

Summation Formula Related To Extended α -Delta Operators

Rajalakshmi R ¹, Dhatchaya V², Gowthami S³, Saraswathi D⁴ and Britto Antony Xavier G⁵

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Abstract

The paper aims to develop higher-order discrete fundamental theorems for delta and alpha-delta operators. By equating the closed and summation forms of inverse operators, we derive novel summation formulae. Additionally, we establish comprehensive summation formulae for generalized exponential and extorial functions incorporating shift values. These advancements contribute to a deeper understanding of the discrete calculus framework and its applications across various mathematical and scientific fields. The results obtained can be applied to improve discrete dynamic systems, enhance computational techniques and provide new insights into the behavior of discrete functions. Our findings pave the way for further exploration and potential breakthroughs in discrete mathematics and its interdisciplinary applications.

Key words: Alpha-delta operators, Extorial Function.

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1 Introduction

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields. The theory of difference

¹Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India.
Email:rajalakshmisathya05@gmail.com

²Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India.
Email:kkvrangan@gmail.com

³Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India.
Email:gowthamibharathi55@gmail.com

⁴Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India.
Email:dswathisaranya@gmail.com

⁵Department of Mathematics, Sacred Heart College, Tirupattur, Tirupattur District, Tamil Nadu, India.
Email:brittoshc@gmail.com

equations, the methods used and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last 15 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many international conferences and numerous special sessions.

Difference Equation

Dynamical system come with many different names. Our particular interesting dynamical system is for the system whose state depends on the input history. In discrete time system, we call such system "Difference Equation"(equivalent to difference equation in continuous time).

Difference equation is an equation involving differences. We can see difference equation from atleast three points of views: as sequence of number, discrete dynamical system and iterated function. It is the same thing but we look at different angle.

2 First Order α -Delta Integration With Several Parameters

Definition 2.1 Let f be a real valued fuction and $\zeta \neq 0, \alpha \neq 1$.The α_1, α_2 -delta operator on f is defined as

$$\Delta_{\alpha_1, \alpha_2} f(\zeta) = f(\zeta + 2) - \alpha_1 f(\zeta + 1) - \alpha_2 f(\zeta). \tag{1}$$

The inverse of α_1, α_2 -delta operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta_{\alpha_1, \alpha_2} g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta), \tag{2}$$

where c is an arbitrary constant.

Theorem 2.2 Let f be a real-valued function, $\Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta)$ and $\alpha_1, \alpha_2 \neq 0$, then $F_0 f(\zeta + m) + F_1 f(\zeta + m - 1) + F_2 f(\zeta + m - 2) + \dots + F_m f(\zeta)$

$$= \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + m + 2) - [\alpha_1 F_m + \alpha_2 F_{m-1}] \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 1) - \alpha^2 F_m \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta),$$

$\forall m \in \mathbb{Z}^+$, where $F_0 = 1, F_1 = \alpha_1$ and $F_m = \alpha_1 F_{m-1} + \alpha_2 F_{m-2}$

Proof: From the definition (2.1) $\Delta_{\alpha_1, \alpha_2} f(\zeta) = f(\zeta + 2) - \alpha_1 f(\zeta + 1) - \alpha_2 f(\zeta)$

Let $g(\zeta) = \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta)$

$$\Delta_{\alpha_1, \alpha_2} g(\zeta) = f(\zeta)$$

$$g(\zeta + 2) - \alpha_1 g(\zeta + 1) - \alpha_2 g(\zeta) = f(\zeta)$$

replace ζ by $\zeta + 1$

$$g(\zeta + 3) - \alpha_1 g(\zeta + 2) - \alpha_2 g(\zeta + 1) = f(\zeta + 1)$$

$$g(\zeta + 3) - \alpha_1 \{f(\zeta) + \alpha_1 g(\zeta + 1) + \alpha_2 g(\zeta)\} - \alpha_2 g(\zeta + 1) = f(\zeta)$$

$$g(\zeta + 3) - \alpha_1 f(\zeta) - \alpha_1^2 g(\zeta + 1) - \alpha_2 \alpha_2 g(\zeta) - \alpha_2 g(\zeta + 1) = f(\zeta + 1)$$

$$g(\zeta + 3) - \alpha_1^2 g(\zeta + 1) - \alpha_2 g(\zeta + 1) - \alpha_1 \alpha_2 g(\zeta) = f(\zeta + 1) + \alpha_1 f(\zeta + 1)$$

$$\Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 3) - \{\alpha_1^2 \alpha_2\} \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 1) - \alpha_1 \alpha_2 \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta) = f(\zeta + 1) + \alpha_1 f(\zeta)$$

