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ORIGINAL RESEARCH **Open Access**

Summation Formula Related To Extended α-Delta Operators

Rajalakshmi R [1], Dhatchaya V[2], Gowthami S[3] and Britto Antony Xavier G[4]

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Abstract

The paper aims to develop higher-order discrete fundamental theorems for delta and alpha-delta operators. By equating the closed and summation forms of inverse operators, we derive novel summation formulae. Additionally, we establish comprehensive summation formulae for generalized exponential and extorial functions incorporating shift values. These advancements contribute to a deeper understanding of the discrete calculus framework and its applications across various mathematical and scientific fields. The results obtained can be applied to improve discrete dynamic systems, enhance computational techniques and provide new insights into the behavior of discrete functions. Our findings pave the way for further exploration and potential breakthroughs in discrete mathematics and its interdisciplinary applications.

Key words: Alpha-delta operators, Extorial Function.

AMS classification: 39A13

1 Introduction

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields. The theory of difference equations, the methods used and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last

[1]Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India. Email:rajalakshmisathya05@gmail.com
[2]Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India. Email:kkvrangan@gmail.com
[3]Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India. Email:gowthamibharathi55@gmail.com
[4]Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India. Email:brittoshc@gmail.com

15 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many international conferences and numerous special sessions.

Difference Equation

Dynamical system come with many different names. Our particular interesting dynamical system is for the system whose state depends on the input history. In discrete time system, we call such system "Difference Equation"(equivalent to difference equation in continuous time).

Difference equation is an equation involving differences. We can see difference equation from atleast three points of views: as sequence of number, discrete dynamical system and iterated function. It is the same thing but we look at different angle.

2 First Order α-Delta Integration With Several Parameters

Definition 2.1 Let f be a real valued fuction and $\zeta \neq 0, \alpha \neq 1$.The α_1, α_2-delta operator on f is defined as

$$\Delta_{\alpha_1,\alpha_2} f(\zeta) = f(\zeta+2) - \alpha_1 f(\zeta+1) - \alpha_2 f(\zeta). \quad (1)$$

The inverse of α_1, α_2-delta operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta_{\alpha_1,\alpha_2} g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta), \quad (2)$$

where c is an arbitrary constant.

Theorem 2.2 Let f be a real-valued function, $\Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta)$ and $\alpha_1, \alpha_2 \neq 0$, then

$$F_0 f(\zeta+m) + F_1 f(\zeta+m-1) + F_2 f(\zeta+m-2) + \cdots + F_m f(\zeta)$$
$$= \Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta+m+2) - [\alpha_1 F_m + \alpha_2 F_{m-1}] \Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta+1) - \alpha^2 F_m \Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta),$$

$\forall m \in \mathbb{Z}^+$, where $F_0 = 1, F_1 = \alpha_1$ and $F_m = \alpha_1 F_{m-1} + \alpha_2 F_{m-2}$

Proof: From the definition (2.1)$\Delta_{\alpha_1,\alpha_2} f(\zeta) = f(\zeta+2) - \alpha_1 f(\zeta+1) - \alpha_2 f(\zeta)$

Let $g(\zeta) = \Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta)$

$\Delta_{\alpha_1,\alpha_2} g(\zeta) = f(\zeta)$

$g(\zeta+2) - \alpha_1 g(\zeta+1) - \alpha_2 g(\zeta) = f(\zeta)$

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replace ζ by $\zeta+1$

$$g(\zeta+3)-\alpha_1 g(\zeta+2)-\alpha_2 g(\zeta+1)=f(\zeta+1)$$

$$g(\zeta+3)-\alpha_1\{f(\zeta)+\alpha_1 g(\zeta+1)+\alpha_2 g(\zeta)\}-\alpha_2 g(\zeta+1)=f(\zeta)$$

$$g(\zeta+3)-\alpha_1 f(\zeta)-\alpha_1^2 g(\zeta+1)-\alpha_2\alpha_2 g(\zeta)-\alpha_2 g(\zeta+1)=f(\zeta+1)$$

$$g(\zeta+3)-\alpha_1^2 g(\zeta+1)-\alpha_2 g(\zeta+1)-\alpha_1\alpha_2 g(\zeta)=f(\zeta+1)+\alpha_1 f(\zeta+1)$$

$$\Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta+3)-\left\{\alpha_1^2\alpha_2\right\}\Delta^{-1}_{\alpha_1\alpha_2} f(\zeta+1)-\alpha_1\alpha_2\Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta)=f(\zeta+1)+\alpha_1 f(\zeta)$$

