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Homomorphisms And Derivations of a Generalized Functional Equation in Various Algebras

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Abstract

In this paper, we investigate the generalized Ulam - Hyers stability of homomorphisms and derivations of a generalized additive functional equation in Banach, Quasi - Banach, C^*, Lie C^*, Jordan C^* Algebras.

Keywords: Functional equation, Generalized Hyers-Ulam stability, Banach Algebras, Quasi -Banach Algebras, C^* Algebras, Lie C^* Algebras, Jordan C^* Algebras.

AMS classification: 39B52, 32B72, 32B82

1 Introduction

In Ulam [26] proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?. In Hyers [7] gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution one can see [2, 6, 17, 21, 23].

During the last seven decades the stability problems of various functional equations in several algebras have been broadly investigated by number of mathematicians and more details about the definitions on all the algebras see [3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 22, 24, 25].

The important Cauchy additive functional equation is

$$\lambda(u+v)=\lambda(u)+\lambda(v) \tag{1}$$

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2456-8686, viii(ii), 2024:105-127
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having solution $\lambda(u) = cu$ (see [1, 8]).

In this paper, we introduce and investigate the generalized Ulam - Hyers stability of homomorphisms and derivations of a generalized additive functional equation

$$\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) = (\alpha + \beta)(\lambda(u) + \lambda(v)) \tag{2}$$

where $\alpha, \beta \neq 0$ in Banach, Quasi - Banach, C^*, Lie C^*, Jordan C^*, Algebras respectively.

Theorem 1.1 Assume V_1 and V_2 are real vector spaces. Suppose that $\lambda : V_1 \to V_2$ satisfies the functional equation (1) then $\lambda : V_1 \to V_2$ satisfies the functional equation (2).

2 Stability Results in Banach Algebras

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ be a Banach algebra with norm $||\cdot||_A$ and $\mathcal{B}$ be a Banach algebra with norm $||\cdot||_B$.

Also, assume a function $\mathcal{N} : \mathcal{A}^2 \to [0, \infty)$ by

$$\mathcal{N}(u,v) = \begin{cases} \pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi \ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi}, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi} + ||u||_{\mathcal{A}}^{2\varpi} + ||v||_{\mathcal{A}}^{2\varpi}\right\}, \end{cases} \quad ; \quad for \quad all \quad u, v \in \mathcal{A}.$$

Homomorphism Stability Result

Theorem 2.1 If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{3}$$

$$\|\lambda(uv) - \lambda(u)\lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{4}$$

2456-8686, viii(ii), 2024:105-127
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and

$$\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v)=0=\lim_{\delta\to\infty}\frac{1}{\gamma^{2\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v) \tag{5}$$

for all $u, v \in \mathcal{A}$ where

$$\nu=\pm 1 \qquad and \qquad \gamma=\alpha+\beta. \tag{6}$$

Then there exists a unique homomorphism function $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}}\le\frac{1}{2\gamma}\sum_{\chi=\frac{1-\nu}{2}}^{\infty}\frac{\eta(\gamma^{\chi\nu}u,\gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{7}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\lambda(\gamma^{\delta\nu}u) \tag{8}$$

for all $u \in \mathcal{A}$.
Proof. Assume $\nu = 1$. Letting (u, v) by (u, u) in (3), we arrive

$$\|2\lambda((\alpha+\beta)u)-2(\alpha+\beta)\lambda(u)\|_{\mathcal{B}}\le\eta(u,u)\Longrightarrow\|\lambda(\gamma u)-\gamma\lambda(u)\|_{\mathcal{B}}\le\frac{1}{2}\eta(u,u) \tag{9}$$

for all $u \in \mathcal{A}$. It follows from above inequality that

$$\left\|\frac{\lambda(\gamma u)}{\gamma}-\lambda(u)\right\|_{\mathcal{B}}\le\frac{1}{2\gamma}\eta(u,u) \tag{10}$$