Again replace ζ by $\zeta + 1$

$$g(\zeta + 3) - \alpha_1 g(\zeta + 2) - \alpha_2 g(\zeta + 1) = f(\zeta + 1)$$

$$g(\zeta + 3) - \alpha_1 \{f(\zeta) + \alpha_1 g(\zeta + 1) + \alpha_2 g(\zeta)\} - \alpha_2 g(\zeta + 1) = f(\zeta + 1)$$

$$g(\zeta + 3) - \alpha_1 f(\zeta) - \alpha_1^2 g(\zeta + 1) - \alpha_2 \alpha_2 g(\zeta) - \alpha_2 g(\zeta + 1) = f(\zeta + 1)$$

$$g(\zeta + 3) - \alpha_1^2 g(\zeta + 1) - \alpha_2 g(\zeta + 1) - \alpha_1 \alpha_2 g(\zeta) = f(\zeta + 1) + \alpha_1 f(\zeta)$$

$$\Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 3) - \{\alpha_1^2 \alpha_2\} \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 1) - \alpha_1 \alpha_2 \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta) = f(\zeta + 1) + \alpha_1 f(\zeta)$$

Again replace ζ by $\zeta + 1$

$$g(\zeta + 4) - \alpha_1^2 g(\zeta + 2) - \alpha_2 g(\zeta + 2) - \alpha_1 \alpha_2 g(\zeta + 1) = f(\zeta + 2) + \alpha_1 f(\zeta + 1)g(\zeta + 4) - \{\alpha_1^2 + \alpha_2\} \{\alpha_1 g(\zeta + 1) + \alpha_2 g(\zeta) + f(\zeta)\} - \alpha_1 \alpha_2 g(\zeta + 1)$$

$$= f(\zeta + 2) + \alpha_1 f(\zeta + 1)$$

$$g(\zeta + 4) - \alpha_1^3 g(\zeta + 1) - \alpha_1^2 \alpha_2 g(\zeta) - \alpha_1^2 f(\zeta) - \alpha_1 \alpha_2 g(\zeta + 1) - \alpha_2^2 g(\zeta) - \alpha_2 f(\zeta) - \alpha_1 \alpha_2 g(\zeta + 1)$$

$$= f(\zeta + 2) + \alpha_1 f(\zeta + 1)$$

$$g(\zeta + 4) - \{\alpha_1^3 + 2\alpha_1 \alpha_2\} g(\zeta + 1) - \{\alpha_1^2 \alpha_2 + \alpha_2^2\} g(\zeta)$$

$$= f(\zeta + 2) + \alpha_1 f(\zeta + 1) + \{\alpha_1^2 + \alpha_2\} f(\zeta)$$

Again replace ζ by $\zeta + 1$

$$g(\zeta + 5) - \{\alpha_1^3 + 2\alpha_1 \alpha_2\} g(\zeta + 2) - \{\alpha_1^2 \alpha_2 + \alpha_2^2\} g(\zeta + 1)$$

$$= f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 5) - \{\alpha_1^3 + 2\alpha_1\alpha_2\} \{\alpha_1g(\zeta + 1) + \alpha_2g(\zeta) + f(\zeta)\} - \{\alpha_1^2\alpha_2 + \alpha_2^2\} g(\zeta + 1) \\ = f(\zeta + 3) + \alpha_1f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 5) - \alpha_1^4g(\zeta + 1) - \alpha_1^3\alpha_2g(\zeta) - \alpha_1^3f(\zeta) - 2\alpha_1^2\alpha_2g(\zeta + 1) - 2\alpha_1\alpha_2^2g(\zeta) - 2\alpha_1\alpha_2f(\zeta) \\ - \{\alpha_1^2\alpha_2 + \alpha_2^2\} g(\zeta + 1) \\ = f(\zeta + 3) + \alpha_1f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 5) - \{\alpha_1^4 + 3\alpha_1^2\alpha_2 + \alpha_2^2\} g(\zeta + 1) - \{\alpha_1^3\alpha_2 + 2\alpha_1\alpha_2^2\} g(\zeta) \\ = f(\zeta + 3) + \alpha_1f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1) + \{\alpha_1^3 + 2\alpha_1\alpha_2\} f(\zeta)$$