Again replace ζ by $\zeta+1$

$$g(\zeta+3)-\alpha_1 g(\zeta+2)-\alpha_2 g(\zeta+1)=f(\zeta+1)$$

$$g(\zeta+3)-\alpha_1\{f(\zeta)+\alpha_1 g(\zeta+1)+\alpha_2 g(\zeta)\}-\alpha_2 g(\zeta+1)=f(\zeta+1)$$

$$g(\zeta+3)-\alpha_1 f(\zeta)-\alpha_1^2 g(\zeta+1)-\alpha_2\alpha_2 g(\zeta)-\alpha_2 g(\zeta+1)=f(\zeta+1)$$

$$g(\zeta+3)-\alpha_1^2 g(\zeta+1)-\alpha_2 g(\zeta+1)-\alpha_1\alpha_2 g(\zeta)=f(\zeta+1)+\alpha_1 f(\zeta)$$

$$\Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta+3)-\left\{\alpha_1^2\alpha_2\right\}\Delta^{-1}_{\alpha_1\alpha_2} f(\zeta+1)-\alpha_1\alpha_2\Delta^{-1}_{\alpha_1,\alpha_2} f(\zeta)=f(\zeta+1)+\alpha_1 f(\zeta)$$

Again replace ζ by $\zeta+1$

$g(\zeta+4)-\alpha_1^2 g(\zeta+2)-\alpha_2 g(\zeta+2)-\alpha_1\alpha_2 g(\zeta+1)=f(\zeta+2)+\alpha_1 f(\zeta+1)g(\zeta+4)-\left\{\alpha_1^2+\alpha_2\right\}\{\alpha_1 g(\zeta+1)+\alpha_2 g(\zeta)+f(\zeta)\}-\alpha_1\alpha_2 g(\zeta+1)$

$$=f(\zeta+2)+\alpha_1 f(\zeta+1)$$

$g(\zeta+4)-\alpha_1^3 g(\zeta+1)-\alpha_1^2\alpha_2 g(\zeta)-\alpha_1^2 f(\zeta)-\alpha_1\alpha_2 g(\zeta+1)-\alpha_2^2 g(\zeta)-\alpha_2 f(\zeta)-\alpha_1\alpha_2 g(\zeta+1)$

$$=f(\zeta+2)+\alpha_1 f(\zeta+1)$$

$$g(\zeta+4)-\left\{\alpha_1^3+2\alpha_1\alpha_2\right\}g(\zeta+1)-\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\}g(\zeta)$$

$$=f(\zeta+2)+\alpha_1 f(\zeta+1)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta)$$

Again replace ζ by $\zeta+1$

$$g(\zeta+5)-\left\{\alpha_1^3+2\alpha_1\alpha_2\right\}g(\zeta+2)-\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\}g(\zeta+1)$$

$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)$$

$$g(\zeta+5)-\left\{\alpha_1^3+2\alpha_1\alpha_2\right\}\{\alpha_1 g(\zeta+1)+\alpha_2 g(\zeta)+f(\zeta)\}-\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\}g(\zeta+1)$$

$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)$$

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$$g(\zeta+5)-\alpha_1^4 g(\zeta+1)-\alpha_1^3\alpha_2 g(\zeta)-\alpha_1^3 f(\zeta)-2\alpha_1^2\alpha_2 g(\zeta+1)-2\alpha_1\alpha_2^2 g(\zeta)-2\alpha_1\alpha_2 f(\zeta)$$
$$-\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\} g(\zeta+1)$$
$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+1)$$

$$g(\zeta+5)-\left\{\alpha_1^4+3\alpha_1^2\alpha_2+\alpha_2^2\right\} g(t+1)-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2\right\} g(\zeta)$$
$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+1)+\left\{\alpha_1^3+2\alpha_1\alpha_2\right\} f(\zeta)$$