for all $u \in \mathcal{A}$. Now replacing u by γu and dividing by γ in (10), we obtain

$$\left\|\frac{\lambda(\gamma^2 u)}{\gamma^2}-\frac{\lambda(\gamma u)}{\gamma}\right\|_{\mathcal{B}}\le\frac{1}{2\gamma^2}\eta(\gamma u,\gamma u) \tag{11}$$

for all $u \in \mathcal{A}$. From (10) and (11), we get

$$\left\|\frac{\lambda(\gamma^2 u)}{\gamma^2}-\lambda(u)\right\|_{\mathcal{B}}\le\frac{1}{2\gamma}\left[\eta(u,u)+\frac{\eta(\gamma u,\gamma u)}{\gamma}\right] \tag{12}$$

for all $u \in \mathcal{A}$. Proceeding further and using induction on a positive integer δ, we

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

have

$$\left\|\frac{\lambda(\gamma^{\delta}u)}{\gamma^{\delta}}-\lambda(u)\right\|_{\mathcal{B}}\leq\frac{1}{2\gamma}\sum_{\chi=0}^{\delta-1}\frac{1}{\gamma^{\chi}}\eta(\gamma^{\chi}u,\gamma^{\chi}u) \tag{13}$$

for all $u\in\mathcal{A}$. It is easy to verify that the sequence $\left\{\frac{\lambda(\gamma^{\delta}u)}{\gamma^{\delta}}\right\}$, is a Cauchy sequence by replacing u by $\gamma^{\epsilon}u$ and dividing by γ^{ϵ} in (13), for any $\epsilon,\delta>0$. Since $\mathcal{B}$ is complete, there exists a mapping $\mathcal{H}(u):\mathcal{A}\to\mathcal{B}$ such that

$$\mathcal{H}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta}}\lambda(\gamma^{\delta}u),\qquad for\quad all\qquad u\in\mathcal{A}.$$

Letting $\delta\to\infty$ in (13), we see that (7) holds for all $u\in\mathcal{A}$. To show that $\mathcal{H}$ satisfies (2), replacing (u,v) by $(\gamma^{\delta}u,\gamma^{\delta}v)$ and dividing by γ^{δ} in (3), we obtain

$$\frac{1}{\gamma^{\delta}}\left\|\lambda(\gamma^{\delta}(\alpha u+\beta v))+\lambda(\gamma^{\delta}(\beta u+\alpha v))-(\alpha+\beta)(\lambda(\gamma^{\delta}u)+\lambda(\gamma^{\delta}v))\right\|_{\mathcal{B}}\frac{1}{\gamma^{n}}\eta(\gamma^{\delta}u,\gamma^{\delta}v)$$

for all $u,v\in\mathcal{A}$. Letting $\delta\to\infty$ in the above inequality and using the definition of $\mathcal{H}(u)$, we see that

$$\mathcal{H}(\alpha u+\beta v)+\mathcal{H}(\beta u+\alpha v)=(\alpha+\beta)(\mathcal{H}(u)+\mathcal{H}(v)).$$

Thus the existence of $\mathcal{H}$ satisfies the additive functional equation (2) for all $u,v\in\mathcal{A}$.

From (4) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}(uv)-\mathcal{H}(u)\mathcal{H}(v)\|_{\mathcal{B}}&=\frac{1}{\gamma^{2\delta}}\left\|\lambda(\gamma^{\delta}u\ \gamma^{\delta}v)-\lambda(\gamma^{\delta}u)\lambda(\gamma^{\delta}v)\right\|_{\mathcal{B}}\\&\leq\frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u,\gamma^{\delta}v)\to0\quad as\ \ \delta\to\infty.\end{aligned}$$

Therefore

$$\mathcal{H}(uv)=\mathcal{H}(u)\mathcal{H}(v)$$

for all $u,v\in\mathcal{A}$. Thus, $\mathcal{H}$ is a algebra homomorphism. To prove existence of $\mathcal{H}$ is unique, we assume $\mathcal{H}'(u)$ be another homomorphism mapping satisfying (2) and (6), then