Again replace ζ by $\zeta + 1$

$$g(\zeta + 6) - \{\alpha_1^4 + 3\alpha_1^2\alpha_2 + \alpha_2^2\} g(\zeta + 1) - \{\alpha_1^3\alpha_2 + 2\alpha_1\alpha_2^2\} g(\zeta + 1) \\ = f(\zeta + 4) + \alpha_1f(\zeta + 3) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 2) + \{\alpha_1^3 + 2\alpha_1\alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 6) - \{\alpha_1^4 + 3\alpha_1^2\alpha_2 + \alpha_2^2\} \{\alpha_1g(\zeta + 1) + \alpha_2g(\zeta) + f(\zeta)\} - \{\alpha_1^3\alpha_2 + 2\alpha_1\alpha_2^2\} g(\zeta + 1) \\ = f(\zeta + 4) + \alpha_1f(\zeta + 3) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 2) + \{\alpha_1^3 + 2\alpha_1\alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 6) - \alpha_1^5g(\zeta + 1) - \alpha_1^4\alpha_2g(\zeta) - \alpha_1^4\alpha_2g(\zeta) - \alpha_1^4f(\zeta) - 3\alpha_1^3\alpha_2g(\zeta + 1) - 3\alpha_1^2\alpha_2g(\zeta) \\ - 3\alpha_1^2\alpha_2f(\zeta) - \alpha_1\alpha_2^2g(\zeta + 1) - \alpha_2^3g(\zeta) - \alpha_2^2f(\zeta) - \{\alpha_1^3\alpha_2 + 2\alpha_1\alpha_2^2\} g(\zeta + 1) \\ = f(\zeta + 4) + \alpha_1f(\zeta + 3) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 2) + \{\alpha_1^3 + 2\alpha_1\alpha_2\} f(\zeta + 1)$$

$$g(\zeta + 6) - \{\alpha_1^5 + 4\alpha_1^3\alpha_2 + 3\alpha_1\alpha_2^2\} g(\zeta + 1) - \{\alpha_1^4\alpha_2 + 3\alpha_1^2\alpha_1^2 + \alpha_2^3\} g(\zeta) \\ = f(\zeta + 4) + \alpha_1f(\zeta + 3) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 2) + \{\alpha_1^3 + 2\alpha_1\alpha_2\} f(\zeta + 1) + \{\alpha_1^4 + 3\alpha_1^2\alpha_2 + \alpha_2^2\} f(\zeta)$$

In general,

$$F_0f(\zeta + ml) + F_1f(\zeta + m - 1) + F_2f(\zeta + m - 2) + \dots + F_mf(\zeta) \\ = \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + m + 2) - [\alpha_1F_m + \alpha_2F_{m-1}] \Delta_{\alpha_1, \alpha_2}^{-1} f(\zeta + 1) - \alpha_2^2 F_m g(\zeta).$$

Example 2.3 Applying $f(\zeta) = 2^\zeta$, $m = 4$, $\alpha_1 = 3$, $\alpha_2 = 4$ and $\zeta = 3$ in theorem (2.2), we get

