

Asymptotic Analysis of a Two-Species Higher Order Rational Difference Model with Cross-Species Dependence

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Abstract

In this study, we analyze the asymptotic stability, persistence, and bounded behavior of the solutions to a system of higher-order rational difference equations. The model consists of two species, where the population of the first species in the n -th and $(n - k)$ -th generations, along with the population of the second species in the n -th and $(n - 1)$ -th generations, influences the growth of the first species. Similarly, the growth of the second species is determined by the population of the second species in the n -th and $(n - k)$ -th generations, along with the population of the first species in the n -th and $(n - 1)$ -th generations.

Key words: Difference equations, System, Boundedness, Persistence, Equilibrium point, Asymptotic behavior, Global asymptotic stability, Bifurcation.

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1 Introduction

Difference equation models are widely applied across various scientific disciplines. In the context of population dynamics, models addressing competitive interactions or predator-prey systems are especially noteworthy. These models are typically formulated as first or second order systems. A special case of these models, which incorporate qualitative characteristics of difference systems with exponential components, have been studied in [11]-[21] and [1]-[5]. In [20, 13, 14, 7], researchers investigated the boundedness, persistence, and global stability properties of various discrete dynamical systems featuring exponential components.

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This study introduces a $(k + 1)$ -th order difference system given by

$$\begin{aligned}x_{n+1} &= \frac{\alpha_1 e^{-x_n} + \beta_1 y_{n-1}}{A_1 + B_1 y_n + C_1 x_{n-k}} \\y_{n+1} &= \frac{\alpha_2 e^{-y_n} + \beta_2 x_{n-1}}{A_2 + B_2 x_n + C_2 y_{n-k}}\end{aligned}\tag{1}$$

$n = 0, 1, 2, \dots$, where the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive real numbers, $k \in \{1, 2, \dots\}$ and the initial conditions $x_0, x_{-1}, \dots, x_{-k}$ and $y_0, y_{-1}, \dots, y_{-k}$ are arbitrary non negative real numbers.

The system (1) represents the situation when a species say x_{n+1} , it's growth depends on the n -th and $(n - k)$ -th generation of x_{n+1} and the n -th and $(n - 1)$ -th previous generations of another species, say y_{n+1} . Whereas the species y_{n+1} is influenced by n -th and $(n - k)$ -th generation of y_{n+1} and the n -th and $(n - 1)$ -th previous generations of x_{n+1} .

This work focuses on analyzing the boundedness, persistence, convergence and asymptotic behavior of the positive solutions of (1).

2 Boundedness

Lemma 2.1 Let $\alpha_1, \alpha_2, \beta_1, \beta_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive real numbers such that $Q = \frac{\beta_1 \beta_2}{A_1 A_2} < 1$. Then every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.

Proof: Let $\{(x_n, y_n)\}$ be an arbitrary solution of system (1). Then it is clear that

$$x_n \geq 0 \quad \text{and} \quad y_n \geq 0 \quad \text{for all} \quad n \geq 1.\tag{2}$$

So every solution of (1) persists.

Also from (1)

$$x_{n+1} \leq \frac{\alpha_1}{A_1} + \frac{\beta_1}{A_1} y_{n-1} \quad \text{and}\tag{3}$$

$$y_{n+1} \leq \frac{\alpha_2}{A_2} + \frac{\beta_2}{A_2} x_{n-1} \quad \text{for all } n \geq 0.\tag{4}$$

Using (3) and (4) we have

$$x_{n+1} \leq P_1 + Qx_{n-3} \tag{5}$$

$$y_{n+1} \leq P_2 + Qy_{n-3} \tag{6}$$

where $P_1 = \frac{\alpha_1}{A_1} + \frac{\alpha_2\beta_1}{A_1A_2}$ and $P_2 = \frac{\alpha_2}{A_2} + \frac{\alpha_1\beta_2}{A_1A_2}$ for all $n \geq 0$.

Consider the system of difference equations

$$\begin{aligned} u_{n+1} &= P_1 + Qu_{n-3} \\ v_{n+1} &= P_2 + Qv_{n-3}, \quad n = 0, 1, 2, \dots \end{aligned} \tag{7}$$

Let (u_n, v_n) be a solution of (7) such that

$$\begin{array}{cccc} u_0 = x_0 & u_{-1} = x_{-1} & u_{-2} = x_{-2} & u_{-3} = x_{-3} \\ v_0 = y_0 & v_{-1} = y_{-1} & v_{-2} = y_{-2} & v_{-3} = y_{-3} \end{array}$$

Then from (7) we obtain $u_1 > 0$ and $v_1 > 0$.

Working inductively it follows that $u_n > 0$ and $v_n > 0$ for all $n \geq 1$.