Again replace ζ by $\zeta+1$

$$g(\zeta+6)-\left\{\alpha_1^4+3\alpha_1^2\alpha_2+\alpha_2^2\right\} g(\zeta+1)-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2\right\} g(\zeta+1)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2\right\} f(\zeta+1)$$

$$g(\zeta+6)-\left\{\alpha_1^4+3\alpha_1^2\alpha_2+\alpha_2^2\right\}\left\{\alpha_1 g(\zeta+1)+\alpha_2 g(\zeta)+f(\zeta)\right\}-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2\right\} g(\zeta+1)$$
$$=f(\zeta+1)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2\right\} f(\zeta+1)$$

$$g(\zeta+6)-\alpha_1^5 g(\zeta+1)-\alpha_1^4\alpha_2 g(\zeta)-\alpha_1^4\alpha_2 g(\zeta)-\alpha_1^4 f(\zeta)-3\alpha_1^3\alpha_2 g(\zeta+1)-3\alpha_1^2\alpha_2 g(\zeta)$$
$$-3\alpha_1^2\alpha_2 f(\zeta)-\alpha_1\alpha_2^2 g(\zeta+1)-\alpha_2^3 g(\zeta)-\alpha_2^2 f(\zeta)-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2\right\} g(\zeta+1)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2\right\} f(\zeta+1)$$

$$g(\zeta+6)-\left\{\alpha_1^5+4\alpha_1^3\alpha_2+3\alpha_1\alpha_2^2\right\} g(\zeta+1)-\left\{\alpha_1^4\alpha_2+3\alpha_1^2\alpha_1^2+\alpha_2^3\right\} g(\zeta)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\} f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2\right\} f(\zeta+1)+\left\{\alpha_1^4+3\alpha_1^2\alpha_2+\alpha_2^2\right\} f(\zeta)$$

In general,

$$F_0 f(\zeta+ml)+F_1 f(\zeta+m-1)+F_2 f(\zeta+m-2)+\cdots+F_m f(\zeta)$$
$$=\Delta_{\alpha_1,\alpha_2}^{-1} f(\zeta+m+2)-[\alpha_1 F_m+\alpha_2 F_{m-1}]\Delta_{\alpha_1,\alpha_2}^{-1} f(\zeta+1)-\alpha^2 F_m g(\zeta).$$

Example 2.3 Applying $f(\zeta)=2^{\zeta}, m=4, \alpha_1=3, \alpha_2=4$ and $\zeta=3$ in theorem (2.2), we get

$$LHS=2^{\zeta+4}+\alpha_1 2^{\zeta+3}+\{\alpha_1^2+\alpha_2\}2^{\zeta+2}+\{\alpha_1^3+2\alpha_1\alpha_2\}2^{\zeta+1}+\{\alpha_1^4+3\alpha_1^2+\alpha_2+\alpha_2^2\}2^{\zeta}$$
$$=2^{3+4}+3(2^{\zeta+3})+\{3^2+4\}2^{3+2}+\{3^3+2(3)(4)\}2^{3+1}+\{3^4+3(3)^2(4)+4^2\}2^3$$
$$=128+192+416+816+1640=3192$$

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$\Delta_{\alpha_1,\alpha_2} f(\zeta) = f(\zeta+2) - \alpha_1 f(\zeta) - \alpha_2 f(\zeta)$

$\Delta_{\alpha_1,\alpha_2} 2^\zeta = 2^{\zeta+2} - \alpha_1 2^{\zeta+1} - \alpha_2 2^\zeta$

$\qquad = 2^\zeta[2^2 - \alpha_1 2 - \alpha_2]$

$$RHS = \Delta^{-1}_{\alpha_1,\alpha_2} 2^{\zeta+6} - [\alpha_1^5 + 4\alpha_1^3\alpha_2 + 3\alpha_1\alpha_2^2]\Delta^{-1}_{\alpha_1,\alpha_2} 2^{\zeta+1} - [\alpha_1^4\alpha_2 + 3\alpha_1^2\alpha_2^2 + \alpha_2^3]\Delta^{-1}_{\alpha_1,\alpha_2} 2^\zeta$$

$$= \frac{2^{3+6}}{4-2(3)-4} - [3^5 + 4(3)^3(4) + 3(3)(4)^2]\frac{2^{3+1}}{4-2(3)-4}$$

$$- [3^4 + 3(3)^2(4)^2 + 4^3]\frac{2^3}{4-2(3)-4}$$

$$= -85.3333 + 2184 + 1093.3333 = 3192.$$

Therefore, $LHS = RHS$.