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

$$\begin{aligned}\|\mathcal{H}(u)-\mathcal{H}'(u)\|_{\mathcal{B}} &= \frac{1}{\gamma^{\epsilon}}\|\mathcal{H}(\gamma^{\epsilon}u)-\mathcal{H}'(\gamma^{\epsilon}u)\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{\epsilon}}\left\{\|\mathcal{H}(\gamma^{\epsilon}u)-\lambda(\gamma^{\epsilon}u)\|_{\mathcal{B}}+\|\lambda(\gamma^{n}u)-\mathcal{H}'(\gamma^{n}u)\|_{\mathcal{B}}\right\} \\ &\leq \frac{2}{2\gamma}\sum_{\chi=0}^{\infty}\frac{1}{\gamma^{(\delta+\epsilon)}}\eta(\gamma^{\delta+\epsilon}u,\gamma^{\delta+\epsilon}u)\to 0 \quad as\ \ \delta\to\infty\end{aligned}$$

for all $u\in\mathcal{A}$. Hence $\mathcal{H}$ is unique. Thus the theorem holds for $\nu=1$.

Letting u by $\frac{u}{\gamma}$ in (9), we get

$$\left\|\lambda(u)-\gamma\lambda\left(\frac{u}{\gamma}\right)\right\|_{\mathcal{B}}\leq\frac{1}{2}\eta\left(\frac{u}{\gamma},\frac{u}{\gamma}\right) \tag{14}$$

for all $u\in\mathcal{A}$. Again setting u by $\frac{u}{\gamma}$ and multiply by γ in (14), we obtain

$$\left\|\gamma\lambda\left(\frac{u}{\gamma}\right)-\gamma^{2}\lambda\left(\frac{u}{\gamma^{2}}\right)\right\|_{\mathcal{B}}\leq\frac{\gamma}{2}\eta\left(\frac{u}{\gamma^{2}},\frac{u}{\gamma^{2}}\right) \tag{15}$$

for all $u\in\mathcal{A}$. From (14) and (15), we achieve

$$\left\|\lambda(u)-\gamma^{2}\lambda\left(\frac{u}{\gamma^{2}}\right)\right\|_{\mathcal{B}}\leq\frac{1}{2}\left[\eta\left(\frac{u}{\gamma},\frac{u}{\gamma}\right)+\gamma\ \eta\left(\frac{u}{\gamma^{2}},\frac{u}{\gamma^{2}}\right)\right] \tag{16}$$

for all $u\in\mathcal{A}$. Proceeding further and using induction on a positive integer δ, we have

$$\left\|\lambda(u)-\gamma^{\delta}\lambda\left(\frac{u}{\gamma^{\delta}}\right)\right\|_{\mathcal{B}}\leq\frac{1}{2}\sum_{\chi=1}^{\delta}\gamma^{\delta-1}\eta\left(\frac{u}{\gamma^{\delta}},\frac{u}{\gamma^{\delta}}\right)=\frac{1}{2\gamma}\sum_{\chi=1}^{\delta}\gamma^{\delta}\eta\left(\frac{u}{\gamma^{\delta}},\frac{u}{\gamma^{\delta}}\right) \tag{17}$$

for all $u\in\mathcal{A}$. The rest of the proof is similar lines to that of case $\nu=1$. Thus, the theorem holds for $\nu=-1$.
Corollary 2.2 Suppose $\lambda:\mathcal{A}\to\mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r}\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}}\\ \|\lambda(uv)-\lambda(u)\lambda(v)\|_{\mathcal{B}}\end{array}\right\}\leq\mathcal{N}(u,v) \tag{18}$$

2456-8686, viii(ii), 2024:105-127
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for all $u, v \in \mathcal{A}$. Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \frac{\pi}{2|1-\gamma|}, & \\ \frac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \frac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \frac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{19}$$

for all $u \in \mathcal{A}$.

