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Stability Analysis of a k-Delayed Rational Difference Equation with Exponential Response

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Abstract

In this paper, we study the asymptotic and boundedness behavior of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha e^{-x_n} + \beta x_{n-1}}{A + Bx_n + Cx_{n-k}} \tag{1}$$

$n = 0, 1, 2, \dots,$ where α, β, A, B, C are positive real numbers, $k \in \{1, 2, \dots\}$, and initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers.

Key words: Equilibrium point, Local asymptotic stability, Global asymptotic stability, Positive solutions, Boundedness, Periodicity.

AMS classification: 39A22, 34D23, 74G25.

1 Introduction

Difference equations are widely recognized as valuable tools for representing discrete-time processes in fields like population biology, economics, and engineering. Within this framework, rational difference equations defined by recurrence relations involving rational functions stand out due to their complex and varied dynamic properties, making them a key area of interest in the study of nonlinear dynamics. Although much attention has been given to understanding the boundedness, periodicity, and stability of lower-order rational difference equations (such as those of first and second order), higher-order rational difference equations have received comparatively little attention, despite the significant theoretical challenges and practical importance they present.

Global stability, boundedness nature and periodic character of the positive solution of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},$$

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$n = 0, 1, 2, \dots$, was investigated by El-Metwally et all [9], where $\alpha > 0$ and $\beta > 0$ are the immigration rate and population growth respectively and the initial conditions x_{-1} and x_0 are arbitrary non negative numbers.

Boundedness and global asymptotic behavior of the solution of the difference equations

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}},$$

$n = 0, 1, 2, \dots$, was studied by Ozturk et all [16], where α and β are positive numbers $k \in \{1, 2, 3, \dots\}$ and the $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$ are arbitrary numbers.

Stability analysis of a nonlinear difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}},\ n = 0, 1, 2, \dots,$$

was established in [2], where α, β and initial conditions are arbitrary positive numbers.

Properties of solutions of various types of second and third order rational difference equations were discussed in [13, 8]. Stability properties and conditions for boundedness of nonlinear difference equation $x_{n+1} = f(x_n)g(x_{n-k})$ was studied in [15] and asymptotic properties of solutions of the difference equation $y_n = \dfrac{f(y_{n-1}, \dots, y_{n-k})}{g(y_{n-1}, \dots, y_{n-k})}, n = 0, 1, 2, \dots$, was studied in [1].

Recent research has highlighted how the inclusion of higher-order terms in difference equations can lead to complex dynamical phenomena such as bifurcations, chaotic dynamics and multi-stability-features that do not typically arise in simpler, lower-order systems.

Motivated by the above studies, we wish to investigate the global attractivity and boundedness of the solutions of the higher order nonlinear autonomous rational difference equation (1). Here the model (1) represents the situation when a species, say x_{n+1}, it's growth depends on the n-th, $(n-1)$-th and $(n-k)$-th generation of x_{n+1}. The combination of theoretical analysis with numerical examples given in this work, reveals how the structure of higher-order rational difference equations affects their long-term behavior.

2 Boundedness and Persistence

In this section, we discuss the boundedness, persistence, invariance and periodic nature of the solutions of equation (1).

Theorem 2.1 Let α, β, A, B, C are positive real numbers such that $A > \beta$. Then following statements are true.

(i). Every solution of (1) is positive, bounded and persists.

(ii). The set $I = \left[0, \dfrac{\alpha}{A-\beta}\right]$ is an invariant set for (1).

(iii). Let $\epsilon > 0$ be any arbitrary positive number and $\{x_n\}$ be an arbitrary solution of (1). Consider the interval J=$\left[0, \dfrac{\alpha + A\epsilon}{A-\beta}\right]$. Then there exist an $n_0 \in \mathbb{N}$ such that $x_n \in J$ for all $n \geq n_0$.