Definition 2.4 Let f be a real valued fuction and $\zeta \neq 0, \alpha \neq 1$. The $\alpha_1, \alpha_2, \alpha_3-$delta operator on f is defined as

$$\Delta_{\alpha_1,\alpha_2,\alpha_3} f(\zeta) = f(\zeta+3) - \alpha_1 f(\zeta+2) - \alpha_2 f(\zeta+1) - \alpha_3 f(\zeta). \tag{3}$$

The inverse of $\alpha_1, \alpha_2, \alpha_3$-Delta operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta_{\alpha_1,\alpha_2,\alpha_3} g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta), \tag{4}$$

where c is an arbitrary constant.

Theorem 2.5 Let f be real-valued function, $\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta)$ exists and $\alpha_1, \alpha_2, \alpha_3 \neq 0$, then

$$F_0 f(\zeta+m) + F_1 f(\zeta+m-1) + F_2 f(\zeta+m-2) + \cdots + F_m f(\zeta)$$

$$= \Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta+m+3) - [\alpha_1 F_m + \alpha_2 F_{m-1} + \alpha_3 F_{m-2}]\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta+2)$$

$$-[\alpha^2 F_m + \alpha_3 F_{m-1}]\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta+1) - \alpha_3 F_m \Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3} f(\zeta),$$

$$\forall m \in \mathbb{Z}^+$$

where $F_0 = 1, F_1 = \alpha_1$ and $F_m = \alpha_1 F_{m-1} + \alpha_2 F_{m-2} + \alpha_3 F_{m-3}$

Proof: From the definiton(2.4)

$$\Delta_{(\alpha_1,\alpha_2,\alpha_3)} f(\zeta) = f(\zeta+3) - \alpha_1 f(\zeta+2) - \alpha_2 f(\zeta+1) - \alpha_3 f(\zeta)$$

$$Let\quad g(\zeta)=\Delta^{-1}_{(\alpha_1,\alpha_2,\alpha_3)}f(\zeta)$$

$$\Delta^{-1}_{(\alpha_1,\alpha_2,\alpha_3)}g(\zeta)=f(\zeta)$$

$$g(\zeta+3)-\alpha_1 g(\zeta+2)-\alpha_2 g(\zeta+1)-\alpha_3 g(\zeta)=f(\zeta)$$

$$g(\zeta+3)=f(\zeta)+\alpha_1 g(\zeta+2)+\alpha_2 g(\zeta+1)+\alpha_3 g(\zeta)$$

replace ζ by $\zeta+1$

$$g(\zeta+4)=f(\zeta+1)+\alpha_1 g(\zeta+3)+\alpha_2 g(\zeta+2)+\alpha_3 g(\zeta+1)\ g(\zeta+4)$$
$$=f(\zeta+1)+\alpha_1\{f(\zeta)+\alpha_1 g(\zeta+2)+\alpha_2 g(\zeta+1)+\alpha_3 g(\zeta)\}$$

$$+\alpha_2 g(\zeta+2)+\alpha_3 g(\zeta+1)$$

$$g(\zeta+4)=f(\zeta+1)+\alpha_1 f(\zeta)+\left\{\alpha_1^2+\alpha_2\right\}g(\zeta+2)+\{\alpha_1\alpha_2+\alpha_3\}g(\zeta+1)+\alpha_1\alpha_3 g(\zeta)$$

replace ζ by $\zeta+1$

$$g(\zeta+5)=f(\zeta+2)+\alpha_1 f(\zeta+1)+\left\{\alpha_1^2+\alpha_2\right\}g(\zeta+3)+\{\alpha_1\alpha_2+\alpha_3\}g(\zeta+2)+\alpha_1\alpha_2 g(\zeta+1)$$