Derivation Stability Result

Theorem 2.3 If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{20}$$

$$\|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \leq \eta(u, v) \tag{21}$$

and

$$\lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta \to \infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{22}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{23}$$

Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi = \frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{24}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{25}$$

for all $u \in \mathcal{A}$.

Proof. By Theorem 2.1, there exists a unique additive mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ satisfying (24). Also, the mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ given by

$$\mathcal{D}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta}}\lambda(\gamma^{\delta}u)$$

for all $u\in\mathcal{A}$. From (21) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned}\|\mathcal{D}(uv)-u\mathcal{D}(v)-\mathcal{D}(u)v\|_{\mathcal{B}}&=\frac{1}{\gamma^{2\delta}}\left\|\lambda(\gamma^{\delta}u\;\gamma^{\delta}v)-\gamma^{\delta}u\lambda(\gamma^{\delta}v)-\lambda(\gamma^{\delta}u)\gamma^{\delta}v\right\|_{\mathcal{B}}\\&\leq\frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u,\gamma^{\delta}v)\to 0\quad as\;\;\delta\to\infty.\end{aligned}$$

Therefore

$$\mathcal{D}(uv)=u\mathcal{D}(v)+\mathcal{D}(u)v$$

for all $u,v\in\mathcal{A}$. Thus, $\mathcal{D}$ is a algebra derivation.

Corollary 2.4 Suppose $\lambda:\mathcal{A}\to\mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{aligned}\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}}\\\|\lambda(uv)-u\lambda(v)-\lambda(u)v\|_{\mathcal{B}}\end{aligned}\right\}\leq\mathcal{N}(u,v)\tag{26}$$

for all $u,v\in\mathcal{A}$. Then there exists a unique derivation function $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}}\leq\begin{cases}\dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma-\gamma^{\varpi}|}, & \varpi\neq 1\\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq 1\\ \dfrac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq 1\end{cases}\tag{27}$$

for all $u\in\mathcal{A}$.

2456-8686, viii(ii), 2024:105-127
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3 Stability Results in Quasi - Banach Algebras

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ be a quasi norm algebra with norm $||\cdot||_A$ and $\mathcal{B}$ be a quasi Banach algebra with norm $||\cdot||_B$.

Homomorphism Stability Result

Theorem 3.1 If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{28}$$

$$\|\lambda(uv) - \lambda(u)\lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{29}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{30}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{31}$$

Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{32}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{33}$$

for all $u \in \mathcal{A}$. Proof. Assume $\nu = 1$. Letting (u, v) by (u, u) in (28), we arrive

$$\|2\lambda((\alpha + \beta)u) - 2(\alpha + \beta)\lambda(u)\|_{\mathcal{B}} \leq \eta(u, u) \Longrightarrow \|\lambda(\gamma u) - \gamma\lambda(u)\|_{\mathcal{B}} \leq \frac{1}{2}\eta(u, u) \tag{34}$$

for all $u \in \mathcal{A}$. It follows from above inequality that

$$\left\| \frac{\lambda(\gamma u)}{\gamma} - \lambda(u) \right\|_{\mathcal{B}} \leq \frac{1}{2\gamma}\eta(u, u) \tag{35}$$

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

for all $u \in \mathcal{A}$. Now replacing u by γu and dividing by γ in (35), we obtain

$$\left\| \frac{\lambda(\gamma^2 u)}{\gamma^2} - \frac{\lambda(\gamma u)}{\gamma} \right\|_{\mathcal{B}} \leq \frac{1}{2\gamma^2} \eta(\gamma u, \gamma u) \tag{36}$$

for all $u \in \mathcal{A}$. From (35) and (36), we get

$$\left\| \frac{\lambda(\gamma^2 u)}{\gamma^2} - \lambda(u) \right\|_{\mathcal{B}} \leq \frac{K}{2\gamma} \left[\eta(u, u) + \frac{\eta(\gamma u, \gamma u)}{\gamma} \right] \tag{37}$$

for all $u \in \mathcal{A}$. Proceeding further and using induction on a positive integer δ, we have

$$\left\| \frac{\lambda(\gamma^\delta u)}{\gamma^\delta} - \lambda(u) \right\|_{\mathcal{B}} \leq \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=0}^{\delta-1} \frac{1}{\gamma^\chi} \eta(\gamma^\chi u, \gamma^\chi u) \tag{38}$$

for all $u \in \mathcal{A}$. The rest of the proof is similar lines to that of Theorem 2.1.

