right)-M_3\left(u_{34}\right); \\ M_4\left(u_{41}-u_{42}+u_{43}-u_{44}\right) &= M_4\left(u_{41}\right)-M_4\left(u_{42}\right)+M_4\left(u_{43}\right)-M_4\left(u_{44}\right);\end{aligned}$$

for all $u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23}, u_{24}, u_{31}, u_{32}, u_{33}, u_{34}, u_{41}, u_{42}, u_{43}, u_{44} \in R$. Hence M_1, M_2, M_3, M_4 are additive functions.

To show that $M_1(u_1), M_2(u_2), M_3(u_3), M_4(u_4)$ are unique. Assume that the another mappings $M_1'(u_1), M_2'(u_2), M_3'(u_3), M_4'(u_4)$ fullfilling the functional equations (8), (9), (10), (11) respectively. Now,

$$\left\|M_1(u_1) - M_1^1(u_1)\right\| \leq \frac{2}{4}\sum_{q=0}^{s-1} \frac{\tau_1(4^{q+l}u_1, 4^{q+l}u_1, 4^{q+l}u_1, 4^{q+l}u_1)}{4^{q+l}} \to 0 \quad as \quad l \to \infty;$$
$$\left\|M_2(u_2) - M_2^1(u_2)\right\| \leq \frac{2}{3}\sum_{q=0}^{s-1} \frac{\tau_2(3^{q+l}u_2, 3^{q+l}u_2, 3^{q+l}u_2, 0)}{3^{q+l}} \to 0 \quad as \quad l \to \infty;$$
$$\left\|M_3(u_3) - M_3^1(u_3)\right\| \leq \sum_{q=0}^{s-1} \frac{\tau_3(2^{q+l}u_3, 2^{q+l}u_3, 0, 0)}{2^{q+l}} \to 0 \quad as \quad l \to \infty;$$
$$\left\|M_4(u_4) - M_4^1(u_4)\right\| \leq \sum_{q=0}^{s-1} \frac{\tau_4(2^{q+l}u_4, 0, 2^{q+l}u_4, 0)}{2^{q+l}} \to 0 \quad as \quad l \to \infty.$$

Thus, we obtain $H_1(u_1) = M_1'(u_1), H_2(u_2) = M_2'(u_2), H_3(u_3) = M_3'(u_3), H_4(u_4) = M_4'(u_4)$ are unique. Hence theorem holds for $p = 1$.

Now substituting

$$\begin{gathered}
(u_{11}, u_{12}, u_{13}, u_{14}) = \left(\tfrac{u_1}{4}, \tfrac{u_1}{4}, \tfrac{u_1}{4}, \tfrac{u_1}{4}\right) \text{ in } (25)\ ; \\
(u_{21}, u_{22}, u_{23}, u_{24}) = \left(\tfrac{u_2}{3}, \tfrac{u_2}{3}, \tfrac{u_2}{3}, 0\right) \text{ in } (26)\ ; \\
(u_{31}, u_{32}, u_{33}, u_{34}) = \left(\tfrac{u_3}{2}, \tfrac{u_3}{2}, 0, 0\right) \text{ in } (27); \\
(u_{41}, u_{42}, u_{43}, u_{44}) = \left(\tfrac{u_4}{2}, 0, \tfrac{u_4}{2}, 0\right) \text{ in } (28)\ ,
\end{gathered}$$

we arrive

$$\left\|H_1(u_1) - 4H_1\left(\frac{u_1}{4}\right)\right\| \leq \tau_1\left(\frac{u_1}{4}, \frac{u_1}{4}, \frac{u_1}{4}, \frac{u_1}{4}\right); u_1 \in R; \tag{45}$$
$$\left\|H_2(u_2) - 3H_2\left(\frac{u_2}{3}\right)\right\| \leq \tau_2\left(\frac{u_2}{3}, \frac{u_2}{3}, \frac{u_2}{3}, 0\right); u_2 \in R; \tag{46}$$
$$\left\|H_3(u_3) - 2H_3\left(\frac{u_3}{2}\right)\right\| \leq \tau_3\left(\frac{u_3}{2}, \frac{u_3}{2}, 0, 0\right); u_3 \in R; \tag{47}$$
$$\left\|H_4(u_4) - 2H_4\left(\frac{u_4}{2}\right)\right\| \leq \tau_4\left(\frac{u_4}{2}, 0, \frac{u_4}{2}, 0\right); u_4 \in R. \tag{48}$$

The rest of the proof is same way to that of case $p = 1$. Thus, the proof is complete.