$$g(\zeta+5)=f(\zeta+2)+\alpha_1 f(\zeta+1)+\left\{\alpha_1^2+\alpha_2\right\}\{f(\zeta)+\alpha_1 g(\zeta+2)+\alpha_2 g(\zeta+1)+\alpha_3 g(\zeta)\}$$
$$+\{\alpha_1\alpha_2+\alpha_3\}g(\zeta+2)+\alpha_1\alpha_2 g(\zeta+1)$$
$$=f(\zeta+2)+\alpha_1 f(\zeta+1)+\alpha_1^2 f(\zeta)+\alpha_1^3 g(\zeta+2)+\alpha_1^2\alpha_2 g(\zeta+1)+\alpha_1^2\alpha_3 g(\zeta)$$
$$+\alpha_2 f(\zeta)+\alpha_1\alpha_2 g(\zeta+2)+\alpha_2^2 g(\zeta+1)+\alpha_2\alpha_3 g(\zeta)+\{\alpha_1\alpha_2+\alpha_3\}g(\zeta+2)$$
$$+\alpha_1\alpha_3 g(\zeta+1)$$

$$g(\zeta+5)=f(\zeta+2)+\alpha_1 f(\zeta+1)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta)+\left\{\alpha_1^3+\alpha_1\alpha_2\right\}g(\zeta+2)$$
$$+\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\}g(\zeta+1)+\left\{\alpha_1^2\alpha_3+\alpha_2\alpha_3\right\}g(\zeta)$$

Again replace ζ by $\zeta+1$

$$g(\zeta+6)=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha 3\right\}g(\zeta+3)\quad +$$
$$\left\{\alpha_1^2\alpha_2+\alpha_2^2\right\}g(\zeta+2)+\left\{\alpha_1^2\alpha_3+\alpha_2\alpha_3\right\}g(\zeta+1)$$
$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)+\left\{\alpha_1^3+2\alpha_1\alpha_2\alpha_3\right\}$$
$$\{f(\zeta)+\alpha_1 g(\zeta+2)+\alpha_2 g(\zeta+1)+\alpha_3 g(\zeta)\}+\left\{\alpha_1^2\alpha_2+\alpha_2^2+\alpha_3\right\}g(\zeta+2)$$
$$+\left\{\alpha_1^2\alpha_3+\alpha_2\alpha_3\right\}g(\zeta+1)$$
$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)+\alpha_1^3 f(\zeta)+\alpha_1^4 g(\zeta+2)$$
$$+\alpha_1^3\alpha_2 g(\zeta+1)+\alpha_1^3+\alpha_1^3\alpha_3 g(\zeta)+2\alpha_1\alpha_2 f(\zeta)+2\alpha_1^2\alpha_2 g(\zeta+2)$$

$$+2\alpha_1\alpha_2^2 g(\zeta+1)+2\alpha_1\alpha_2\alpha_3 g(\zeta)+\alpha_3 f(\zeta)+\alpha_1\alpha_3 g(\zeta+2)+\alpha_2\alpha_3 g(\zeta+1)$$
$$+\alpha_3^2 g(\zeta)+\left\{\alpha_1^2\alpha_2+\alpha_2^2+\alpha_1\alpha_3\right\}g(\zeta+2)+\left\{\alpha_1^2\alpha_3+\alpha_2\alpha_3\right\}g(\zeta+1)$$
$$g(\zeta+6)=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha_3\right\}f(\zeta)$$
$$+\left\{\alpha_1^4+2\alpha_1^2\alpha_2+\alpha_1\alpha_3+\alpha_1^2\alpha_2+\alpha_2^2+\alpha_1\alpha_3\right\}g(\zeta+2)$$
$$+\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2+\alpha_2\alpha_3+\alpha_1^2\alpha_3+\alpha_2\alpha_3\right\}g(\zeta+1)$$
$$+\left\{\alpha_1^3\alpha_3+2\alpha_1\alpha_2\alpha_3+\alpha_3^2\right\}g(\zeta)$$
$$g(\zeta+6)-\left\{\alpha_1^4+3\alpha_1^2\alpha_2+2\alpha_1\alpha_3+\alpha_2^2\right\}g(\zeta+2)-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2\alpha_3+\alpha_3^2\right\}g(\zeta)$$
$$=f(\zeta+3)+\alpha_1 f(\zeta+2)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+1)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha_3\right\}f(\zeta)$$