$$\begin{aligned}
 LHS &= 2^{\zeta+4} + \alpha_1 2^{\zeta+3} + \{\alpha_1^2 + \alpha_2\} 2^{\zeta+2} + \{\alpha_1^3 + 2\alpha_1 \alpha_2\} 2^{\zeta+1} + \{\alpha_1^4 + 3\alpha_1^2 \alpha_2 + \alpha_2 + \alpha_2^2\} 2^\zeta \\
 &= 2^{3+4} + 3(2^{\zeta+3}) + \{3^2 + 4\} 2^{3+2} + \{3^3 + 2(3)(4)\} 2^{3+1} + \{3^4 + 3(3)^2(4) + 4^2\} 2^3 \\
 &= 128 + 192 + 416 + 816 + 1640 = 3192 \\
 \Delta_{\alpha_1, \alpha_2} f(\zeta) &= f(\zeta + 2) - \alpha_1 f(\zeta) - \alpha_2 f(\zeta) \\
 \Delta_{\alpha_1, \alpha_2} 2^\zeta &= 2^{\zeta+2} - \alpha_1 2^{\zeta+1} - \alpha_2 2^\zeta \\
 &= 2^\zeta [2^2 - \alpha_1 2 - \alpha_2] \\
 RHS &= \Delta_{\alpha_1, \alpha_2}^{-1} 2^{\zeta+6} - [\alpha_1^5 + 4\alpha_1^3 \alpha_2 + 3\alpha_1 \alpha_2^2] \Delta_{\alpha_1, \alpha_2}^{-1} 2^{\zeta+1} - [\alpha_1^4 \alpha_2 + 3\alpha_1^2 \alpha_2^2 + \alpha_2^3] \Delta_{\alpha_1, \alpha_2}^{-1} 2^\zeta \\
 &= \frac{2^{3+6}}{4 - 2(3) - 4} - [3^5 + 4(3)^3(4) + 3(3)(4)^2] \frac{2^{3+1}}{4 - 2(3) - 4} \\
 &\quad - [3^4 + 3(3)^2(4)^2 + 4^3] \frac{2^3}{4 - 2(3) - 4} \\
 &= -85.3333 + 2184 + 1093.3333 = 3192.
 \end{aligned}$$

Therefore, $LHS = RHS$.

Definition 2.4 Let f be a real valued function and $\zeta \neq 0, \alpha \neq 1$. The $\alpha_1, \alpha_2, \alpha_3$ -delta operator on f is defined as

$$\Delta_{\alpha_1, \alpha_2, \alpha_3} f(\zeta) = f(\zeta + 3) - \alpha_1 f(\zeta + 2) - \alpha_2 f(\zeta + 1) - \alpha_3 f(\zeta). \quad (3)$$

The inverse of $\alpha_1, \alpha_2, \alpha_3$ -Delta operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta_{\alpha_1, \alpha_2, \alpha_3} g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta), \quad (4)$$

where c is an arbitrary constant.

Theorem 2.5 Let f be real-valued function, $\Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta)$ exists and $\alpha_1, \alpha_2, \alpha_3 \neq 0$, then

$$\begin{aligned}
 &F_0 f(\zeta + m) + F_1 f(\zeta + m - 1) + F_2 f(\zeta + m - 2) + \dots + F_m f(\zeta) \\
 &= \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta + m + 3) - [\alpha_1 F_m + \alpha_2 F_{m-1} + \alpha_3 F_{m-2}] \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta + 2) \\
 &\quad - [\alpha^2 F_m + \alpha_3 F_{m-1}] \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta + 1) - \alpha_3 F_m \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} f(\zeta),
 \end{aligned}$$

$$\forall m \in \mathbb{Z}^+$$

where $F_0 = 1, F_1 = \alpha_1$ and $F_m = \alpha_1 F_{m-1} + \alpha_2 F_{m-2} + \alpha_3 F_{m-3}$

Proof: From the definition(2.4)

$$\Delta_{(\alpha_1, \alpha_2, \alpha_3)} f(\zeta) = f(\zeta + 3) - \alpha_1 f(\zeta + 2) - \alpha_2 f(\zeta + 1) - \alpha_3 f(\zeta)$$

$$\text{Let } g(\zeta) = \Delta_{(\alpha_1, \alpha_2, \alpha_3)}^{-1} f(\zeta)$$

$$\Delta_{(\alpha_1, \alpha_2, \alpha_3)}^{-1} g(\zeta) = f(\zeta)$$

$$g(\zeta + 3) - \alpha_1 g(\zeta + 2) - \alpha_2 g(\zeta + 1) - \alpha_3 g(\zeta) = f(\zeta)$$