Moreover again from (5) and (6), for all $n \geq 1$

$$\begin{aligned} u_n &= r_1Q^{\frac{n}{4}} + r_2(-1)^nQ^{\frac{n}{4}} + r_3Q^{\frac{n}{4}} \cos\left(\frac{n\pi}{4}\right) + r_4Q^{\frac{n}{4}} \sin\left(\frac{n\pi}{4}\right) + \frac{P_1}{1-Q} \\ v_n &= s_1Q^{\frac{n}{4}} + s_2(-1)^nQ^{\frac{n}{4}} + s_3Q^{\frac{n}{4}} \cos\left(\frac{n\pi}{4}\right) + s_4Q^{\frac{n}{4}} \sin\left(\frac{n\pi}{4}\right) + \frac{P_2}{1-Q} \end{aligned} \tag{8}$$

where r_1, r_2, r_3, r_4 and s_1, s_2, s_3, s_4 are constants defined by $x_0, x_{-1}, x_{-2}, x_{-3}$ and $y_0, y_{-1}, y_{-2}, y_{-3}$ respectively.

Using (5),(6) and (7) we can prove that

$$x_n \leq u_n \quad \text{and} \quad y_n \leq v_n \quad \text{for} \quad n = 1, 2, 3, \dots \tag{9}$$

Then from (2),(8) and (9) we obtain that every solution of (1) is bounded. This completes the proof of the lemma.

Lemma 2.2 Let $\alpha_1, \alpha_2, \beta_1, \beta_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive real numbers such that $Q = \frac{\beta_1\beta_2}{A_1A_2} < 1$. Then the following statements are true.

- (a) The set $\left[0, \frac{P_1}{1-Q}\right] \times \left[0, \frac{P_2}{1-Q}\right]$ is an invariant set for system (1).

- (b) Let ϵ be an arbitrary positive number and $\{(x_n, y_n)\}$ be an arbitrary solution of (1) and consider the intervals

$$J_1 = \left[0, \frac{P_1 + \epsilon}{1 - Q}\right] \quad \text{and} \quad J_2 = \left[0, \frac{P_2 + \epsilon}{1 - Q}\right]. \quad (10)$$

Then there exist an $m \in \mathbb{N}$ such that $x_n \in J_1$ and $y_n \in J_2$ for all $n \geq m$.

Proof:

- (a) Let $I_1 = \left[0, \frac{P_1}{1 - Q}\right]$ and $I_2 = \left[0, \frac{P_2}{1 - Q}\right]$. Let $x_0, x_{-1}, \dots, x_{-k} \in I_1$ and $y_0, y_{-1}, \dots, y_{-k} \in I_2$. Then from (1) we have

$$\begin{aligned} 0 \leq x_1 &= \frac{\alpha_1 e^{-x_0} + \beta_1 y_{-1}}{A_1 + B_1 y_0 + C_1 x_{-k}} \\ &\leq \frac{\alpha_1}{A_1} + \frac{\beta_1}{A_1} y_{-1} \\ &\leq \frac{\alpha_1}{A_1} + \frac{\beta_1 \left[\frac{\alpha_2}{A_2} + \frac{\alpha_1 \beta_2}{A_1 A_2} \right]}{A_1 [1 - Q]} \\ &= \frac{\alpha_1}{A_1} + \frac{\alpha_2 \beta_1}{A_1 A_2 [1 - Q]} + \frac{\alpha_1 Q}{A_1 [1 - Q]} \\ &= \frac{\alpha_1}{A_1 [1 - Q]} + \frac{\alpha_2 \beta_1}{A_1 A_2 [1 - Q]} \\ &= \frac{P_1}{1 - Q}. \end{aligned}$$

So, $x_1 \in I_1$. Similarly, $y_1 \in I_2$.

Working inductively we get $x_n \in I_1, y_n \in I_2 \quad \forall n$.

This completes the proof of part(a).

- (b) Let $\epsilon > 0$ be given and let $\{(x_n, y_n)\}$ be an arbitrary solution of (1). Then by Lemma (2.1) we get

$$\begin{aligned} 0 \leq l_1 = \liminf x_n, \quad 0 \leq l_2 = \liminf y_n \\ L_1 = \limsup x_n < \infty, \quad \text{and} \quad L_2 = \limsup y_n < \infty. \end{aligned} \quad (11)$$

It follows from (1) and (11) that

$$L_1 \leq \frac{\alpha_1 e^{-l_1} + \beta_1 L_2}{A_1 + B_1 l_2 + C_1 l_1} < \frac{\alpha_1}{A_1} + \frac{\beta_1}{A_1} L_2 \quad \text{and} \quad (12)$$

$$L_2 \leq \frac{\alpha_2 e^{-l_2} + \beta_2 L_1}{A_2 + B_2 l_1 + C_2 l_2} < \frac{\alpha_2}{A_2} + \frac{\beta_2}{A_2} L_1. \quad (13)$$

Combining (11),(12) and (13) we get

$$0 \leq L_1 \leq \frac{P_1}{1-Q} \quad \text{and} \quad 0 \leq L_2 \leq \frac{P_2}{1-Q}.$$

Thus there exists an $m \in \mathbb{N}$ such that $x_n < L_1 + \epsilon$ and $y_n < L_2 + \epsilon$ for all $n \geq m$. This implies $x_n \in J_1$ and $y_n \in J_2$ for all $n \geq m$. This completes the proof of part(b).