Again replace ζ by $\zeta+1$

$$g(\zeta+7)-\left\{\alpha_1^4+3\alpha_1^2\alpha_2+2\alpha_1\alpha_3+\alpha_2^2\right\}g(\zeta+3)$$
$$-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2+\alpha_1^2\alpha_3+2\alpha_2\alpha_3\right\}g(\zeta+2)-\left\{\alpha_1^3\alpha_3+2\alpha_1\alpha_2\alpha_3+\alpha 3^2\right\}g(\zeta+1)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha_3\right\}f(\zeta+1)$$
$$g(\zeta+7)-\left\{\alpha_1^4+3\alpha_1^2+2\alpha_1\alpha_3+\alpha_2^2\right\}\left\{f(\zeta)+\alpha_1 g(\zeta+2)+\alpha_2 g(\zeta+1)+\alpha_3 g(\zeta)\right\}$$
$$-\left\{\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2+\alpha_1^2\alpha_3+2\alpha_1\alpha_3\right\}g(\zeta+2)-\left\{\alpha_1^3\alpha_3\alpha_3+2\alpha_1\alpha_2\alpha_3+\alpha_3^2\right\}g(\zeta+1)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha_3\right\}f(\zeta+1)$$
$$g(\zeta+7)-\left\{\alpha_1^5+3\alpha_1^3\alpha_2+2\alpha_1^2\alpha_3+\alpha_1\alpha_2^2+\alpha_1^3\alpha_2+2\alpha_1\alpha_2^2+\alpha_1^2\alpha_3+2\alpha_2\alpha_3\right\}g(\zeta+2)$$
$$-\left\{\alpha_1^4\alpha_2+3\alpha_1^2\alpha_2^2+2\alpha_1\alpha_2\alpha_3+\alpha_2^3+\alpha_1^3\alpha_3+2\alpha_1\alpha_2\alpha_3+\alpha_3^2\right\}g(\zeta+1)$$
$$-\left\{\alpha_1^4\alpha_3+3\alpha_1^2\alpha_2\alpha_3+2\alpha_1\alpha_3^2+\alpha_1^2\alpha_3\right\}g(\zeta)$$
$$=f(\zeta+4)+\alpha_1 f(\zeta+3)+\left\{\alpha_1^2+\alpha_2\right\}f(\zeta+2)+\left\{\alpha_1^3+2\alpha_1\alpha_2+\alpha_3\right\}f(\zeta+1)$$
$$+\left\{\alpha_1^4+3\alpha_1^2\alpha_2+2\alpha_1\alpha_3+\alpha_2^2\right\}f(\zeta)$$
$$F_0 f(\zeta+4)+F_1 f(\zeta+3)+\{\alpha_1F_1+\alpha_2F_2\}f(\zeta+2)+\{\alpha_1F_2+\alpha_2F_1+\alpha_3F_0\}f(\zeta+1)$$
$$+\{\alpha_1F_3+\alpha_2F_2+\alpha_3F_1\}f(\zeta)$$
$$=g(\zeta+7)-\{\alpha_1F_4+\alpha_2F_3+\alpha_3F_2\}-\{\alpha_2F_4+\alpha_3F_3\}g(\zeta+1)-\alpha_3F_4g(\zeta)$$

In general

$$F_0 f(\zeta+m)+F_1 f(\zeta+m-1)+F_2 f(\zeta+m-2)+\cdots+F_m f(\zeta)$$
$$=g(\zeta+m+3)-\sum_{r=1}^{3}\alpha_r F_{m-(r-1)}g(\zeta+n-1)-\sum_{r=2}^{3}\alpha_r F_{m-(r-2)}g(\zeta+n-2)-\alpha_3F_3g(\zeta).$$

Example 2.6 Applying $f(\zeta)=2^{\zeta}, m=4, \alpha_1=3, \alpha_2=4, \alpha_3=5, \zeta=2$ in theorem (2.5), we get