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Corollary 3.2 Let η, ρ be positive integers and assume the mappings $H_1, H_2, H_3, H_4 : R \to S$ managing the inequalities

$$\|H_1(u_{11}+u_{12}+u_{13}+u_{14}) - \{H_1(u_{11})+H_1(u_{12})+H_1(u_{13})+H_1(u_{14})\}\| \leq \begin{cases} (i)\ \ \eta; \\ \eta\sum_{r=1}^{4}\|u_{1r}\|^{\rho}; \\ \eta\sum_{r=1}^{4}\|u_{1r}\|^{\rho_r}; \\ \eta\prod_{r=0}^{4}\|u_{1r}\|^{\rho}; \\ \eta\prod_{r=0}^{4}\|u_{1r}\|^{\rho_r}; \end{cases} \tag{49}$$

$$\|H_2(u_{21}+u_{22}+u_{23}-u_{24}) - \{H_2(u_{21})+H_2(u_{22})+H_2(u_{23})-H_2(u_{24})\}\| \leq \begin{cases} (i)\ \ \eta; \\ \eta\sum_{r=1}^{4}\|u_{2r}\|^{\rho}; \\ \eta\sum_{r=1}^{4}\|u_{2r}\|^{\rho_r}; \end{cases} \tag{50}$$

$$\|H_3(u_{31}+u_{32}-u_{33}-u_{34}) - \{H_3(u_{31})+H_3(u_{32})-H_3(u_{33})-H_3(u_{34})\}\| \leq \begin{cases} (i)\ \ \eta; \\ \eta\sum_{r=1}^{4}\|u_{3r}\|^{\rho}; \\ \eta\sum_{r=1}^{4}\|u_{3r}\|^{\rho_r}; \end{cases} \tag{51}$$

$$\|H_4(u_{41}-u_{42}+u_{43}-u_{44}) - \{H_4(u_{41})-H_4(u_{42})+H_4(u_{43})-H_4(u_{44})\}\| \leq \begin{cases} (i)\ \ \eta; \\ \eta\sum_{r=1}^{4}\|u_{4r}\|^{\rho}; \\ \eta\sum_{r=1}^{4}\|u_{1r}\|^{\rho_r}; \end{cases} \tag{52}$$

for all $u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23}, u_{24}, u_{31}, u_{32}, u_{33}, u_{34}, u_{41}, u_{42}, u_{43}, u_{44} \in R$. Then there exists a unique mappings $M_1, M_2, M_3, M_4 : R \to S$ such that

$$\|H_1(u_1) - M_1(u_1)\| \leq \begin{cases} (i) \ \ \frac{\eta}{|3|}; \\ \frac{\eta 4\|u_1\|^{\rho}}{|4-4^{\rho}|}, \rho \neq 1; \\ \sum_{k=1}^{4} \frac{\eta\|u_1\|^{\rho_k}}{|4-4^{\rho_k}|}, \rho_k \neq 1; \\ \frac{\eta\|u\|^{4\rho}}{|4-4^{4\rho}|}, 4\rho \neq 1; \\ \frac{\eta\|u\|^{\rho_1+\rho_2+\rho_3+\rho_4}}{|4-4^{\rho_1+\rho_2+\rho_3+\rho_4}|}, \rho_1+\rho_2+\rho_3+\rho_4 \neq 1; u_1 \in R \end{cases} \tag{53}$$

$$\|H_2(u_2) - M_2(u_2)\| \leq \begin{cases} (i) \ \ \frac{\eta}{|2|}; \\ \frac{\eta 3\|u_2\|^{\rho}}{|3-3^{\rho}|}, \rho \neq 1; \\ \sum_{k=1}^{3} \frac{\eta\|u_2\|^{\rho_k}}{|3-3^{\rho_k}|}, \rho_k \neq 1; u_2 \in R; \end{cases} \tag{54}$$

$$\|H_3(u_3) - M_3(u_3)\| \leq \begin{cases} (i) \ \ |\eta|; \\ \frac{\eta 2\|u_3\|^{\rho}}{|2-2^{\rho}|}, \rho \neq 1; \\ \sum_{k=1}^{2} \frac{\eta\|u_3\|^{\rho_k}}{|2-2^{\rho_k}|}, \rho_k \neq 1; u_3 \in R \end{cases} \tag{55}$$

$$\|H_4(u_4) - M_4(u_4)\| \leq \begin{cases} (i) \ \ |\eta|; \\ \frac{\eta 2\|u_4\|^{\rho}}{|2-2^{\rho}|}, \rho \neq 1; \\ \sum_{k=1}^{2} \frac{\eta\|u_4\|^{\rho_k}}{|2-2^{\rho_k}|}, \rho \neq 1; u_4 \in R \end{cases} \tag{56}$$

Proof. Let us take the right hand side of (16); (17); (18); (19) as the right hand of (49); (50); (51); (52), respectively in Theorem 3.1, we obtain our needed result.

4 Conclusion

According to Table 1.1 and after, tasting the food on 4 hotels, we gave some suggestions about the hotels to my friends based on Food, Quality of food, Serves attitude and Issuing tips.

We liked first two hotels H_1 and H_2 because the food was tasty and they served very well. So normally people will go to hotels H_1 and H_2 in more number.

Also third Hotel H_3 was self-serviced, so many of them did not like it. The food was not tasty in the fourth Hotel H_4 and no one liked it and none of them will go there.

By Corollary 3.2 for conditions (i) of (53); (54); (55); (56), we get the better possible upper bound stability analysis in first two hotels only that is in (53); (54). In rest of the situations that is in (55); (56), we cannot get the upper bound.

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