Lemma 2.3 If $Q = \frac{\beta_1 \beta_2}{A_1 A_2} < 1$, then the equilibrium point of system (1) is bounded.

Proof: Let (\bar{x}, \bar{y}) be an equilibrium point of system (1). Then we have

$$\bar{x} = \frac{\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}}{A_1 + B_1 \bar{y} + C_1 \bar{x}} \quad \text{and} \quad \bar{y} = \frac{\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}}{A_2 + B_2 \bar{x} + C_2 \bar{y}}. \quad (14)$$

From (14) it is clear that

$$\bar{x} \geq 0 \quad \text{and} \quad \bar{y} \geq 0 \quad (15)$$

and

$$\bar{x} \leq \frac{\alpha_1}{A_1} + \frac{\beta_1}{A_1} \bar{y}, \quad \bar{y} \leq \frac{\alpha_2}{A_2} + \frac{\beta_2}{A_2} \bar{x}. \quad (16)$$

From (15) and (16) we get

$$0 \leq \bar{x} \leq \frac{P_1}{1-Q} \quad \text{and} \quad 0 \leq \bar{y} \leq \frac{P_2}{1-Q}.$$

This completes the proof of the Lemma.

3 Global Behavior

The following Lemma is a minor adaptation of Theorem 1.16 from [8]. To aid the reader, we present it here without proof.

Lemma 3.1 Let $[a, b]$ and $[c, d]$ be intervals of real numbers. Let $\varrho : [a, b] \times [c, d] \times [c, d] \rightarrow [a, b]$ and $\zeta : [a, b] \times [a, b] \times [c, d] \rightarrow [c, d]$ be continuous functions. Consider the difference system

$$\begin{aligned}x_{n+1} &= \varrho(x_n, x_{n-k}, y_n, y_{n-1}) \\y_{n+1} &= \zeta(x_n, x_{n-1}, y_n, y_{n-k}) \quad n = 0, 1, \dots\end{aligned}\tag{17}$$

such that the initial values $x_{-k}, x_{-k+1}, \dots, x_0 \in [a, b]$, $y_{-k}, y_{-k+1}, \dots, y_0 \in [c, d]$. Suppose that the following statements are true.

1. If $\varrho(u_1, u_2, v_1, v_2)$ is non-increasing in u_1 , is non-increasing in u_2 , is non-increasing in v_1 and is non-decreasing in v_2 .
2. $\zeta(u_1, u_2, v_1, v_2)$ is non-increasing in u_1 , is non-decreasing in u_2 , is non-increasing in v_1 and is non-increasing in v_2 .
3. If $(m, M, r, R) \in [a, b]^2 \times [c, d]^2$ satisfies the systems

$$\begin{aligned}M &= \varrho(m, m, r, R) & m &= \varrho(M, M, R, r) \\R &= \zeta(m, M, r, r) & r &= \zeta(M, m, R, R)\end{aligned}$$

then $m=M$ and $r=R$.

Then there exists a unique equilibrium solution (\bar{x}, \bar{y}) of system (17) with $\bar{x} \in [a, b]$, $\bar{y} \in [c, d]$. Also every solution of system (17) converges to (\bar{x}, \bar{y}) as n approaches ∞ .

Proposition 3.2 Consider the system (1) such that $Q = \frac{\beta_1\beta_2}{A_1A_2} < 1$. Also let $A_1 > \alpha_1$, $A_2 > \alpha_2$ and

$$\frac{\left[\frac{B_1P_1}{1-Q} + \beta_1 \right] \left[\frac{B_2P_2}{1-Q} + \beta_2 \right]}{[A_1 - \alpha_1][A_2 - \alpha_2]} < 1.\tag{18}$$

Then the system (1) has a unique positive equilibrium (\bar{x}, \bar{y}) . Also every positive solution of system (1) converges to (\bar{x}, \bar{y}) . In otherwords, the unique positive equilibrium (\bar{x}, \bar{y}) of the system (1) is a global attractor.

Proof: Let $f : I_1 \times I_1 \times I_2 \times I_2 \rightarrow I_1$ and $g : I_1 \times I_1 \times I_2 \times I_2 \rightarrow I_2$ be continuous functions such that

$$f(u_1, u_2, v_1, v_2) = \frac{\alpha_1 e^{-u_1} + \beta_1 v_2}{A_1 + B_1 v_1 + C_1 u_2}$$

and

$$g(u_1, u_2, v_1, v_2) = \frac{\alpha_2 e^{-v_1} + \beta_2 u_2}{A_2 + B_2 u_1 + C_2 v_2}$$

where $I_1 = \left[0, \frac{P_1}{1-Q}\right]$ and $I_2 = \left[0, \frac{P_2}{1-Q}\right]$.