$$g(\zeta + 3) = f(\zeta) + \alpha_1 g(\zeta + 2) + \alpha_2 g(\zeta + 1) + \alpha_3 g(\zeta)$$

replace ζ by $\zeta + 1$

$$g(\zeta + 4) = f(\zeta + 1) + \alpha_1 g(\zeta + 3) + \alpha_2 g(\zeta + 2) + \alpha_3 g(\zeta + 1) \quad g(\zeta + 4)$$

$$= f(\zeta + 1) + \alpha_1 \{f(\zeta) + \alpha_1 g(\zeta + 2) + \alpha_2 g(\zeta + 1) + \alpha_3 g(\zeta)\}$$

$$+ \alpha_2 g(\zeta + 2) + \alpha_3 g(\zeta + 1)$$

$$g(\zeta + 4) = f(\zeta + 1) + \alpha_1 f(\zeta) + \{\alpha_1^2 + \alpha_2\} g(\zeta + 2) + \{\alpha_1 \alpha_2 + \alpha_3\} g(\zeta + 1) + \alpha_1 \alpha_3 g(\zeta)$$

replace ζ by $\zeta + 1$

$$g(\zeta + 5) = f(\zeta + 2) + \alpha_1 f(\zeta + 1) + \{\alpha_1^2 + \alpha_2\} g(\zeta + 3) + \{\alpha_1 \alpha_2 + \alpha_3\} g(\zeta + 2) + \alpha_1 \alpha_2 g(\zeta + 1)$$

$$g(\zeta + 5) = f(\zeta + 2) + \alpha_1 f(\zeta + 1) + \{\alpha_1^2 + \alpha_2\} \{f(\zeta) + \alpha_1 g(\zeta + 2) + \alpha_2 g(\zeta + 1) + \alpha_3 g(\zeta)\}$$

$$+ \{\alpha_1 \alpha_2 + \alpha_3\} g(\zeta + 2) + \alpha_1 \alpha_2 g(\zeta + 1)$$

$$= f(\zeta + 2) + \alpha_1 f(\zeta + 1) + \alpha_1^2 f(\zeta) + \alpha_1^3 g(\zeta + 2) + \alpha_1^2 \alpha_2 g(\zeta + 1) + \alpha_1^2 \alpha_3 g(\zeta)$$

$$+ \alpha_2 f(\zeta) + \alpha_1 \alpha_2 g(\zeta + 2) + \alpha_2^2 g(\zeta + 1) + \alpha_2 \alpha_3 g(\zeta) + \{\alpha_1 \alpha_2 + \alpha_3\} g(\zeta + 2)$$

$$+ \alpha_1 \alpha_3 g(\zeta + 1)$$

$$g(\zeta + 5) = f(\zeta + 2) + \alpha_1 f(\zeta + 1) + \{\alpha_1^2 + \alpha_2\} f(\zeta) + \{\alpha_1^3 + \alpha_1 \alpha_2\} g(\zeta + 2)$$

$$+ \{\alpha_1^2 \alpha_2 + \alpha_2^2\} g(\zeta + 1) + \{\alpha_1^2 \alpha_3 + \alpha_2 \alpha_3\} g(\zeta)$$

Again replace ζ by $\zeta + 1$

$$g(\zeta + 6) = f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1) + \{\alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3\} g(\zeta + 3) \quad +$$

$$\{\alpha_1^2 \alpha_2 + \alpha_2^2\} g(\zeta + 2) + \{\alpha_1^2 \alpha_3 + \alpha_2 \alpha_3\} g(\zeta + 1)$$

$$= f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{\alpha_1^2 + \alpha_2\} f(\zeta + 1) + \{\alpha_1^3 + 2\alpha_1 \alpha_2 \alpha_3\}$$

$$\{f(\zeta) + \alpha_1 g(\zeta + 2) + \alpha_2 g(\zeta + 1) + \alpha_3 g(\zeta)\} + \{\alpha_1^2 \alpha_2 + \alpha_2^2 + \alpha_3\} g(\zeta + 2)$$