Corollary 3.2 Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha+\beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - \lambda(u)\lambda(v)\|_{\mathcal{B}} \end{array}\right\} \leq \mathcal{N}(u, v) \tag{39}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{K^{\delta-1}\pi}{2|1-\gamma|}, & \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{K^{\delta-1}3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{40}$$

for all $u \in \mathcal{A}$.

Derivation Stability Result

Theorem 3.3 If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{41}$$

$$\|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \leq \eta(u, v) \tag{42}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{43}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \textit{and} \qquad \gamma = \alpha + \beta. \tag{44}$$

Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{45}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{46}$$

for all $u \in \mathcal{A}$.

Proof. The proof is similar to that of Theorem 2.3.

Corollary 3.4 Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \end{array}\right\} \leq \mathcal{N}(u, v) \tag{47}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$

satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{K^{\delta-1}\pi}{2|1-\gamma|}, & \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma-\gamma^{\varpi}|}, & \varpi\neq 1 \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq 1 \\ \dfrac{K^{\delta-1}3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq 1 \end{cases} \tag{48}$$

for all $u \in \mathcal{A}$.

4 Stability Results in C^* Algebras

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ be a C^*- algebra with norm $||\cdot||_A$ and $\mathcal{B}$ be a C^*- algebra with norm $||\cdot||_B$.
Homomorphism Stability Result

Theorem 4.1 If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0,\infty)$ are functions satisfying the triple inequalities

$$\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}} \leq \eta(u,v) \tag{49}$$
$$\|\lambda(uv)-\lambda(u)\lambda(v)\|_{\mathcal{B}} \leq \eta(u,v) \tag{50}$$
$$\|\lambda(u^*)-\lambda(u)^*\|_{\mathcal{B}} \leq \eta(u) \tag{51}$$

and

$$\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v)=0=\lim_{\delta\to\infty}\frac{1}{\gamma^{2\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\eta(\gamma^{\delta\nu}u) \tag{52}$$

for all $u, v \in \mathcal{A}$ where

$$\nu=\pm 1 \qquad \textit{and} \qquad \gamma=\alpha+\beta. \tag{53}$$

Then there exists a unique C^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma}\sum_{\chi=\frac{1-\nu}{2}}^{\infty}\frac{\eta(\gamma^{\chi\nu}u,\gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{54}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{55}$$

for all $u \in \mathcal{A}$.
Proof. By Theorem 2.1, there exists a unique homomorphism mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying (54). From (51) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}(u^*) - \mathcal{H}(u)^*\|_{\mathcal{B}} &= \frac{1}{\gamma^{\delta}} \left\|\lambda(\gamma^{\delta} u^*) - \lambda(\gamma^{\delta} u)^*\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{\delta}} \eta(\gamma^{\delta} u) \to 0 \quad as \;\; \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{H}(u^*) = \mathcal{H}(u)^*$$

for all $u \in \mathcal{A}$. Thus, $\mathcal{H}$ is a C^*-algebra homomorphism.

Corollary 4.2 Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r}\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha+\beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - \lambda(u)\lambda(v)\|_{\mathcal{B}}\end{array}\right\} \leq \mathcal{N}(u, v) \tag{56}$$

and

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \pi ||u||_{\mathcal{A}}^{\varpi} \tag{57}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique C^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi ||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi ||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi ||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{58}$$

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

for all $u \in \mathcal{A}$.