Let now $(m, M, r, R) \in I_1^2 \times I_2^2$ satisfies the following,

$$m = \frac{\alpha_1 e^{-M} + \beta_1 r}{A_1 + B_1 R + C_1 M} \quad M = \frac{\alpha_1 e^{-m} + \beta_1 R}{A_1 + B_1 r + C_1 m} \quad (19)$$

$$r = \frac{\alpha_2 e^{-R} + \beta_2 m}{A_2 + B_2 M + C_2 R} \quad R = \frac{\alpha_2 e^{-r} + \beta_2 M}{A_2 + B_2 m + C_2 r} \quad (20)$$

$$m \leq M, \quad r \leq R. \quad (21)$$

From relation (19), we have

$$A_1[M - m] + B_1[Mr - Rm] = \alpha_1 [e^{-m} - e^{-M}] + \beta_1[R - r].$$

This implies

$$\begin{aligned} [A_1 + B_1 r][M - m] - B_1 m[R - r] &= \alpha_1 [e^{-m} - e^{-M}] + \beta_1[R - r] \\ [A_1 + B_1 r][M - m] - B_1 m[R - r] &= \alpha_1 e^{-(m+M)} [e^M - e^m] + \beta_1[R - r]. \end{aligned} \quad (22)$$

Furthermore, there exists a $\theta_1, m \leq \theta_1 \leq M$ such that

$$e^M - e^m = e^{\theta_1} [M - m]. \quad (23)$$

Using (22) and (23) we get

$$[A_1 + B_1 r][M - m] = \alpha_1 e^{-(m+M-\theta_1)} [M - m] + [B_1 m + \beta_1][R - r].$$

This implies

$$[A_1 - \alpha_1][M - m] \leq [B_1 m + \beta_1][R - r].$$

Hence,

$$[M - m] \leq \frac{\left[\frac{B_1 P_1}{1 - Q} + \beta_1 \right]}{[A_1 - \alpha_1]} [R - r]. \quad (24)$$

From relation (20), we have

$$A_2[R - r] + B_2[mR - Mr] = \alpha_2 [e^{-r} - e^{-R}] + \beta_2[M - m].$$

This implies

$$\begin{aligned} [A_2 + B_2 m][R - r] - B_2 r[M - m] &= \alpha_2 [e^{-r} - e^{-R}] + \beta_2[M - m] \\ [A_2 + B_2 m][R - r] - B_2 r[M - m] &= \alpha_2 e^{-(r+R)} [R - r] + \beta_2[M - m]. \end{aligned} \quad (25)$$

Furthermore, there exists a $\theta_2, r \leq \theta_2 \leq R$ such that

$$e^R - e^r = e^{\theta_2} [R - r]. \quad (26)$$

Using (25) and (26) we get

$$[A_2 + B_2 m][R - r] = \alpha_2 e^{-(r+R-\theta_2)} [R - r] + [B_2 r + \beta_2][M - m].$$

This implies

$$[A_2 - \alpha_2][R - r] \leq [B_2 r + \beta_2][M - m].$$

Hence,

$$[R - r] \leq \frac{\left[\frac{B_2 P_2}{1 - Q} + \beta_2 \right]}{[A_2 - \alpha_2]} [M - m]. \quad (27)$$

Relations (24) and (27) imply that

$$[M - m] \leq \frac{\left[\frac{B_1 P_1}{1 - Q} + \beta_1 \right] \left[\frac{B_2 P_2}{1 - Q} + \beta_2 \right]}{[A_1 - \alpha_1][A_2 - \alpha_2]} [M - m] \quad (28)$$

and

$$[R - r] \leq \frac{\left[\frac{B_1 P_1}{1 - Q} + \beta_1 \right] \left[\frac{B_2 P_2}{1 - Q} + \beta_2 \right]}{[A_1 - \alpha_1][A_2 - \alpha_2]} [R - r]. \quad (29)$$

Therefore from (18),(28) and (29), we have $M = m$ and $R = r$. Consequently by Lemma 3.1 , the system (1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of system (1) tends to (\bar{x}, \bar{y}) . This completes the proof.

Proposition 3.3 Consider the system (1) where $Q = \frac{\beta_1\beta_2}{A_1A_2} < 1$ and the condition (18) hold true. Also suppose that $A_1 > \alpha_1, A_2 > \alpha_2$ and

$$\begin{aligned} & \frac{\alpha_1}{A_1} + \frac{\alpha_2}{A_2} + \frac{\alpha_1\alpha_2}{A_1A_2} + \frac{\beta_1\beta_2}{A_1A_2} + \frac{[B_1B_2 + C_1C_2]}{A_1^2A_2^2} \left[\alpha_1 + \frac{\beta_1P_2}{1-Q} \right] \left[\alpha_2 + \frac{\beta_2P_1}{1-Q} \right] \\ & + \left(\frac{B_1\beta_2}{A_1^2A_2} + \frac{C_1}{A_1^2} + \frac{\alpha_2C_1}{A_1^2A_2} \right) \left[\alpha_1 + \frac{\beta_1P_2}{1-Q} \right] + \left(\frac{B_2\beta_1}{A_1A_2^2} + \frac{C_2}{A_2^2} + \frac{\alpha_1C_2}{A_1A_2^2} \right) \left[\alpha_2 + \frac{\beta_2P_1}{1-Q} \right] < 1 \end{aligned} \tag{30}$$

then the unique positive equilibrium (\bar{x}, \bar{y}) of the system (1) is globally asymptotically stable.