Proof:

(i). Let $\{x_n\}$ be an arbitrary positive solution of (1).
Clearly,

$$x_n \geq 0 \qquad \text{for} \qquad n = 1, 2, \cdots . \tag{2}$$

Further from (1),

$$x_{n+1} \leq \frac{\alpha}{A} + \frac{\beta}{A} x_{n-1} \quad \text{for} \quad n \geq 1. \tag{3}$$

Consider the non-homogeneous difference equation

$$u_{n+1} = \frac{\alpha}{A} + \frac{\beta}{A} u_{n-1} \quad \text{for} \quad n \geq 1. \tag{4}$$

Let $\{u_n\}$ be a solution of (4) such that

$$u_0 = x_0, u_{-1} = x_{-1}, \cdots , u_{-k} = x_{-k}. \tag{5}$$

Then

$$u_n = r_1 \left(\frac{\beta}{A}\right)^{n/2} + r_2 (-1)^n \left(\frac{\beta}{A}\right)^{n/2} + \frac{\alpha}{A-\beta} \tag{6}$$

where r_1, r_2 are constants.
Using (3)-(5) we can prove by induction that

$$x_n \leq u_n, \qquad \text{for} \qquad n = -k, -k+1, \cdots \tag{7}$$

Then from (2),(6) and (7) we obtain that every solution of (1) is bounded.

(ii). Let $\{x_n\}$ be a positive solution of (1) with initial values $x_0, x_1, \cdots , x_k$ such that

$x_0, x_1, \cdots, x_k \in I$. Then from (1) we get

$$0 \leq x_{k+1} = \frac{\alpha e^{-x_k} + \beta x_{k-1}}{A + Bx_k + Cx_0} \leq \frac{\alpha}{A} + \frac{\beta}{A} x_{k-1}$$
$$\leq \frac{\alpha}{A} + \frac{\beta}{A}\left(\frac{\alpha}{A-\beta}\right)$$
$$= \frac{\alpha}{A-\beta}.$$

Therefore, $x_{k+1} \in I$.

Working inductively we have, $x_n \in I$ for all n.

(iii). Let $l = \liminf x_n$ and $L = \limsup x_n$. Then,

$$L \leq \frac{\alpha e^{-l} + \beta L}{A + (B+C)l} \leq \frac{\alpha}{A} + \frac{\beta}{A} L.$$

Thus, $L \leq \dfrac{\alpha}{A-\beta}$. Since, $L = \limsup x_n$, for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$x_n < L + \epsilon \leq \frac{\alpha}{A-\beta} + \epsilon < \frac{\alpha + A\epsilon}{A-\beta} \qquad \text{for all} \quad n \geq n_0.$$

Therefore, $x_n \in \left[0, \dfrac{\alpha + A\epsilon}{A-\beta}\right]$ for all $n \geq n_0$.

Lemma 2.2 Suppose $A > \alpha + \beta$, then equation (1) has no solution with prime period two.

Proof: Assume that there exist distinct positive real numbers ϕ and ψ such that

$$\cdots, \phi, \psi, \phi, \psi, \cdots$$

forms a period 2 solution for (1).

Suppose k is odd. Then $x_{n+1} = x_{n-k}$ and ϕ and ψ satisfy the system

$$\phi = \frac{\alpha e^{-\psi} + \beta\phi}{A + B\psi + C\phi} \qquad \text{and} \qquad \psi = \frac{\alpha e^{-\phi} + \beta\psi}{A + B\phi + C\psi}.$$

This imples

$$A\phi + B\phi\psi + C\phi^2 = \alpha e^{-\psi} + \beta\phi \tag{8}$$
$$A\psi + B\phi\psi + C\psi^2 = \alpha e^{-\phi} + \beta\psi. \tag{9}$$

Taking the differences of above two equations we get

$$A[\phi - \psi] + C\left[\phi^2 - \psi^2\right] = \alpha\left[e^{-\psi} - e^{-\phi}\right] + \beta[\phi - \psi]. \tag{10}$$

Then by Mean Value Theorem, there exists a θ_1 in between ϕ and ψ such that

$$e^{-\psi} - e^{-\phi} = e^{-\theta_1}[\phi - \psi].$$

Substituting above relation in (10) we get

$$\left[A + C(\phi + \psi) - \alpha e^{-\theta_1} - \beta\right][\phi - \psi] = 0. \tag{11}$$

But since $A > \alpha+\beta$, the first term in the above product is always positive. Therefore, the relation (11) is true only if $\phi = \psi$, a contradiction.
Suppose k is even. Then $x_n = x_{n-k}$ and ϕ and ψ satisfy the system

$$\phi = \frac{\alpha e^{-\psi} + \beta\phi}{A + (B + C)\psi} \qquad \text{and} \qquad \psi = \frac{\alpha e^{-\phi} + \beta\psi}{A + (B + C)\phi}.$$