Derivation Stability Result

Theorem 4.3 If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the triple inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{59}$$

$$\|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \leq \eta(u, v) \tag{60}$$

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \eta(u) \tag{61}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{62}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \textit{and} \qquad \gamma = \alpha + \beta. \tag{63}$$

Then there exists a unique C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{64}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{65}$$

for all $u \in \mathcal{A}$.

Proof. The proof is similar to that of Theorem 4.1.

Corollary 4.4 Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{aligned} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \end{aligned}\right\} \leq \mathcal{N}(u, v) \tag{66}$$

and

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \pi||u||_{\mathcal{A}}^{\varpi} \tag{67}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{68}$$

for all $u \in \mathcal{A}$.

5 Stability Results in Lie C^* Algebras

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ be a Lie C^*- algebra with norm $|| \cdot ||_A$ and $\mathcal{B}$ be a Lie C^*- algebra with norm $|| \cdot ||_B$.

Homomorphism Stability Result

Theorem 5.1 If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{69}$$

$$\|\lambda([uv]) - [\lambda(u), \lambda(v)]\|_{\mathcal{B}} \leq \eta(u, v) \tag{70}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}}\eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}}\eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) \tag{71}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{72}$$

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

Then there exists a unique Lie C^*- homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{73}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{74}$$

for all $u \in \mathcal{A}$.
Proof. By Theorem 2.1, there exists a unique homomorphism mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying (73). From (70) and definition of $\mathcal{H}$, we achieve

$$\|\mathcal{H}([uv]) = [\mathcal{H}(u), \mathcal{H}(v)]\|_{\mathcal{B}} = \tfrac{1}{\gamma^{2\delta}} \left\|\lambda([\gamma^{\delta}u\ \gamma^{\delta}v]) - [\lambda(\gamma^{\delta}u), \lambda(\gamma^{\delta}v)]\right\|_{\mathcal{B}}$$

$$\leq \tfrac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta}u, \gamma^{\delta}v) \to 0 \quad as\ \ \delta \to \infty.$$ Therefore

$$\mathcal{H}([uv]) = [\mathcal{H}(u), \mathcal{H}(v)]$$

for all $u \in \mathcal{A}$. Thus, $\mathcal{H}$ is a Lie C^*- algebra homomorphism.
Corollary 5.2 Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda([uv]) - [\lambda(u), \lambda(v)]\|_{\mathcal{B}} \end{array}\right\} \leq \mathcal{N}(u, v) \tag{75}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique Lie C^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{76}$$

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

for all $u \in \mathcal{A}$.

Derivation Stability Result

Theorem 5.3 If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{77}$$

$$\|\lambda([uv]) - [\lambda(u)v] - [u, \lambda(v)]\|_{\mathcal{B}} \leq \eta(u, v) \tag{78}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) \tag{79}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{80}$$

Then there exists a unique Lie C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{81}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{82}$$

for all $u \in \mathcal{A}$.
Proof. By Theorem 2.3, there exists a unique additive mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying (81). Also, the mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta}} \lambda(\gamma^{\delta}u)$$

for all $u \in \mathcal{A}$. From (78) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned}\|\mathcal{D}([uv]) - [\mathcal{D}(u), v] - [u, \mathcal{D}(v)]\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda([\gamma^{\delta}u\ \gamma^{\delta}v]) - [\lambda(\gamma^{\delta}u), \gamma^{\delta}v] - [\gamma^{\delta}u, \lambda(\gamma^{\delta}v)]\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta}u, \gamma^{\delta}v) \to 0 \quad as \ \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{D}([uv]) = [\mathcal{D}(u), v] + [u, \mathcal{D}(v)]$$

for all $u, v \in \mathcal{A}$. Thus, $\mathcal{D}$ is a Lie C^*- algebra derivation.

Corollary 5.4 Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda([uv]) - [\lambda(u)v] - [u, \lambda(v)]\|_{\mathcal{B}} \end{array}\right\} \leq \mathcal{N}(u, v) \tag{83}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique Lie C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{84}$$

for all $u \in \mathcal{A}$.