Proof: The linearized system of (1) about (\bar{x}, \bar{y}) is

$$\begin{aligned} x_{n+1} &= R_1x_n + R_2y_n + R_3y_{n-1} + R_4x_{n-k} \\ y_{n+1} &= T_1x_n + T_2y_n + T_3x_{n-1} + T_4y_{n-k} \end{aligned} \tag{31}$$

where

$$\begin{aligned} R_1 &= \frac{-\alpha_1e^{-\bar{x}}}{A_1 + B_1\bar{y} + C_1\bar{x}} & R_2 &= \frac{-B_1[\alpha_1e^{-\bar{x}} + \beta_1\bar{y}]}{[A_1 + B_1\bar{y} + C_1\bar{x}]^2} \\ R_3 &= \frac{\beta_1}{A_1 + B_1\bar{y} + C_1\bar{x}} & R_4 &= \frac{-C_1[\alpha_1e^{-\bar{x}} + \beta_1\bar{y}]}{[A_1 + B_1\bar{y} + C_1\bar{x}]^2} \\ T_1 &= \frac{-B_2[\alpha_2e^{-\bar{y}} + \beta_2\bar{x}]}{[A_2 + B_2\bar{x} + C_2\bar{y}]^2} & T_2 &= \frac{-\alpha_2e^{-\bar{y}}}{A_2 + B_2\bar{x} + C_2\bar{y}} \\ T_3 &= \frac{\beta_2}{A_2 + B_2\bar{x} + C_2\bar{y}} & T_4 &= \frac{-C_2[\alpha_2e^{-\bar{y}} + \beta_2\bar{x}]}{[A_2 + B_2\bar{x} + C_2\bar{y}]^2} \end{aligned}$$

Notice that the system (31) is equivalent to the following system

$$W_{n+1} = JW_n,$$

where

$$J = \begin{pmatrix} R_1 & R_2 & 0 & R_3 & 0 & 0 & \cdots & 0 & R_4 & 0 \\ T_1 & T_2 & 0 & T_3 & 0 & 0 & \cdots & 0 & 0 & T_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix} \text{ and } W_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ \cdots \\ x_{n-k} \\ y_{n-k} \end{pmatrix}.$$

Then the characteristic equation corresponding to the matrix J is

$$\begin{aligned} \lambda^{2k+2} - (R_1 + T_2)\lambda^{2k+1} + (R_1T_2 - R_2T_1)\lambda^{2k} \\ - (R_2T_3 + R_3T_1)\lambda^{2k-1} - R_3T_3\lambda^{2k-2} - (R_4 + T_4)\lambda^{k+1} \\ + (R_1T_4 + R_4T_2)\lambda^k + R_4T_4 = 0 \end{aligned} \quad (32)$$