This imples

$$A\phi + B\phi\psi + C\phi\psi = \alpha e^{-\psi} + \beta\phi \tag{12}$$
$$A\psi + B\phi\psi + C\phi\psi = \alpha e^{-\phi} + \beta\psi. \tag{13}$$

Taking the differences of above two equations we get

$$A[\phi - \psi] = \alpha\left[e^{-\psi} - e^{-\phi}\right] + \beta[\phi - \psi]. \tag{14}$$

Then by Mean Value Theorem, there exists a θ_2 in between ϕ and ψ such that

$$e^{-\psi} - e^{-\phi} = e^{-\theta_2}[\phi - \psi].$$

Substituting above relation in (14) we get

$$\left[A - \alpha e^{-\theta_2} - \beta\right][\phi - \psi] = 0. \tag{15}$$

But since $A > \alpha+\beta$, the first term in the above product is always positive. Therefore, the relation (15) is true only if $\phi = \psi$, a contradiction.
Hence, there does not exist a period 2 solution for (1).

3 Global Behaviour

In this section we discuss about the stability of solutions of the equation (1). Here we provide the conditions for the solutions of (1) to be globally and locally asymptotically stable.
The equilibrium points of equation (1) are the solutions of the equation

$$\bar{x} = \frac{\alpha e^{-\bar{x}} + \beta\bar{x}}{A + (B+C)\bar{x}}. \tag{16}$$

Set $g(x) = \dfrac{\alpha e^{-x} + \beta x}{A + (B+C)x} - x.$
Then we get
$g(0) = \dfrac{\alpha}{A} > 0,\ \lim\limits_{x\to\infty} g(x) = -\infty$ and

$$g'(x) = \frac{\beta}{[A + (B+C)x]} - \frac{\alpha e^{-x}}{[A + (B+C)x]} - \frac{(B+C)[\alpha e^{-x} + \beta x]}{[A + (B+C)x]^2} - 1. \tag{17}$$

Let z be a solution of $g(x) = 0$. Clearly $z \neq 0$ and
$g(z) = 0 \implies z = \dfrac{\alpha e^{-z}}{[A + (B+C)z]} + \dfrac{\beta z}{[A + (B+C)z]}.$
Then (17) becomes,

$$g'(z) = \frac{-\alpha e^{-z}}{z[A + (B+C)z]} - \frac{\alpha e^{-z}}{[A + (B+C)z]} - \frac{(B+C)[\alpha e^{-z} + \beta z]}{[A + (B+C)z]^2} < 0.$$

Therefore, there exists an $\epsilon > 0$ such that $g'(x) < 0 \quad$ for all $x \in (z-\epsilon, z+\epsilon)$. Hence, g is decreasing on the interval $(z-\epsilon, z+\epsilon)$. Suppose g has roots greater than the root z. let z_1 be the smallest root of g such that $z_1 > z$. From the argument above, we can show that there exists an ϵ_1 such that g is decreasing in the interval $(z_1-\epsilon_1, z_1+\epsilon_1)$. Since, $g(z+\epsilon) < 0, g(z_1-\epsilon_1) > 0$ and g is continuous we see that g must have a root in the interval $(z+\epsilon, z_1-\epsilon_1)$. This is a contradiction, since z_1 is the smallest root of g such that $z_1 > z$. Similarly, we can prove that g has no solution in $(0, z)$. Therefore, the equation $g(x) = 0$ must have a unique solution. Thus equation (1) has a unique positive equilibrium.

Theorem 3.1 Let α, β, A, B, C are positive real numbers such that $A > \beta$. Then the equilibrium point of (1) is bounded .
Proof: Since the equilibrium point of (1) is positive, we have $\bar{x} > 0$.
Then

$$0 < \bar{x} = \frac{\alpha e^{-\bar{x}} + \beta\bar{x}}{A + (B+C)\bar{x}} \leq \frac{\alpha}{A} + \frac{\beta}{A}\bar{x}.$$

This implies

$$0 < \bar{x} \leq \frac{\alpha}{A-\beta}.$$

Therefore the equilibrium point is bounded.