$LHS=f(\zeta+4)+\alpha_1 f(\zeta+3)+[\alpha_1^2+\alpha_2]f(\zeta+2)+[\alpha_1^3+2\alpha_1\alpha_2+\alpha_3]f(\zeta+1)$

$$+[\alpha_1^4+3\alpha_1^2\alpha_2+2\alpha_1\alpha_3+\alpha_2^2]f(\zeta)$$

$=2^{\zeta+4}+\alpha_1 2^{\zeta+3}+[\alpha_1^2+\alpha_2]2^{\zeta+2}+[\alpha_1^3+2\alpha_1\alpha_2+\alpha_3]2^{\zeta+1}+[\alpha_1^4+3\alpha_1^2\alpha_2+2\alpha_1\alpha_3+\alpha_2^2]2^{\zeta}$

$=2^{2+4}+(3)2^{2+3}+[3^2+4]2^{2+2}+[3^3+2(3)(4)+5]2^{2+1}$

$$+[3^4+3(3)^2(4)+2(3)(5)+4^2]2^2$$

$=64+96+208+448+940=1756$

$RHS=\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3}2^{\zeta+7}-[\alpha_1F_4+\alpha_2F_3+\alpha_3F_2]\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3}2^{\zeta+2}-[\alpha_2F_4+\alpha_3F_3]\Delta^{-1}_{\alpha_1,\alpha_2\alpha_3}2^{\zeta+1}$

$$-\alpha_3F_4\Delta^{-1}_{\alpha_1,\alpha_2,\alpha_3}2^{\zeta}$$

$$=\frac{2^{\zeta+7}}{8-4\alpha_1-2\alpha_2-\alpha_3}-[235\alpha_1+56\alpha_2+13\alpha_3]\frac{2^{\zeta+2}}{8-4\alpha_1-2\alpha_2-\alpha_3}$$

$$-[235\alpha_2+56\alpha_3]\frac{2^{\zeta+1}}{8-4\alpha_1-2\alpha_2-\alpha_3}-235\alpha_3\frac{2^{\zeta}}{8-4\alpha_1-2\alpha_2-\alpha_3}$$

$$=\frac{2^{2+7}}{8-4(3)-2(4)-5}-[(3)235+(4)56+(5)13]\frac{2^{2+2}}{8-4(3)-2(4)-5}$$

$$-[4(235)+5(56)]\frac{2^{2+1}}{8-4(3)-2(4)-5}-5(235)\frac{2^2}{8-4(3)-2(4)-5}$$

$=-30.11764706+935.5294118(0.941176471)+1220(0.470588235)$

$+1175(-1.411764706)$

$=-30.11764706+935.5294118+574.1176471+276.4705882=1756$

Therefore, $LHS=RHS$.

Theorem 2.7 Let f be real-valued function, $\Delta^{-1}_{\alpha_1,\alpha_1,\cdots,\alpha_n} f(\zeta)$ exists and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \neq 0$, then

$$F_0 f(\zeta+m) + F_1 f(\zeta+m-1) + F_1 f(\zeta+m-1) + F_2 f(\zeta+m-2) \cdots + F_m f(\zeta)$$
$$= g(\zeta+m+n) - \sum_{r=1}^{n} \alpha_r F_{m-(r-1)} g(\zeta+n-1) - \sum_{r=2}^{n} \alpha_r F_{m-(r-2)} g(\zeta+n-2) - \cdots$$
$$- \sum_{r=n}^{n} \alpha_r F_{m-(r-n)} g(\zeta)$$

where $F_0 = 1, F_1 = \alpha_1$ *and* $F_m = \sum_{r=1}^{m} \alpha_r F_{n-r}$.

3 Conclusion

From this work, We developed the theory of discrete version of fundamental theorems using α-delta, $\alpha(h)$-delta, α-nabla and $\alpha(h)$-nabla operators. This theory is subsequently applied to establish several higher order fundamental theorems. Also we derived fundamental theorems using first order α-delta integration with several parameters. Our findings are validated with suitable examples in the field of discrete fractional calculus.

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