Now, by using (30) and Lemma (2.3) we have

$$\begin{aligned}
 & |R_1 + T_2| + |R_1 T_2 - R_2 T_1| + |R_2 T_3 + R_3 T_1| + |R_3 T_3| + |R_4 + T_4| + |R_1 T_4 + R_4 T_2| + |R_4 T_4| \\
 & \leq |R_1| + |T_2| + |R_1||T_2| + |R_2||T_1| + |R_2||T_3| + |R_3||T_1| + |R_3||T_3| \\
 & \quad + |R_4| + |T_4| + |R_1||T_4| + |R_4||T_2| + |R_4||T_4| \\
 & \leq \frac{\alpha_1 e^{-\bar{x}}}{A_1 + B_1 \bar{y} + C_1 \bar{x}} + \frac{\alpha_2 e^{-\bar{y}}}{A_2 + B_2 \bar{x} + C_2 \bar{y}} + \frac{\alpha_1 \alpha_2 e^{-(\bar{x} + \bar{y})}}{[A_1 + B_1 \bar{y} + C_1 \bar{x}][A_2 + B_2 \bar{x} + C_2 \bar{y}]} \\
 & \quad + \frac{B_1 B_2 [\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}][\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}]^2 [A_2 + B_2 \bar{x} + C_2 \bar{y}]^2} + \frac{B_1 \beta_2 [\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}]^2 [A_2 + B_2 \bar{x} + C_2 \bar{y}]} \\
 & \quad + \frac{\beta_1 B_2 [\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}][A_2 + B_2 \bar{x} + C_2 \bar{y}]^2} + \frac{\beta_1 \beta_2}{[A_1 + B_1 \bar{y} + C_1 \bar{x}][A_2 + B_2 \bar{x} + C_2 \bar{y}]} \\
 & \quad + \frac{C_2 [\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}]}{[A_2 + B_2 \bar{x} + C_2 \bar{y}]^2} + \frac{C_1 [\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}]^2} + \frac{C_2 \alpha_1 e^{-\bar{x}} [\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}][A_2 + B_2 \bar{x} + C_2 \bar{y}]^2} \\
 & \quad + \frac{C_1 \alpha_2 e^{-\bar{y}} [\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}]^2 [A_2 + B_2 \bar{x} + C_2 \bar{y}]} + \frac{C_1 C_2 [\alpha_1 e^{-\bar{x}} + \beta_1 \bar{y}][\alpha_2 e^{-\bar{y}} + \beta_2 \bar{x}]}{[A_1 + B_1 \bar{y} + C_1 \bar{x}]^2 [A_2 + B_2 \bar{x} + C_2 \bar{y}]^2} \\
 & \leq \frac{\alpha_1}{A_1} + \frac{\alpha_2}{A_2} + \frac{\alpha_1 \alpha_2}{A_1 A_2} + \frac{B_1 B_2}{A_1^2 A_2^2} [\alpha_1 + \beta_1 \bar{y}][\alpha_2 + \beta_2 \bar{x}] + \frac{B_1 \beta_2}{A_1^2 A_2} [\alpha_1 + \beta_1 \bar{y}] \\
 & \quad + \frac{B_2 \beta_1}{A_1 A_2^2} [\alpha_2 + \beta_2 \bar{x}] + \frac{\beta_1 \beta_2}{A_1 A_2} + \frac{C_2}{A_2^2} [\alpha_2 + \beta_2 \bar{x}] + \frac{C_1}{A_1^2} [\alpha_1 + \beta_1 \bar{y}] \\
 & \quad + \frac{\alpha_1 C_2}{A_1 A_2^2} [\alpha_2 + \beta_2 \bar{x}] + \frac{\alpha_2 C_1}{A_1^2 A_2} [\alpha_1 + \beta_1 \bar{y}] + \frac{C_1 C_2}{A_1^2 A_2^2} [\alpha_1 + \beta_1 \bar{y}][\alpha_2 + \beta_2 \bar{x}] \\
 & = \frac{\alpha_1}{A_1} + \frac{\alpha_2}{A_2} + \frac{\alpha_1 \alpha_2}{A_1 A_2} + \frac{\beta_1 \beta_2}{A_1 A_2} + \frac{[B_1 B_2 + C_1 C_2]}{A_1^2 A_2^2} [\alpha_1 + \beta_1 \bar{y}][\alpha_2 + \beta_2 \bar{x}] \\
 & \quad + \left(\frac{B_1 \beta_2}{A_1^2 A_2} + \frac{C_1}{A_1^2} + \frac{\alpha_2 C_1}{A_1^2 A_2} \right) [\alpha_1 + \beta_1 \bar{y}] + \left(\frac{B_2 \beta_1}{A_1 A_2^2} + \frac{C_2}{A_2^2} + \frac{\alpha_1 C_2}{A_1 A_2^2} \right) [\alpha_2 + \beta_2 \bar{x}] \\
 & \leq \frac{\alpha_1}{A_1} + \frac{\alpha_2}{A_2} + \frac{\alpha_1 \alpha_2}{A_1 A_2} + \frac{\beta_1 \beta_2}{A_1 A_2} + \frac{[B_1 B_2 + C_1 C_2]}{A_1^2 A_2^2} \left[\alpha_1 + \frac{\beta_1 P_2}{1 - Q} \right] \left[\alpha_2 + \frac{\beta_2 P_1}{1 - Q} \right] \\
 & \quad + \left(\frac{B_1 \beta_2}{A_1^2 A_2} + \frac{C_1}{A_1^2} + \frac{\alpha_2 C_1}{A_1^2 A_2} \right) \left[\alpha_1 + \frac{\beta_1 P_2}{1 - Q} \right] + \left(\frac{B_2 \beta_1}{A_1 A_2^2} + \frac{C_2}{A_2^2} + \frac{\alpha_1 C_2}{A_1 A_2^2} \right) \left[\alpha_2 + \frac{\beta_2 P_1}{1 - Q} \right] \\
 & < 1. \tag{33}
 \end{aligned}$$

Therefore, from relation (33) and by Theorem 1.6 of [8] all the roots of equation (32) satisfy $|\lambda| < 1$. Hence by Linearized Stability Theorem the unique equilibrium point (\bar{x}, \bar{y}) of system (1) is locally asymptotically stable. Using Proposition (3.2), it can be seen that (\bar{x}, \bar{y}) is globally asymptotically stable. This completes the proof of the Proposition.

4 Numerical Analysis

This part presents several computational illustrations to confirm the analytical findings and reinforce the theoretical analysis. These cases demonstrate various qualitative dynamics of the model described by (1).