A modest modification of Theorem 1.15 in [10] is the following theorem.

Theorem 3.2 Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-k}), \quad n = 0, 1, 2, \cdots \tag{18}$$

where $k \in \{1, 2, 3, \cdots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that $f : [a, b] \times [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties,

(a) $f(u, v, w)$ is non-increasing in u.
(b) $f(u, v, w)$ is non-decreasing in v.
(c) $f(u, v, w)$ is non-increasing in w.
(d) If (m, M) is a solution of the system $m = f(M, m, M)$ and $M = f(m, M, m)$ then $m = M$.

Then equation (18) has a unique positive equilibrium $\bar{x}$ and every solution of equation (18) converges to $\bar{x}$.

Theorem 3.3 Consider the difference equation (1) such that $A > \alpha + \beta$. Then (1) has a unique positive equilibrium $\bar{x}$ and every positive solution of (1) tends to the

unique positive equilibrium of (1) as $n \to \infty$.
Proof: Consider the continuous function $f : I \times I \times I \to I$ defined by

$$f(x, y, z) = \frac{\alpha e^{-x} + \beta y}{A + Bx + Cz}$$

where $I = \left[0, \dfrac{\alpha}{A-\beta}\right]$. In light of Theorem 2.1, this function is well defined. Also the function is non-increasing in x and z and is non-decreasing in y.
Let m, M be positive real numbers satisfying

$$m = \frac{\alpha e^{-M} + \beta m}{A + (B+C)M} \qquad \text{and} \qquad M = \frac{\alpha e^{-m} + \beta M}{A + (B+C)m} \tag{19}$$

This implies

$$\begin{aligned} Am + BmM + CmM &= \alpha e^{-M} + \beta m \\ AM + BmM + CmM &= \alpha e^{-m} + \beta M \end{aligned}$$

Taking difference we get

$$[A - \beta][M - m] = \alpha \left[e^{-m} - e^{-M}\right] \tag{20}$$

There exists ζ between m and M such that

$$e^{-m} - e^{-M} = e^{-\zeta}[M - m].$$

From (20)

$$\begin{aligned} [A - \beta][M - m] &= \alpha e^{-\zeta}[M - m] \\ &\leq \alpha[M - m]. \end{aligned}$$

This implies

$$[A - \beta]|M - m| \leq \alpha|M - m|.$$

Thus

$$[A - \alpha - \beta]|M - m| \leq 0.$$

Since $A > \alpha + \beta$, we have $M = m$. Hence by Theorem 3.2, equation (1) has a unique equilibrium point $\bar{x}$ and every positive solution of (1) converges to $\bar{x}$.

Theorem 3.4 Consider the difference equation (1) such that the condition $A > \alpha+\beta$ holds true. Also suppose that

$$\frac{\alpha+\beta}{A} + \frac{\alpha(B+C)}{A(A-\beta)} < 1. \tag{21}$$

Then the unique equilibrium point $\bar{x}$ of (1) is globally asymptotically stable.
Proof: The characteristic equation corresponding to (1) is

$$\lambda^{k+1} - P_1\lambda^k - P_2\lambda^{k-1} - P_3 = 0 \tag{22}$$

where $P_1 = \dfrac{-\alpha e^{-\bar{x}}}{[A+(B+C)\bar{x}]} - \dfrac{B\left[\alpha e^{-\bar{x}} + \beta\bar{x}\right]}{[A+(B+C)\bar{x}]^2}$, $\quad P_2 = \dfrac{\beta}{[A+(B+C)\bar{x}]}$
and $P_3 = \dfrac{-C[\alpha e^{-\bar{x}} + \beta\bar{x}]}{[A+(B+C)\bar{x}]^2}$.
Then