$$\begin{aligned}
 & + \{ \alpha_1^2 \alpha_3 + \alpha_2 \alpha_3 \} g(\zeta + 1) \\
 & = f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 1) + \alpha_1^3 f(\zeta) + \alpha_1^4 g(\zeta + 2) \\
 & + \alpha_1^3 \alpha_2 g(\zeta + 1) + \alpha_1^3 + \alpha_1^3 \alpha_3 g(\zeta) + 2\alpha_1 \alpha_2 f(\zeta) + 2\alpha_1^2 \alpha_2 g(\zeta + 2) \\
 & + 2\alpha_1 \alpha_2^2 g(\zeta + 1) + 2\alpha_1 \alpha_2 \alpha_3 g(\zeta) + \alpha_3 f(\zeta) + \alpha_1 \alpha_3 g(\zeta + 2) + \alpha_2 \alpha_3 g(\zeta + 1) \\
 & + \alpha_3^2 g(\zeta) + \{ \alpha_1^2 \alpha_2 + \alpha_2^2 + \alpha_1 \alpha_3 \} g(\zeta + 2) + \{ \alpha_1^2 \alpha_3 + \alpha_2 \alpha_3 \} g(\zeta + 1) \\
 g(\zeta + 6) & = f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 1) + \{ \alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3 \} f(\zeta) \\
 & + \{ \alpha_1^4 + 2\alpha_1^2 \alpha_2 + \alpha_1 \alpha_3 + \alpha_1^2 \alpha_2 + \alpha_2^2 + \alpha_1 \alpha_3 \} g(\zeta + 2) \\
 & + \{ \alpha_1^3 \alpha_2 + 2\alpha_1 \alpha_2^2 + \alpha_2 \alpha_3 + \alpha_1^2 \alpha_3 + \alpha_2 \alpha_3 \} g(\zeta + 1) \\
 & + \{ \alpha_1^3 \alpha_3 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_3^2 \} g(\zeta) \\
 g(\zeta + 6) & - \{ \alpha_1^4 + 3\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_3 + \alpha_2^2 \} g(\zeta + 2) - \{ \alpha_1^3 \alpha_2 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_3^2 \} g(\zeta) \\
 & = f(\zeta + 3) + \alpha_1 f(\zeta + 2) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 1) + \{ \alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3 \} f(\zeta)
 \end{aligned}$$

Again replace ζ by $\zeta + 1$

$$\begin{aligned}
 g(\zeta + 7) & - \{ \alpha_1^4 + 3\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_3 + \alpha_2^2 \} g(\zeta + 3) \\
 & - \{ \alpha_1^3 \alpha_2 + 2\alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_3 + 2\alpha_2 \alpha_3 \} g(\zeta + 2) - \{ \alpha_1^3 \alpha_3 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_3^2 \} g(\zeta + 1) \\
 & = f(\zeta + 4) + \alpha_1 f(\zeta + 3) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 2) + \{ \alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3 \} f(\zeta + 1) \\
 g(\zeta + 7) & - \{ \alpha_1^4 + 3\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_3 + \alpha_2^2 \} \{ f(\zeta) + \alpha_1 g(\zeta + 2) + \alpha_2 g(\zeta + 1) + \alpha_3 g(\zeta) \} \\
 & - \{ \alpha_1^3 \alpha_2 + 2\alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_3 + 2\alpha_1 \alpha_3 \} g(\zeta + 2) - \{ \alpha_1^3 \alpha_3 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_3^2 \} g(\zeta + 1) \\
 & = f(\zeta + 4) + \alpha_1 f(\zeta + 3) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 2) + \{ \alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3 \} f(\zeta + 1) \\
 g(\zeta + 7) & - \{ \alpha_1^5 + 3\alpha_1^3 \alpha_2 + 2\alpha_1^2 \alpha_3 + \alpha_1 \alpha_2^2 + \alpha_1^3 \alpha_2 + 2\alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_3 + 2\alpha_2 \alpha_3 \} g(\zeta + 2) \\
 & - \{ \alpha_1^4 \alpha_2 + 3\alpha_1^2 \alpha_2^2 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_2^3 + \alpha_1^3 \alpha_3 + 2\alpha_1 \alpha_2 \alpha_3 + \alpha_3^2 \} g(\zeta + 1) \\
 & - \{ \alpha_1^4 \alpha_3 + 3\alpha_1^2 \alpha_2 \alpha_3 + 2\alpha_1 \alpha_3^2 + \alpha_1^2 \alpha_3 \} g(\zeta) \\
 & = f(\zeta + 4) + \alpha_1 f(\zeta + 3) + \{ \alpha_1^2 + \alpha_2 \} f(\zeta + 2) + \{ \alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3 \} f(\zeta + 1) \\
 & + \{ \alpha_1^4 + 3\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_3 + \alpha_2^2 \} f(\zeta)
 \end{aligned}$$