Example 4.1 Consider the system (1) with parameters $\alpha_1 = 2, \beta_1 = 4, \alpha_2 = 3, \beta_2 = 2.4, A_1 = 3, A_2 = 3.6, B_1 = 2.5, B_2 = 0.5, C_1 = 0.3, C_2 = 1.3$ for $k = 4$. Let the initial values be $x_0 = 7.5, x_{-1} = 7.7, x_{-2} = 8.7, x_{-3} = 4.4, x_{-4} = 4.1$, and $y_0 = 5.1, y_{-1} = 5.1, y_{-2} = 6.1, y_{-3} = 2.6, y_{-4} = 2.3$. Figure 1 shows that the difference system is bounded and persists as given in Lemma 2.1 and 2.2. Moreover for the given parametric values the solution of the system is eventually 2 - periodic.

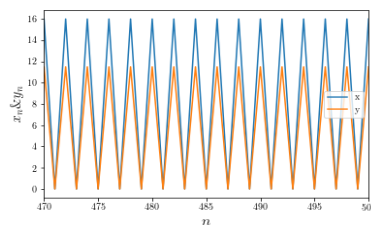


Figure 1: Plot of (x_n, y_n)

Example 4.2 Following figures represent the bifurcation diagrams of system (1) for $k = 4$ and with the initial values given as $(x_0, y_0) = (4.1, 2.3), (x_1, y_1) = (4.4, 2.6), (x_2, y_2) = (8.7, 6.1), (x_3, y_3) = (7.7, 5.1), (x_4, y_4) = (7.5, 5.1)$ in each case.

- (a) Figure 2 represents the bifurcation diagram of (1) as β_1 as the bifurcation parameter. Here $\alpha_1 = 2, \alpha_2 = 3, \beta_2 = 2.4, A_1 = 3, A_2 = 3.6, B_1 = 2.5, B_2 = 0.5, C_1 = 0.3, C_2 = 1.3$
- (b) Figure 3 represents the bifurcation diagram of (1) as β_2 as the bifurcation parameter. Here $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 2.4, A_1 = 3, A_2 = 3.6, B_1 = 2.5, B_2 = 0.5, C_1 = 0.3, C_2 = 1.3$.
- (c) Figure 4 represents the bifurcation diagram of (1) as A_1 as the bifurcation parameter. Here $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 2.4, \beta_2 = 3, A_2 = 3.6, B_1 = 2.5, B_2 = 0.5, C_1 = 0.3, C_2 = 1.3$.
- (d) Figure 5 represents the bifurcation diagram of (1) as C_2 as the bifurcation parameter. Here $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 2.4, \beta_2 = 3, A_1 = 3, A_2 = 3.6, B_1 = 2.5, B_2 = 0.5, C_1 = 0.3$.

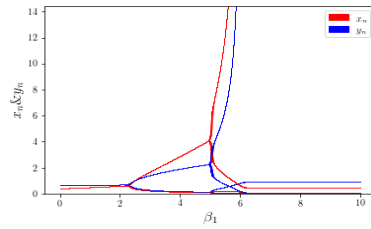


Figure 2: Bifurcaion diagram of (1) with β_1 as bifurcation parameter

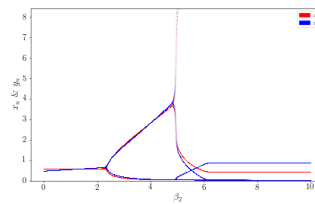


Figure 3: Bifurcaion diagram of (1) with β_2 as bifurcation parameter

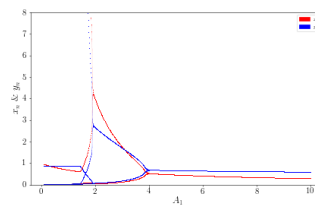


Figure 4: Bifurcaion diagram of (1) with A_1 as bifurcation parameter

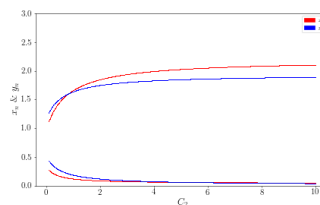


Figure 5: Bifurcaion diagram of (1) with C_2 as bifurcation parameter

5 Conclusion

In this study, we have analyzed the asymptotic behavior of a two-species higher-order rational difference model that incorporates cross-species dependence. Our primary focus was on the asymptotic stability, persistence, and boundedness of the solutions to the system. By developing a $(k + 1)$ -th order difference system model, we have examined how the population dynamics of one species, at different generations, influence the growth of the other species, and vice versa. We have shown that under appropriate conditions, the solutions of the system remain bounded and exhibit persistence over time, ensuring that neither species goes extinct under normal conditions. Furthermore, our analysis has provided insights into the equilibrium points of the system, identifying conditions for the global asymptotic stability of these points. Numerical analysis have been introduced to illustrate the system's qualitative behavior. These simulations reveal the intricate dynamics of the species, demonstrating different patterns of convergence, oscillations, and other phenomena influenced by the interplay between the species' populations across generations. In conclusion, this work provides a theoretical framework for analyzing the behavior of higher-order difference systems, particularly systems that involve delayed interactions between species.