$$\begin{aligned}
|P_1| + |P_2| + |P_3| &= \frac{\alpha e^{-\bar{x}}}{[A+(B+C)\bar{x}]} + \frac{B\left[\alpha e^{-\bar{x}} + \beta\bar{x}\right]}{[A+(B+C)\bar{x}]^2} + \frac{\beta}{[A+(B+C)\bar{x}]} + \frac{C[\alpha e^{-\bar{x}} + \beta\bar{x}]}{[A+(B+C)\bar{x}]^2} \\
&\leq \frac{\alpha}{A} + \frac{B\alpha}{A^2} + \frac{B\beta\bar{x}}{A^2} + \frac{\beta}{A} + \frac{C\alpha}{A^2} + \frac{C\beta\bar{x}}{A^2} \\
&\leq \frac{\alpha+\beta}{A} + \frac{\alpha(B+C)}{A^2} + \frac{\beta(B+C)}{A^2}\bar{x} \\
&\leq \frac{\alpha+\beta}{A} + \frac{\alpha(B+C)}{A^2} + \frac{\beta(B+C)}{A^2}\left(\frac{\alpha}{A-\beta}\right), \qquad \text{From Theorem 3.1} \\
&\leq \frac{\alpha+\beta}{A} + \frac{\alpha(B+C)}{A^2}\left(1 + \frac{\beta}{A-\beta}\right) \\
&\leq \frac{\alpha+\beta}{A} + \frac{\alpha(B+C)}{A(A-\beta)} \\
&< 1
\end{aligned} \tag{23}$$

Therefore, all the roots of the equation (22) satisfy $|\lambda| < 1$. Then the equilibrium point $\bar{x}$ is locally asymptotically stable. By using Theorem 3.3, we see that $\bar{x}$ is globally asymptotically stable. This completes the proof of the theorem.

4 Numerical Illustrations

In this section, we present several numerical examples to validate our theoretical findings and reinforce our mathematical analysis.

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Example 4.1 Consider the difference equation (1) with $\alpha = 7, \beta = 4, A = 15, B = 2, C = 3.5$ and $k = 8$. Let the initial values be $x_0 = 0.6, x_{-1} = 1.72, x_{-2} = 0.05, x_{-3} = 2.98, x_{-4} = 0.59, x_{-5} = 2.6, x_{-6} = 0.8, x_{-7} = 2.95$ and $x_{-8} = 2.0$. Then Figure 1 shows that the solution of (1) converges to the equilibrium point 0.37061236234514. The parameters also satisfies the conditions stated in the Theorems 3.3 and 3.4. So the equilibrium point is a global attractor for every solution $\{x_n\}$ of (1).

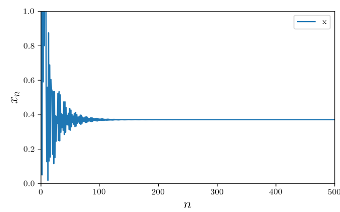

Figure 1: Plot of x_n

Example 4.2 Figure 2 represents the bifurcation diagram of the difference equation (1) with β as the bifurcation parameter. Here $\alpha = 2, A = 3, B = 2.5, C = 0.3$ and $k = 5$. Let the initial values be $x_5 = 8.5, x_4 = 7.5, x_3 = 7.7, x_2 = 8.7, x_1 = 4.4$ and $x_0 = 4.1$. Then diagram 2 shows that for values of β less than approximately 2, the difference equation (1) is stable. As β increases beyond 2, a bifurcation occurs, where the single stable state splits into two.

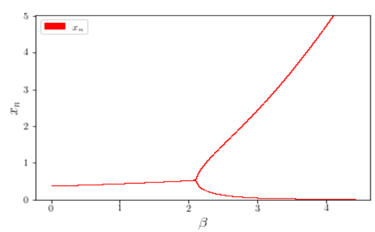

Figure 2: Bifurcaion diagram of (1) with β as bifurcation parameter where $\alpha = 2.0, A = 3.0, B = 2.5, C = 0.3$ and $k = 5.0$.

5 Conclusion

In this paper, we have investigated the dynamic behavior of the higher-order nonlinear autonomous rational difference equation (1), focusing on its boundedness, persistence, invariance, periodicity, and stability properties. The conditions established for local and global asymptotic stability, demonstrates when the equilibrium points act as global attractor for the model. Our analysis provides a comprehensive framework for understanding the complex long-term behavior of a model, which is essential for modeling real-world processes. These findings also lay the groundwork for future studies aimed at exploring even more intricate behaviors, such as chaotic dynamics and multi-stability, which are essential for accurately capturing the full spectrum of behaviors in real-world models.

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