$$\begin{aligned}
 F_0 f(\zeta + 4) & + F_1 f(\zeta + 3) + \{ \alpha_1 F_1 + \alpha_2 F_2 \} f(\zeta + 2) + \{ \alpha_1 F_2 + \alpha_2 F_1 + \alpha_3 F_0 \} f(\zeta + 1) \\
 & + \{ \alpha_1 F_3 + \alpha_2 F_2 + \alpha_3 F_1 \} f(\zeta) \\
 & = g(\zeta + 7) - \{ \alpha_1 F_4 + \alpha_2 F_3 + \alpha_3 F_2 \} - \{ \alpha_2 F_4 + \alpha_3 F_3 \} g(\zeta + 1) - \alpha_3 F_4 g(\zeta)
 \end{aligned}$$

In general

$$F_0 f(\zeta + m) + F_1 f(\zeta + m - 1) + F_2 f(\zeta + m - 2) + \dots + F_m f(\zeta)$$

$$= g(\zeta + m + 3) - \sum_{r=1}^3 \alpha_r F_{m-(r-1)} g(\zeta + n - 1) - \sum_{r=2}^3 \alpha_r F_{m-(r-2)} g(\zeta + n - 2) - \alpha_3 F_3 g(\zeta).$$

Example 2.6 Applying $f(\zeta) = 2^\zeta, m = 4, \alpha_1 = 3, \alpha_2 = 4, \alpha_3 = 5, \zeta = 2$ in theorem (2.5), we get

$$\begin{aligned} LHS &= f(\zeta + 4) + \alpha_1 f(\zeta + 3) + [\alpha_1^2 + \alpha_2] f(\zeta + 2) + [\alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3] f(\zeta + 1) \\ &\quad + [\alpha_1^4 + 3\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_3 + \alpha_2^2] f(\zeta) \\ &= 2^{\zeta+4} + \alpha_1 2^{\zeta+3} + [\alpha_1^2 + \alpha_2] 2^{\zeta+2} + [\alpha_1^3 + 2\alpha_1 \alpha_2 + \alpha_3] 2^{\zeta+1} + [\alpha_1^4 + 3\alpha_1^2 \alpha_2 + \\ &\quad 2\alpha_1 \alpha_3 + \alpha_2^2] 2^\zeta \\ &= 2^{2+4} + (3)2^{2+3} + [3^2 + 4]2^{2+2} + [3^3 + 2(3)(4) + 5]2^{2+1} \\ &\quad + [3^4 + 3(3)^2(4) + 2(3)(5) + 4^2]2^2 \\ &= 64 + 96 + 208 + 448 + 940 = 1756 \\ RHS &= \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} 2^{\zeta+7} - [\alpha_1 F_4 + \alpha_2 F_3 + \alpha_3 F_2] \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} 2^{\zeta+2} - [\alpha_2 F_4 + \alpha_3 F_3] \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} 2^{\zeta+1} \\ &\quad - \alpha_3 F_4 \Delta_{\alpha_1, \alpha_2, \alpha_3}^{-1} 2^\zeta \\ &= \frac{2^{\zeta+7}}{8 - 4\alpha_1 - 2\alpha_2 - \alpha_3} - [235\alpha_1 + 56\alpha_2 + 13\alpha_3] \frac{2^{\zeta+2}}{8 - 4\alpha_1 - 2\alpha_2 - \alpha_3} \\ &\quad - [235\alpha_2 + 56\alpha_3] \frac{2^{\zeta+1}}{8 - 4\alpha_1 - 2\alpha_2 - \alpha_3} - 235\alpha_3 \frac{2^\zeta}{8 - 4\alpha_1 - 2\alpha_2 - \alpha_3} \\ &= \frac{2^{2+7}}{8 - 4(3) - 2(4) - 5} - [(3)235 + (4)56 + (5)13] \frac{2^{2+2}}{8 - 4(3) - 2(4) - 5} \\ &\quad - [4(235) + 5(56)] \frac{2^{2+1}}{8 - 4(3) - 2(4) - 5} - 5(235) \frac{2^2}{8 - 4(3) - 2(4) - 5} \\ &= -30.11764706 + 935.5294118(0.941176471) + 1220(0.470588235) \\ &\quad + 1175(-1.411764706) \\ &= -30.11764706 + 935.5294118 + 574.1176471 + 276.4705882 = 1756 \end{aligned}$$