References

- [1] Dilip DS, Adem Kilicman and Sibi C. Babu, Asymptotic and Boundedness Behavior of a Rational Difference Equation, *Journal of Difference Equations and Applications*, 25(3), 305-312, 2019.
- [2] Dilip DS and Tony Philip, Global Stability, Periodicity and Boundedness Behavior of a Difference Equation, *Southeast Asian Bulletin of Mathematics*, 44(3), 315-323, 2020.
- [3] Dilip DS , Smitha Mary Mathew and E M Elsayed, Asymptotic and Boundedness Behaviour of a Second Order Difference Equation, *Journal of Computational Mathematica*, 4(2), 68-77, 2020.
- [4] Dilip DS and Smitha Mary Mathew, Dynamics of a Second Order Nonlinear Difference System with Exponents, *Journal of the Egyptian Mathematical Society*, 29:10, (2021).

- [5] Dilip DS and Sibi C Babu, Rectangular p -periodicity and p_q -periodicity of a rational difference system, Examples and counterexamples, 1, 100004, (2021).
- [6] De Vault R, Kosmala W, Ladas G and Schultz SW, Global Behavior of $y_{n+1} = \frac{p + y_{n-k}}{qy_n + y_{n-k}}$, Nonlinear Analysis, 47, 4743 - 4751(2001).
- [7] El-Metwally H, Grove EA, Ladas G, Levins R and Radin M, On the Difference Equation $x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}$, Nonlinear Analysis, 47(7), 4623 - 4634(2001).
- [8] Grove EA and Ladas G, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC, 2005.
- [9] Khan AQ and Qureshi MN, Behaviour of an Exponential system of Difference Equations, Discrete Dynamics in Nature and Society, 2014, DOI: [dx.doi.org/10.1155/2014/607281](https://doi.org/10.1155/2014/607281).
- [10] Michael A Radin, Periodic Character and Patterns of Recursive Sequences, Springer, 2018.
- [11] Muhammad Naeem Qureshi, Qadeer Khan A and Qamar Din, Asymptotic Behavior of a Nicholson-Bailey Model, Advances in Difference Equations, 2014:62, (2014).
- [12] Papaschinopoulos G, Schinas CJ and Stefanidou G, On a k -Order System of Lyness Type Difference Equations, Advances in Difference Equations, 31272, (2007).
- [13] Ozturk I, Bozkurt F and Ozen S, On the Difference Equation $y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}$, Applied Mathematics and Computations, 181, 1387 - 1393 (2006).
- [14] Ozturk I, Bozkurt F, Ozen S, Global asymptotic behaviour of the difference equation $y_{n+1} = \frac{\alpha e^{-(ny_n + (n-k)y_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}}$, Applied Mathematics Letters, 22, 595-599(2009).
- [15] Papaschinopoulos G, Radin MA and Schinas CJ, On the system of Two Difference Equations of Exponential Form, Mathematical and Computer Modelling, 54, 2969 - 2977(2011).
- [16] Papaschinopoulos G and Schinas CJ, On the Dynamics of two Exponential Type Systems of Difference Equations, Computers and Mathematics with Applications, 64, 2326 - 2334(2012).

- [17] Papaschinopoulos G, Ellina G and Papadopoulos KB, Asymptotic Behavior of the Positive Solutions of an Exponential Type System of Difference Equations, *Applied Mathematics and Computation*, 245 , 181 - 190(2014).
- [18] Papaschinopoulos G, Fotiades n and Schinas CJ, On a System of Difference Equations Including Negative Exponential Terms, *Journal of Difference Equations and Applications*, 20(5-6), 717 - 732, (2014).
- [19] Psarros N and Papaschinopoulo G, Long-term Behavior of Positive Solutions of an Exponentially Self-regulating System of Difference Equations, *International Journal of Biomathematics*, 10(3), 1750045, (2017).
- [20] Papaschinopoulos G, Schinas CJ and Ellina G, On the Dynamics of the Solutions of a Biological Model, *Journal of Difference Equations and Applications*, 20(5 - 6), 694 - 705, (2014).
- [21] Tilman D and Wedin D, Oscillations and Chaos in the Dynamics of a Perennial Grass, *Letters to Nature*, 353 , 653 - 655(1991).
- [22] Saber N. Elaydi, *Discrete Chaos with Applications in Science and Engineering*, Chapman & Hall/CRC Press, 2007.
- [23] Sibi C. Babu, Dilip DS and Smitha Mary Mathew, Behavior of solutions of a discrete population model with mutualistic interaction, *Computational and Mathematical Biophysics*, 2024; 12:20230121.
- [24] Sibi C. Babu and Dilip DS, On the Discrete Population Models Without and With the Influence of $(k + 1)^{th}$ Previous Generation of Another Species, *Int.J.Appl.Comput.Math* 11,55 (2025).