6 Stability Results in Jordan C^* Algebras

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ be a JC^*- algebra with norm $||\cdot||_A$ and $\mathcal{B}$ be a JC^*- algebra with norm $||\cdot||_B$.

Homomorphism Stability Result

Theorem 6.1 If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{85}$$

$$\|\lambda(u \circ v) - \lambda(u) \circ \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{86}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{87}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{88}$$

Then there exists a unique JC^*- homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{89}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{90}$$

for all $u \in \mathcal{A}$.

Proof. By Theorem 2.1, there exists a unique homomorphism mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying (89). From (86) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned} \|\mathcal{H}(u \circ v) - \mathcal{H}(u) \circ \mathcal{H}(v)\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta}u \circ \gamma^{\delta}v) - \lambda(\gamma^{\delta}u) \circ \lambda(\gamma^{\delta}v)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta}u, \gamma^{\delta}v) \to 0 \quad as \;\; \delta \to \infty. \end{aligned}$$

Therefore

$$\mathcal{H}(u \circ v) = \mathcal{H}(u) \circ \mathcal{H}(v)$$

for all $u \in \mathcal{A}$. Thus, $\mathcal{H}$ is a JC^*- algebra homomorphism.

Corollary 6.2 Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{aligned} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(u \circ v) - \lambda(u) \circ \lambda(v)\|_{\mathcal{B}} \end{aligned}\right\} \leq \mathcal{N}(u, v) \tag{91}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique JC^*- algebra homomorphism function

2456-8686, viii(ii), 2024:105-127
https://doi.org/10.26524/cm200

$\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{92}$$

for all $u \in \mathcal{A}$.

Derivation Stability Result

Theorem 6.3 If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{93}$$

$$\|\lambda(u \circ v) - \lambda(u) \circ v - u \circ \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{94}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) \tag{95}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \textit{and} \qquad \gamma = \alpha + \beta. \tag{96}$$

Then there exists a unique JC^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{97}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{98}$$

for all $u \in \mathcal{A}$.
Proof. By Theorem 2.3, there exists a unique additive mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$

satisfying (97). Also, the mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta}} \lambda(\gamma^{\delta} u)$$

for all $u \in \mathcal{A}$. From (94) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned} \|\mathcal{D}(u \circ v) - \mathcal{D}(u) \circ v - u \circ \mathcal{D}(v)]\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta} u \circ \gamma^{\delta} v]) - \lambda(\gamma^{\delta} u) \circ \gamma^{\delta} v - \gamma^{\delta} u \circ \lambda(\gamma^{\delta} v)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta} u, \gamma^{\delta} v) \to 0 \quad as \;\; \delta \to \infty. \end{aligned}$$

Therefore

$$\mathcal{D}(u \circ v) = \mathcal{D}(u) \circ v + u \circ \mathcal{D}(v)$$

for all $u, v \in \mathcal{A}$. Thus, $\mathcal{D}$ is a $JC^{*}-$ algebra derivation.

Corollary 6.4 Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that

$$\left.\begin{aligned} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(u \circ v) - \lambda(u) \circ v - u \circ \lambda(v)\|_{\mathcal{B}} \end{aligned}\right\} \leq \mathcal{N}(u, v) \quad (99)$$

for all $u, v \in \mathcal{A}$. Then there exists a unique $JC^{*}-$ algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi ||u||_{\mathcal{A}}^{\varpi}}{|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi ||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \\ \dfrac{3\pi ||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \quad (100)$$

for all $u \in \mathcal{A}$.

7 Conclusion

In this paper, we provide the generalized Ulam - Hyers stability of homomorphisms and derivations of a generalized additive functional equation in Banach Algebra, Quasi - Banach Algebra, C^*- Algebra, Lie C^*- Algebra, Jordan C^* Algebra.

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