Therefore, $LHS = RHS$.

Theorem 2.7 Let f be real-valued function, $\Delta_{\alpha_1, \alpha_2, \dots, \alpha_n}^{-1} f(\zeta)$ exists and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \neq 0$, then

$$\begin{aligned}
 &F_0 f(\zeta + m) + F_1 f(\zeta + m - 1) + F_1 f(\zeta + m - 1) + F_2 f(\zeta + m - 2) \cdots + F_m f(\zeta) \\
 &= g(\zeta + m + n) - \sum_{r=1}^n \alpha_r F_{m-(r-1)} g(\zeta + n - 1) - \sum_{r=2}^n \alpha_r F_{m-(r-2)} g(\zeta + n - 2) - \cdots \\
 &\quad - \sum_{r=n}^n \alpha_r F_{m-(r-n)} g(\zeta)
 \end{aligned}$$

where $F_0 = 1, F_1 = \alpha_1$ and $F_m = \sum_{r=1}^m \alpha_r F_{m-r}$.

3 Conclusion

From this work, We developed the theory of discrete version of fundamental theorems using α -delta, $\alpha(h)$ -delta, α -nabla and $\alpha(h)$ -nabla operators. This theory is subsequently applied to establish several higher order fundamental theorems. Also we derived fundamental theorems using first order α -delta integration with several parameters. Our findings are validated with suitable examples in the field of discrete fractional calculus.

References

- [1] Agarwal RP, Difference Equations and Inequalities, Theory, Methods and Applications, 2/e, Revised and Expanded, Marcel Dekker, New York, 2000.
- [2] Britto Antony Xavier G, Vasantha kumar SU, and Mohan B, Sum of two dimensional Fibonacci sequence by solutions of second order difference equations, Ser.A:Appl.math.inform.and Mech.Vol.8,2 (2016), 131-138.
- [3] Falcon S and Plaza A, "On the Fibonacci k -numbers", Chaos, Solitons and Fractals, vol.32, no.5, pp. 1615-1624, 2007.
- [4] Ferreira RAC and Torres DFM, Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics, 5(1) (2011), 110-121.
- [5] Jerzy Popenda and Blazej Szmanda, On the Oscillation of Solutions of Certain Difference Equations, Demonstratio Mathematica, XVII(1), (1984), 153 - 164.

- [6] Maria Susai Manuel M, Britto Antony Xavier G and Thandapani.E, Theory of Generalized Difference Operator and Its Applications, Far East Journal of Mathematical Sciences, 20(2) (2006), 163 - 171.
- [7] Maria Susai Manuel M, Chandrasekar V and Britto Antony Xavier G, Solutions and Applications of Certain Class of α -Difference Equations, International Journal of Applied Mathematics, 24(6) (2011), 943-954.
- [8] Maria Susai Manuel M, Chandrasekar V and Britto Antony Xavier G, Theory of Generalized α -Difference Operator and its Applications in Number Theory, Advances in Differential Equations and Control Processes, 9(2) (2012), 141-155.
- [9] Susai Manuel M, Britto Antony Xavier G, Chandrasekar V and Pugalarasu R, Theory and application of the Generalized Difference Operator of the n^{th} kind(Part I), Demonstratio Mathematica, 45(1)(2012), 95-106.